VOLUME OF INTERSECTION OF SIX SPHERES: A SPECIAL CASE OF PRACTICAL INTEREST

L. Chkhartishvili [1], S. G. Narasimhan [2]

[1] Georgian Technical University
Department of Engineering Physics
Tbilisi, Georgia
chkharti2003@yahoo.con
[2] Shanmugha Arts, Science, Technology & Research Academy
(SASTRA University)
School of Mechanical Engineering
Thanjavur, India
gnaras15@gmail.com

Accepted April 30, 2015

Abstract

A number of scientific and technical problems need the calculation of the intersection volume of spheres with different radii and distances between centers. Present work aims to summarize previous attempts made in this direction and solve the six-sphere problem for a special case of practical interest.

1. Problem of intersection volume of spheres

1.1. A single sphere

The problem of intersection volume of spheres is reduced to a trivial problem of classical geometry [1]: determination of the volume V of the sphere with given radius R_1, which as a whole is contained inside the intersection of all other spheres:

$$V(R_1) = \frac{4\pi R_1^3}{3}. \qquad (1)$$

1.2. Two spheres

In the engineering practice, e.g., for determining the moments of inertia of machine parts [2], the volume of intersection of two spheres with given radii R_1 and R_2, and distance D_{12} between their centers are calculated numerically. As for the general solution of this geometrical problem, it was done by us analytically [3] in relation to the physical problem of calculation of crystalline band structures within an approximation of quasi-classical type. The matrix elements of the secular equation determining the band structure were expressed by the volume of intersection of two spheres and then the corresponding formula was obtained:

$$V(R_1, R_2, D_{i12}) = \qquad (2)$$

$$= \frac{4\pi R_1^3}{3}, \qquad D_{i12} < R_2 - R_1$$

$$= \frac{4\pi R_2^3}{3}, \qquad D_{i12} < R_1 - R_2$$

$$= \frac{\pi(R_1 + R_2 - D_{12})^2((R_1 + R_2 + D_{12})^2 - 4(R_1^2 - R_1R_2 + R_2^2))}{12D_{i12}} \qquad |R_1 - R_2| \le D_{12} \le R_1 + R_2$$

$$= 0 \qquad D_{12} > R_1 + R_2.$$

One can see that, $V(R_1, R_2, D_{i12})$ is a piecewise algebraic function. It and its partials derivative $\partial V(R_1, R_2, D_{i12}) / \partial D_{12}$ in the argument D_{12} are continuous at both of boundaries of analyticity $D_{i12} = |R_1 - R_2|$ and $D_{12} = R_1 + R_2$. As for the partial derivatives of higher-orders, they are discontinuous functions.

1.3. Three spheres

It seems that, before 1997 there were made no attempts to determine the volume of the intersection of three spheres in general – as an explicit function of their radii R_1, R_2, R_3 and distances D_{12}, D_{13}, D_{23} between centers. Although, it was considered a several cases, when spheres' parameters are related to each other in a special way [2, 4, 5]; and Gibson & Scheraga [6, 7] summarizing the previous efforts performed in this direction had generalized Powell's formula [4] for the spheres of equal radii. The expressions were derived not only for the case of spheres of unequal size that insect in pair of points (**Figure 1**), but also for other types of triple intersection. However, we should note that these expressions being analytical at the same time are only implicit representations of the function $V(R_1, R_2, R_3, D_{i12}, D_{13}, D_{23})$.

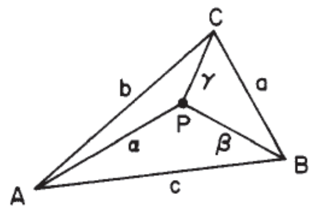
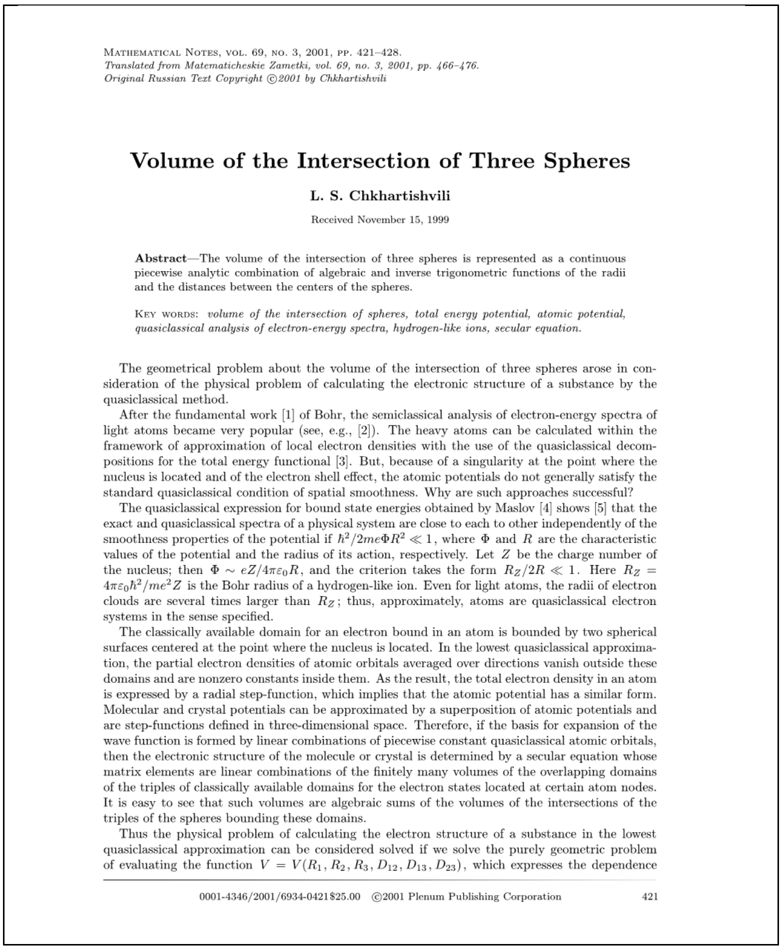


Figure 1. Representation of tetrahedron formed by centers of spheres at points A, B, C and point P, which is one of two points common to surfaces of all three spheres [6]. Here: $R_1 \equiv \alpha$, $R_2 \equiv \beta$, $R_3 \equiv \gamma$, $D_{12} \equiv c$, $D_{13} \equiv b$, $D_{23} \equiv a$.

