BOUNDS ON COMPLEMENTARY 3-DOMINATION NUMBER IN GRAPHS

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ABSTRACT
In Graph theory, a **dominating set** for a graph G is a subset S of its vertices such that every vertex in $V-S$ is adjacent to atleast one vertex in S. The minimum cardinality of a dominating set is called the domination number and is denoted by (G). A dominating set S of a graph G is said to be a **complementary 3-dominating set** of G if for every vertex in S has atleast three neighbors in $V-S$. The minimum cardinality of a complementary 3-dominating set is the complementary 3-domination number γ_3' of a graph G. In this paper we determine complementary 3-domination number for some standard graphs and obtain some results concerning this parameter.

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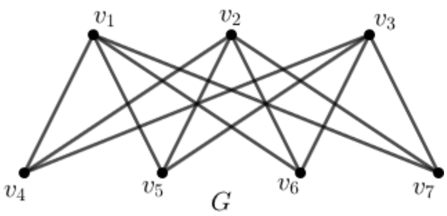
1. Introduction