In 1997, developing the method of analysis of crystalline electronic energy spectrum based on the quasi-classically calculated characteristics of atomic orbitals, Chkhartishvili had demonstrated [8] that the energy and configuration characteristics of quasi-classical electronic states of crystals can be expressed by the analytical combinations of elliptic integrals. In particular, the volume of intersection of three spheres was expressed by the linear combination of elliptic integrals. In 2001, Chkhartishvili had found out [9] that (**Figures 2** and **3**), the integrand part standing under the square root is a perfect square. This finding allowed to conduct the integration in elementary functions and express the required volume as a piecewise analytic combination of algebraic and inverse trigonometric functions.

Figure 2. “Three apples”. Cover of 1995 August 21 issue of magazine ‘Chemical & Engineering News’.



MATHEMATICAL NOTES, VOL. 69, NO. 3, 2001, PP. 421–428.
Translated from Matematicheskie Zametki, vol. 69, no. 3, 2001, pp. 466–476.

Volume of the Intersection of Three Spheres

L. S. Chkhartishvili

Received November 15, 1999

Abstract—The volume of the intersection of three spheres is represented as a continuous piecewise analytic combination of algebraic and inverse trigonometric functions of the radii and the distances between the centers of the spheres.

KEY WORDS: *volume of the intersection of spheres, total energy potential, atomic potential, quasiclassical analysis of electron-energy spectra, hydrogen-like ions, secular equation.*

The geometrical problem about the volume of the intersection of three spheres arose in consideration of the physical problem of calculating the electronic structure of a substance by the quasiclassical method.

After the fundamental work [1] of Bohr, the semiclassical analysis of electron-energy spectra of light atoms became very popular (see, e.g., [2]). The heavy atoms can be calculated within the framework of approximation of local electron densities with the use of the quasiclassical decompositions for the total energy functional [3]. But, because of a singularity at the point where the nucleus is located and of the electron shell effect, the atomic potentials do not generally satisfy the standard quasiclassical condition of spatial smoothness. Why are such approaches successful?

The quasiclassical expression for bound state energies obtained by Maslov [4] shows [5] that the exact and quasiclassical spectra of a physical system are close to each to other independently of the smoothness properties of the potential if $\hbar^2/2me\Phi R^2 \ll 1$, where Φ and R are the characteristic values of the potential and the radius of its action, respectively. Let Z be the charge number of the nucleus; then $\Phi \sim eZ/4\pi\varepsilon_0 R$, and the criterion takes the form $R_Z/2R \ll 1$. Here $R_Z = 4\pi\varepsilon_0\hbar^2/me^2Z$ is the Bohr radius of a hydrogen-like ion. Even for light atoms, the radii of electron clouds are several times larger than R_Z; thus, approximately, atoms are quasiclassical electron systems in the sense specified.

The classically available domain for an electron bound in an atom is bounded by two spherical surfaces centered at the point where the nucleus is located. In the lowest quasiclassical approximation, the partial electron densities of atomic orbitals averaged over directions vanish outside these domains and are nonzero constants inside them. As the result, the total electron density in an atom is expressed by a radial step-function, which implies that the atomic potential has a similar form. Molecular and crystal potentials can be approximated by a superposition of atomic potentials and are step-functions defined in three-dimensional space. Therefore, if the basis for expansion of the wave function is formed by linear combinations of piecewise constant quasiclassical atomic orbitals, then the electronic structure of the molecule or crystal is determined by a secular equation whose matrix elements are linear combinations of the finitely many volumes of the overlapping domains of the triples of classically available domains for the electron states located at certain atom nodes. It is easy to see that such volumes are algebraic sums of the volumes of the intersections of the triples of the spheres bounding these domains.

Thus the physical problem of calculating the electron structure of a substance in the lowest quasiclassical approximation can be considered solved if we solve the purely geometric problem of evaluating the function $V = V(R_1, R_2, R_3, D_{12}, D_{13}, D_{23})$, which expresses the dependence

0001-4346/2001/6934-0421$25.00

Figure 3. First page of paper (Math. Notes, 2001, 69, 3, 421-428) bearing explicit analytical solution of problem of intersection volume of three spheres.

In particular, when intersection region is bounded by parts of all three spheres or they intersect in pair of points (**Figure 4**):

$$V(R_i, R_j, R_k; D_{ij}, D_{ik}, D_{jk}) = V_i + V_j + V_k \quad (3)$$

$$i, j, k, = 1,2,3,$$

where

$$V_i = \quad (4)$$

$$= \frac{H(2d_{ij}d_{ik} - (d_{ij}^2 + d_{ik}^2)\cos t_i)}{3\sin t_i} -$$

$$- \frac{d_{ij}(3R_i^2 - d_{ij}^2)}{3} \arccos \frac{d_{ik} - d_{ij}\cos t_i}{r_{ij}\sin t_i} -$$

$$- \frac{d_{ik}(3R_i^2 - d_{ik}^2)}{3} \arccos \frac{d_{ij} - d_{ik}\cos t_i}{r_{ik}\sin t_i} +$$

$$+ \frac{2R_i^3}{3} \arccos \frac{d_{ij}d_{ik} - R_i^2\cos t_i}{r_{ij}r_{ik}}.$$

Here H is the half-length of common chord connecting the pair of points, where all three spheres intersect:

$$H^2 = \quad (5)$$

$$= \frac{(R_i^2 R_j^2 + R_k^2 D_{ij}^2)(D_{ik}^2 + D_{jk}^2 - D_{ij}^2)}{2(D_{ij}^2 D_{ik}^2 + D_{ij}^2 D_{jk}^2 + D_{ik}^2 D_{jk}^2) - (D_{ij}^4 + D_{ik}^4 + D_{jk}^4)} +$$

$$+\frac{(R_i^2R_k^2+R_j^2D_{ik}^2)(D_{ij}^2+D_{jk}^2-D_{ik}^2)}{2(D_{ij}^2D_{ik}^2+D_{ij}^2D_{jk}^2+D_{ik}^2D_{jk}^2)-(D_{ij}^4+D_{ik}^4+D_{jk}^4)}+$$

$$+\frac{(R_j^2R_k^2+R_i^2D_{jk}^2)(D_{ij}^2+D_{ik}^2-D_{jk}^2)}{2(D_{ij}^2D_{ik}^2+D_{ij}^2D_{jk}^2+D_{ik}^2D_{jk}^2)-(D_{ij}^4+D_{ik}^4+D_{jk}^4)}-$$

$$-\frac{R_i^2D_{jk}^4+R_j^2D_{ik}^4+R_k^2D_{ij}^4+D_{ij}^2D_{ik}^2D_{jk}^2}{2(D_{ij}^2D_{ik}^2+D_{ij}^2D_{jk}^2+D_{ik}^2D_{jk}^2)-(D_{ij}^4+D_{ik}^4+D_{jk}^4)},$$

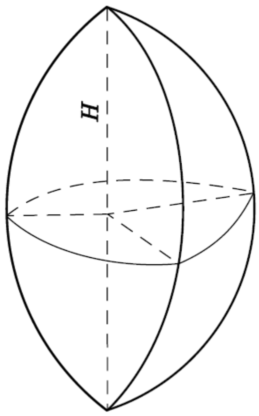


Figure 4. Intersection region, when it is bounded by parts of all three spheres or they intersect in pair of points.

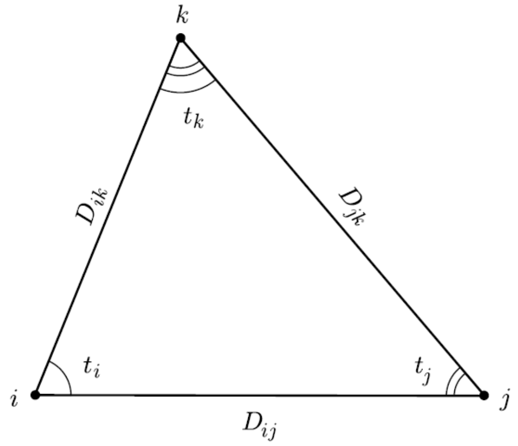


Figure 5. Line segments linking centers of spheres and angles between them.

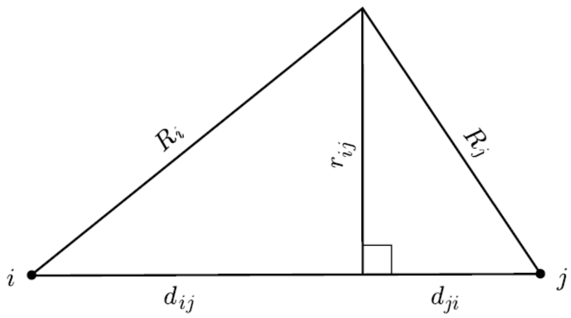


Figure 6. Radius of intersection circle between two spheres and distances between that circle and centers of spheres.

t_i is the angle between line segments linking center of i -sphere with j - and k -spheres (**Figure 5**),

$$\cos t_i = \frac{D_{ij}^2 + D_{ik}^2 - D_{jk}^2}{2D_{ij}D_{ik}} , \tag{6}$$

$r_{ij} = r_{ji}$ are the radius of the intersection circle between i - and j -spheres (**Figure 6**),

$$r_{ij} = r_{ji} = \frac{\sqrt{2(R_i^2 R_j^2 + R_i^2 D_{ij}^2 + R_j^2 D_{ij}^2) - (R_i^4 + R_j^4 + D_{ij}^4)}}{2D_{ij}} , \tag{7}$$

and d_{ij} and d_{ij} are the distances between that circle and centers of i - and j -spheres, respectively (**Figure 6**),

$$d_{ij} = \frac{R_i^2 - R_j^2 + D_{ij}^2}{2D_{ij}} \tag{8}$$

and

$$d_{ji} = \frac{R_j^2 - R_i^2 + D_{ij}^2}{2D_{ij}} . \tag{9}$$

We can see that, the $V(R_1, R_2, R_3, D_{i12}, D_{13}, D_{23})$ is a piecewise combination of algebraic and inverse trigonometric functions. It and its partials derivatives $\partial V(R_1, R_2, D_{i12}) / \partial D_{12}$, $\partial V(R_1, R_2, D_{i12}) / \partial D_{13}$, $\partial V(R_1, R_2, D_{i12}) / \partial D_{23}$, respectively, in arguments D_{12} , D_{13} , D_{23} are continuous at corresponding boundaries of analyticity. As for the partial derivatives of higher-orders, they are discontinuous functions.

In some cases, the intersection volume between three spheres is reduced to that of a pair of spheres or even a sphere from these three. The general constructing principles of an algorithm of numerical calculations, according to the obtained formulas, have been formulated in the Annex of our Monograph [10].

Above solution of the geometrical problem was provoked by the development of a quasi-classical theory of substance [11]. Within this approach, matrix elements of the secular equation, determining electronic structure of a substance, are analytically expressed by linear combinations of three-center volume integrals, each of which is reduced to a intersection volume of threes spheres. The distances between centers of spheres are the distances between atomic sites of the substance structure; and the radii of spheres are the inner or outer radii of radial layers of quasi-homogeneity of overlapping electron densities of pairs of constituent atoms and electric field potential formed by a third atom. Quasi-classical method, including obtained analytical expression of $V(R_1, R_2, R_3, D_{i12}, D_{13}, D_{23})$, was successfully tested, e.g., for boron nitride (BN) structural modifications [12].

The application of this expression seems to be useful in the old intersecting spheres model of molecules developed by Antoci [13 – 16]. Relatively recently, same formula was utilized [17] to calculate the total energy of protein structures used in biotechnology.

1.4. More than three spheres and applications

Cases of more than spheres are too difficult to be solved in general. Only attempt of such kind for four, six and twelve spheres belongs to Lustig [5]. All other available reports describe the numerical algorithms.

A fast computer algorithm was presented [18] for complete analytical calculation of van der Waals volumes. This algorithm computes overlaps of any order, not only second- and third-order atomic spheres overlaps like previously suggested analytical algorithms giving insufficient numerical approximations of the exact van der Waals volumes. It was noted that, practical situations frequently involve six-order overlaps. Computed volumes of 63 chemicals were compared with Monte Carlo measured values.

Frequently, packings of spheres serve as useful models of the geometry of many physical systems and the description of the void region (not occupied by the spheres), in general, composed of disconnected cavities is crucial. An algorithm for decomposing void space into cavities and determining the exact volumes of such cavities in 3D packings of monodisperse and polydisperse spheres was presented in [19].

In [20], it was presented a simple algorithm for the calculation of the volume of a union of spheres of different radii based on the ideas of [18]. Analytical formulas for atomic volumes were derived. This could be achieved without explicit calculation of multiple intersections of the overlapping atoms. Such ideas were implemented for the calculation of the occupied volume inside the polyhedra defined by power Voronoi diagram. This allows calculating the required values for spheres with different radii. Algorithm was applied to the calculation of the solvation shell volume for complex solute molecules in molecular dynamics models of solutions.

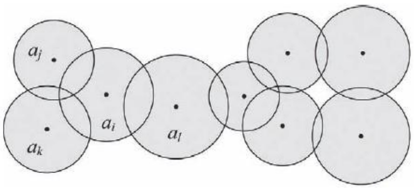


Figure 7. Beta-decomposition idea for 2D molecule [21].

Figure 8. All of 3D space volume primitives
together (including 3D atomic shells) [21].

The mass properties such as the volume of the union of the atoms – given in 3D space set of spherical balls called atoms – are important for many disciplines, particularly for computational and structural chemistry and biology. In [21], algorithms that compute the mass properties of both the union of atoms and their offsets both correctly and efficiently have been proposed. They employ the beta-decomposition approach (**Figure 7**) to decompose the target mass property – volume into a set of primitives (**Figure 8**) – using the simplexes of the beta-complex. Then, the volume is computed by appropriately summing up the volume corresponding to each simplex.

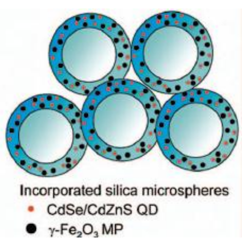


Figure 9. Magnetic and fluorescent silica microspheres fabricated by incorporating maghemite (g-Fe_2O_3) nanoparticles and CdSe / CdZnS core / shell quantum dots into silica shell around preformed silica microspheres of sizes ~ 500 nm [24].

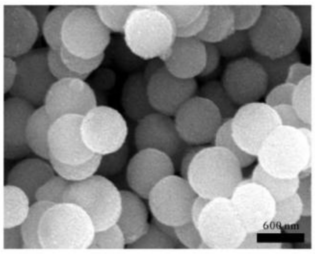


Figure 10. Scanning electron microscopy image of monodisperse hollow single crystalline magnetite microspheres synthesized at temperature 180 °C by one-step process through a template-free hydrothermal method [25].

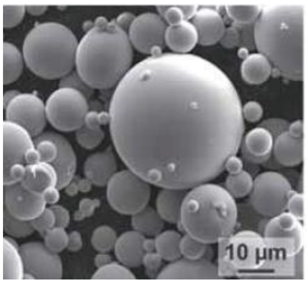


Figure 11. Scanning electron microscopy image of gas-atomized Fe-based powder with nominal composition $Fe_{74}Mo_4P_{10}C_{7.5}B_{2.5}Si_2$. The powder particle size extends from few up to 50 μm and particle surfaces are very smooth [26].

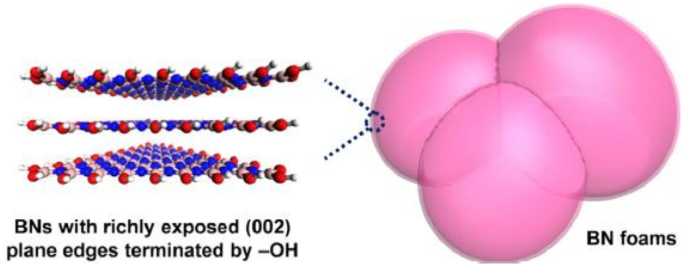


Figure 12. Schematic illustration of the preparation of BN foams with richly exposed plane edges that terminated with hydroxyl groups [27].

In general, determination of intersection volume of an union of spheres is important for morphological studies of porous / powdered materials (**Figures 9** – **12**) – see, e.g., [22 – 27].

2. Determination of constant orientation workspaces of 6-DOF parallel manipulator "Stewart platform"

Determining the volume of workspace of a Stewart platform is an important technical problem in the field of robotics because currently the 6-DOF parallel manipulator "Stewart platform" (**Figure 13**) is widely industrialized (one of its principal applications is flight simulator). The results of geometrical optimization of different workspaces of this robot [28] have to be used in various applications and also helpful in a better analysis of the Stewart platform itself. So far determination of workspaces of a 6-DOF parallel manipulator has been done only through analytical and numerical methods.

Figure 13. Stewart platform.

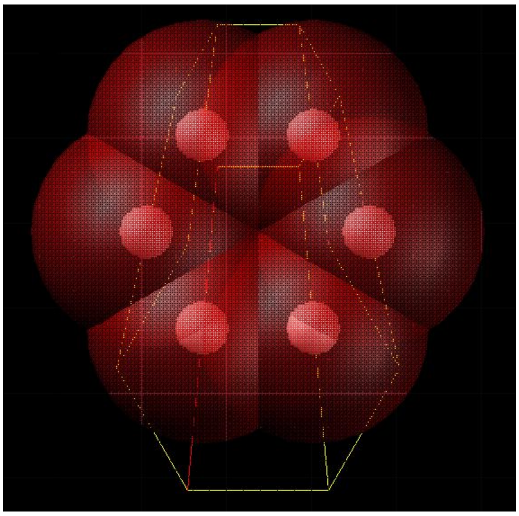


Figure 14. Stewart platform with workspace (big) spheres and void (small) spheres inside them.

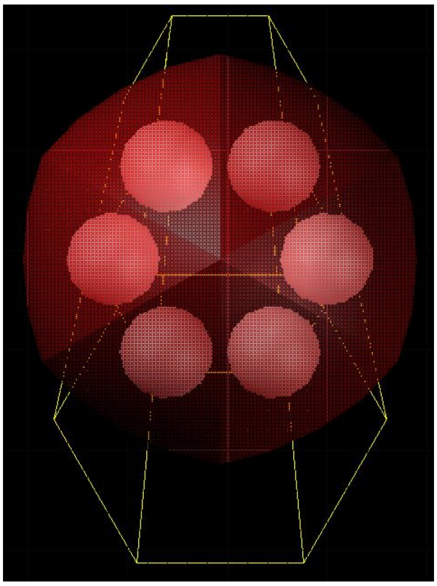


Figure 15. Final volume of intersection of six spheres, whose value needs to be find out.

In the geometrical approach, one has to try to find the workspace of the mobile platform (regular hexagon), resting on all the six legs. For each leg, its workspace is a part of a sphere whose size depends upon the dimensions of the leg or a void sphere presented inside each of these spheres (**Figure 14**). The net volume of workspace of a platform is the sum of these parts. We have found out the various combinations in which these spheres will intersect. In **Figure 15**, it is shown a specific case in combinations of spheres of the legs to achieve the theoretical workspace of the platform.

So, in general we need to calculate the volume of intersection of six spheres to determine the workspace volume. On the system under the consideration, the constraints are: (1) equal radii for all six spheres, and (2) equal distances between centers of nearest pairs of spheres, so that all centers of the spheres are in the same plane and are the vertices of a regular hexagon.

Figure 16. Intersection of six spheres
and void spheres of different radii.

Apart from the intersection of six spheres, which is the theoretical workspace, due to the combination of two legs, to which the mobile platform is attached, another (void) sphere crops up, which intersects with this six sphere intersection. Then, the real workspace, through which the platform moves, is the six sphere intersection minus the void sphere intersection. Thus, there are many specific cases, where to compute the final volume we have to find out the volume of intersection of the intersection of six spheres of equal radii with another sphere of a different radius. In the **Figure 16**, the volume of intersection of six spheres and the spheres of different radii, whose intersection with the former should be calculated, are shown together.

3. Volume of intersection of six spheres in special case

3.1. Volume of intersection of six spheres of equal radii centered at vertices of regular hexagon

Thus, we consider six spheres with equal radii of R centered at the vertices of a regular hexagon with sides of D. Denote their intersection volume by the function of these two variables: $V = V(R,D)$. From the **Figure 18**, we can see that at $R \leq D$ a pair of spheres centered at the ends of a hexagon's large diagonal do not intersect and then the required volume identically equals to zero: $V(R,D) \equiv 0$.

The volume $V(R,D) > 0$ only at $R > D$. Let's analyze this case. Assume that centers of all the spheres are placed in the xOy plane, i.e., the Oz axis is perpendicular to the hexagon plane. When its center coincides with the origin O, at the altitude of z the intersection of a sphere with the plane parallel to the xOy plane is a circle of radius

$$R(z) = \sqrt{R^2 - z^2} \quad (10)$$

(**Figure 17**). At $z \neq 0$, spheres are intersected if $R(z) > D$, which is a condition analogous to the condition at $z = 0$.

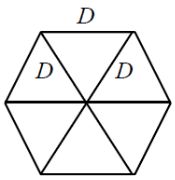


Figure 17. Regular hexagon with sides of D.

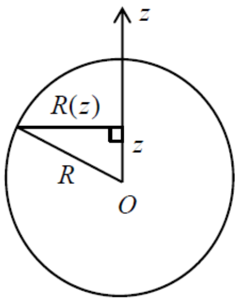


Figure 18. Radius $R(z)$ of circle of intersection of sphere with plane parallel to hexagon plane at altitude of z.

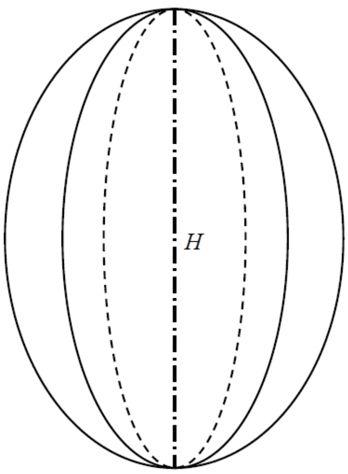


Figure 19. Intersection of six identical spheres.

The intersection of six identical spheres is a convex figure bounded by six identical fragments of all the spheres (**Figure 19**), for which Oz is the axis of symmetry of order of 6 and xOy is the plane of mirror reflection. The coordinates of the vertices of the figure are the solutions of the equation $R(z) = D$:

$$z_{\pm} = \pm\sqrt{R^2 - D^2} \quad (11)$$

and, therefore, the height of the intersection figure is

$$H = z_{+} - z_{-} = 2\sqrt{R^2 - D^2}\,. \quad (12)$$

Let $S(z)$ denotes the cross section area of the figure at the altitude of z. It is evident that the volume can be calculated as the following integral:

$$V(R,D)=\int_{z_-}^{z_+} dz\,S(z)=2\int_0^{\sqrt{R^2-D^2}} dz\,S(z). \tag{13}$$

In this section, there are six line-segments that connect the spheres' endpoints with the central point placed at the Oz axis, which divide this cross section into six identical parts. If we denote their areas by $s(z)$ we will have

$$V(R,D)=12\int_0^{\sqrt{R^2-D^2}} dz\,s(z). \tag{14}$$

Each of these plane figures are bounded by an arc of a circle of radius $R(z)$ and the two sides of an equilateral triangle, the third side of which is a chord subtending the said arc (**Figure 20**). Consequently, the area $s(z)$ is equal to the sum of areas of the circle-segment $s_{Segment}(z)$ of radius $R(z)$ and the equilateral triangle $s'_{Triangle}(z)$, one side of which coincides with the chord bounding the segment. Denote the side of the triangle, i.e., chord length, by $X(z)$. The area of the circle-segment can be calculated as the difference between the areas of the corresponding sector $s_{Sector}(z)$ and isosceles triangle $s''_{Triangle}(z)$ with slopes of $R(z)$ and base of $X(z)$. Thus we can write down:

$$\begin{aligned} s(z) &= s_{Segment}(z)+s'_{Triangle}(z)= \\ &=(s_{Sector}(z)-s''_{Triangle}(z))+s'_{Triangle}(z)= \\ &=s_{Sector}(z)-(s''_{Triangle}(z)-s'_{Triangle}(z)). \end{aligned} \tag{15}$$

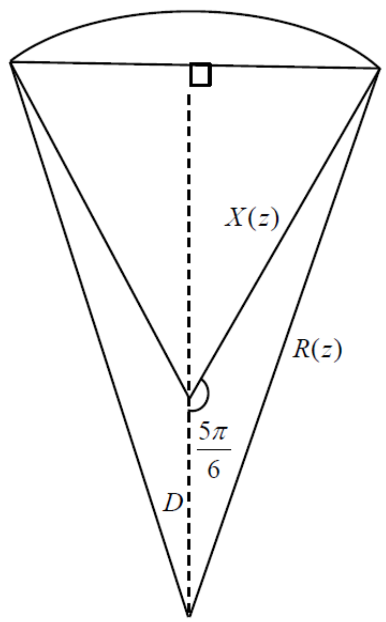


Figure 20. One of six identical parts of cross section of intersection of six identical spheres at altitude of z (I).

According to the law of cosines,

$$R^2(z)=D^2+X^2(z)-2DX(z)\cos\frac{5\pi}{6} \tag{16}$$

and then

$$X(z)=\sqrt{R^2-\frac{1}{4}D^2-z^2}-\frac{\sqrt{3}}{2}D. \qquad (17)$$

Since the base of the isosceles triangle equals to a side of the equilateral triangle $X(z)$, and the difference in heights of these triangles at any altitude is D, the difference of their areas is easily calculated:

$$s''_{Triangle}(z)-s'_{Triangle}(z)=\frac{1}{2}DX(z)=\frac{1}{2}D\left(\sqrt{R^2-\frac{1}{4}D^2-z^2}-\frac{\sqrt{3}}{2}D\right). \qquad (18)$$

It is convenient to calculate separately its contribution in the required integral:

$$V_{Triangles}=6D\int_0^{\sqrt{R^2-D^2}}dz\left(\sqrt{R^2-\frac{1}{4}D^2-z^2}-\frac{\sqrt{3}}{2}D\right)= \qquad (19)$$

$$=\frac{3}{4}D(4R^2-D^2)\arcsin 2\sqrt{\frac{R^2-D^2}{4R^2-D^2}}-\frac{3}{2}D^2\sqrt{3(R^2-D^2)}.$$

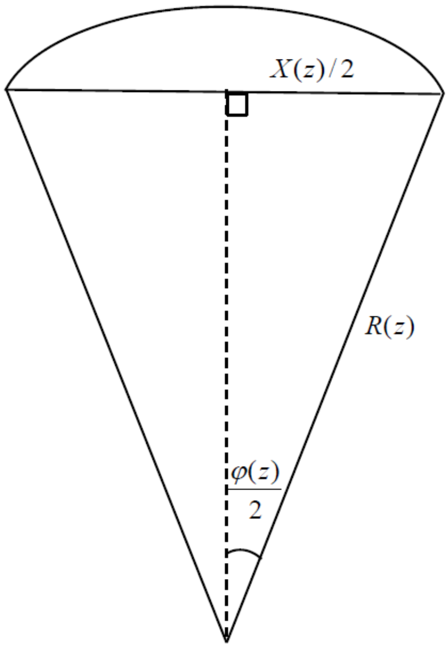


Figure 21. One of six identical parts of cross section of intersection of six identical spheres at altitude of z (II).

From **Figure 21**, we find the sector angle $\varphi(z)$:

$$\varphi(z)=2\arcsin\frac{1}{2}\frac{\sqrt{R^2-\frac{1}{4}D^2-z^2}-\frac{\sqrt{3}}{2}D}{\sqrt{R^2-z^2}}. \qquad (20)$$

Thus, area of the sector is

$$s_{Sector}(z)=\frac{1}{2}\varphi(z)R^2(z)=(R^2-z^2)\arcsin\frac{1}{2}\frac{\sqrt{R^2-\frac{1}{4}D^2-z^2}-\frac{\sqrt{3}}{2}D}{\sqrt{R^2-z^2}}. \qquad (21)$$

As for the corresponding contribution in the required integral, it equals to

$$V_{Sector} = 12 \int_0^{\sqrt{R^2-z^2}} dz\,(R^2 - z^2)\arcsin\frac{1}{2}\frac{\sqrt{R^2 - \frac{1}{4}D^2 - z^2} - \frac{\sqrt{3}}{2}D}{\sqrt{R^2 - z^2}} = \quad (22)$$

$$= 8R^3 \arcsin\sqrt{\frac{R^2 - D^2}{4R^2 - D^2}} - \frac{1}{4}D(12R^2 + D^2)\arcsin 2\sqrt{\frac{R^2 - D^2}{4R^2 - D^2}} - \frac{1}{2}D^2\sqrt{3(R^2 - D^2)}$$

and we get

$$V(R,D) = V_{Sector} - V_{Triangles} = \quad (23)$$

$$= 8R^3 \arcsin\sqrt{\frac{R^2 - D^2}{4R^2 - D^2}} - \frac{(12R^2 - D^2)D}{2}\arcsin 2\sqrt{\frac{R^2 - D^2}{4R^2 - D^2}} + D^2\sqrt{3(R^2 - D^2)}\,.$$

As expected the resulting formula gives 0 when $R = D$,

$$V(R,D)\big|_{R=D} = 0\,, \quad (24)$$

while at $R >> D$ asymptotically tends to the volume of sphere:

$$V(R,D)\big|_{R>>D} \approx \frac{4\pi R^3}{3}\,. \quad (25)$$

Finally, the problem's solution is the following expression:

$$V(R,D) = \quad (26)$$

$$= 0 \qquad R \le D$$

$$= 8R^3 \arcsin\sqrt{\frac{R^2 - D^2}{4R^2 - D^2}} - \left(6R^2 - \frac{D^2}{2}\right)D\arcsin 2\sqrt{\frac{R^2 - D^2}{4R^2 - D^2}} + D^2\sqrt{3(R^2 - D^2)} \qquad R \ge D\,.$$

3.2. Volume of intersection of six spheres of equal radii with spheres of different radii

Is the intersection of the intersection of six spheres of equal radii, centered at vertices of a regular hexagon, with the sphere of a different radius, centered in the hexagon plane, same as the intersection of two spheres of unequal radii? In general, it is not. But, for some constrains for geometric parameters, the standard formula of the volume of intersection of two spheres will work.

As above R is the radius of a sphere, one of six spheres with equal radii centered at the vertices of a regular hexagon, D is the side of that regular hexagon, and $R > D$ is the condition of existence of the intersection between these six spheres. For given R and D, the shortest distance X from the center of the regular hexagon to the surface of a sphere with radius R can be calculated as:

$$X = \sqrt{R^2 - \frac{D^2}{4}} - \frac{\sqrt{3}\,D}{2}\,. \quad (27)$$

Let's introduce two additional geometric parameters: r – the radius of the sphere that intersects with one of six spheres with radius of R and d – the distance between centers of spheres with radii R and r. Note that, here we consider the symmetric case, when centers S and s of two spheres with radii R and r, respectively, and center O of regular hexagon are located at same line (**Figure 22**).

Asymmetric cases also can be considered analytically, but relations will be significantly complicated.

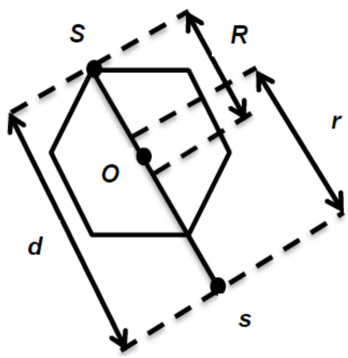


Figure 22. Radii and distances between centers of two intersected spheres.

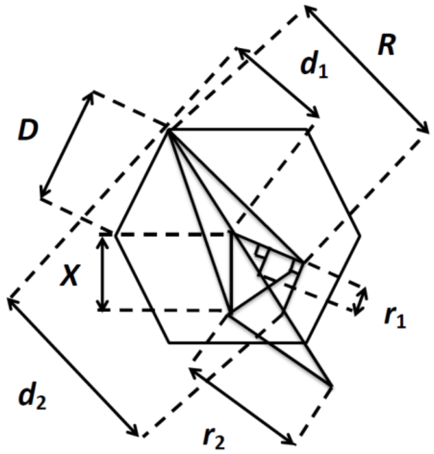


Figure 23. Critical radii and critical distances between centers of two intersected spheres.

From the **Figure 23**, one can see that there are two critical distances, d_1 and d_2, between centers of spheres with radii R and r:

$$d_1 = D \tag{28}$$

and

$$d_2 = D + \frac{2X}{\sqrt{3}}, \tag{29}$$

and correspondingly two critical values r_1 and r_2 of the sphere with radius r:

$$r_1 = \frac{d - D}{2} \tag{30}$$

and

$$r_2 = \sqrt{(d - D)^2 - \sqrt{3}\,(d - D)X + X^2}\,. \tag{31}$$

There are three different cases:

– when $0 \le d < d_1$ general formula for intersection of two spheres does not work;
– when $d_1 \le d < d_2$ general formula for intersection of two spheres works if $0 < r \le r_1$; and
– when $d_2 \le d < \infty$ general formula for intersection of two spheres works if $r_1 < r \le r_2$.

When the formula works, the volume $V(R,r,d)$ of intersection of two spheres is continuous piecewise algebraic function of variables R, r and d:

$$V(R,r,d) = \tag{32}$$

$$= \frac{4\pi r^3}{3} \qquad 0 \le d < R - r$$

$$= \frac{\pi(R+r-d)^2((R+r+d)^2 - 4(R^2 - Rr + r^2))}{12d} \qquad |R-r| \leq d < R+r$$

$$= 0 \qquad R+r \leq d < \infty .$$

Note that, the condition $0 \leq d < r - R$, when the volume of intersection of two spheres coincides with the volume $4\pi R^3/3$ of a sphere of radius R is excluded in the system under the consideration because the intersection of six spheres is only a part of a sphere with radius R.

4. Conclusions

Above described general procedure on how to determine the volume of intersection of six spheres with all the constraints was successfully applied to volume computation in a number of specific cases in the field of parallel manipulators.

Regarding the numerical application of the formula for intersection volume of two spheres in the Stewart platform problem, it has proven that in ~ 30 % of tested cases it was found to be applicable. In the rest ~ 70 % of cases, if d is between d_1 and d_2, r was found to be greater than r_1 (which is violation of a condition) and if d is itself greater than d_2, then r was greater than r_2 (violation of other condition), what makes these cases indeterminable.

References

1. A. D. Alexandrov, N. Yu. Netsvetaev. Geometry. 1990, Moscow: Nauka.
2. M. V. Favorin. Moments of Inertia of Bodies. Reference Book. 1977, Moscow: Mashinostroenie.
3. L. Chkhartishvili. Quasi-classical calculation of the crystalline band structure. Trans. Georg. Tech. Univ., 1996, 3 (411), 45-52.
4. M. J. D. Powell. The volume internal to three intersecting hard spheres. Mol. Phys., 1964, 7, 6, 591-592.
5. R. Lustig. Surface and volume of three, four, six and twelve hard fused spheres. Mol. Phys., 1985, 55, 2, 305-317.
6. K. D. Gibson, H. A. Scheraga. Volume of the intersection of three spheres of unequal size. A simplified formula. J. Phys. Chem., 1987, 91, 15, 4121-4122.
7. K. D. Gibson, H. A. Scheraga. Erratum: Volume of the intersection of three spheres of unequal size. A simplified formula (Journal of Physical Chemistry (1987) 91, (4121)). J. Phys. Chem., 1987, 91, 24, 6326-6326.
8. L. Chkhartishvili. Method of the analysis of crystalline electronic energy spectrum based on the quasi-classically calculated characteristics of atomic orbitals. Trans. Georg. Tech. Univ., 1997, 3 (414) – GTU 75 Anniv. Ed., 205-213.
9. L. S. Chkhartishvili. Volume of the intersection of three spheres. Math. Notes, 2001, 69, 3, 421-428.
10. L. Chkhartishvili. Iterative and Transcendental Solutions of Algebraic Equations (Monograph). 2012, Saarbruken: Palmarium Acad. Publ.
11. L. Chkhartishvili. Quasi-Classical Theory of Substance. 2004, Tbilisi: Tech. Univ. Press.
12. L. Chkhartishvili. Quasi-Classical Method of Calculation of Substance Structural Parameters and Electron Energy Spectrum. 2006, Tbilisi: Tbilisi Univ. Press.

13. S. Antoci. The use of discontinuous trial functions in the computation of the electronic structure of molecules: Calculation of the ESCA spectrum of tetrathiofulvalence (TTF). J. Chem. Phys., 1975, 63, 2, 697-701.
14. S. Antoci, L. Mihich. Removal of the discontinuities of the trial function in the computation of the electronic structure of molecules by the intersecting spheres model: Evaluation of the equilibrium geometry of N_2 and H_2O. J. Chem. Phys., 1976, 64, 4, 1442-1445.
15. S. Antoci. An asymptotically exact form of the intersecting spheres model of molecules: Formulation. J. Chem. Phys., 1976, 65, 1, 253-256.
16. S. Antoci, L. Barino. An asymptotically exact form of the intersecting spheres model of molecules: Computation of the total energy and the ESCA spectra of H_2^+, Li_2, N_2, F_2 and H_2O. J. Chem. Phys., 1976, 65, 1, 257-260.
17. A. Shirvani, 2009 – *Private Communication*.
18. M. Petitjean. On the analytical calculation of van der Waals surfaces and volumes: Some numerical aspects. J. Comput. Chem., 1994, 15, 5, 507-523.
19. S. Sastry, D. S. Corti, P. G. Debenedetti, F. H. Stillinger. Statistical geometry of particle packings. I. Algorithm for exact determination of connectivity, volume, and surface areas of void space in monodisperse and polydisperse sphere packings. Phys. Rev. E, 1997, 56, 5, 5524-5532.
20. V. P. Voloshin, A. V. Anikeenko, N. N. Medvedev, A. Geiger. An algorithm for the calculation of volume and surface of unions of spheres. Application for salvation shells. In: Proc. 8th Int. Symp. Voronoi Diag. Sci. & Eng., 2011, 5988932, 170-176.
21. D.-S. Kim, J. Ryu, H. Shin, Y. Cho. Beta-decomposition for the volume and area of the union of three-dimensional balls and their offsets. J. Comput. Chem., 2012, 33, 1252-1273.
22. B. T. Holland, Ch. F. Blanford, A. Stein. Synthesis of macroporous minerals with highly ordered three-dimensional arrays of spheroidal voids. Science, 1998, 281, 5376, 538-540.
23. A. Fernandez–Nieves, D. R. Link, D. Rudhart, D. A. Weitz. Electro-optics on bipolar nematic liquid crystal droplets. Phys. Rev. Lett., 2004, 92, 105503, 1-4.
24. N. Insin, J. B. Tracy, H. Lee, J. P. Zimmer, R. M. Westervelt, M. G. Bawendi. Incorporation of iron oxide nanoparticles and quantum dots into silica microspheres. ACS Nano, 2008, 2, 2, 197-202.
25. F. Marquez, T. Campo, M. Cotto, R. Polanco, R. Roque, P. Fierro, J. M. Sanz, E. Elizalde, C. Morant. Synthesis and characterization of monodisperse magnetite hollow microspheres. Soft Nano Sci. Lett., 2011, 1, 1, 25-32.
26. S. Pauly, L. Lober, R. Petters, M. Stoica, S. Scudino, U. Kuhn, J. Eckert. Processing metallic glasses by selective laser melting. Mater. Today, 2013, 16, 1-2, 37-41.
27. Q. Weng, Y. Ide, X. Wang, X. Wang, Ch. Zhanga, X. Jiang, Y. Xue, P. Dai, K. Komaguchi, Y. Bando, D. Golberg. Design of BN porous sheets with richly exposed (002) plane edges and their application as TiO_2 visible light sensitizer. Nano Energy, 2015, 2015, 16, 19-27.
28. S. G. Narasimhan, S. Raghuraman, K. Venkatasubramanian, A. K. Dash. Determination of constant orientation workspace of a Stewart platform by geometrical method. Appl. Mech. & Mater., 2015, 813-814, 997-1001.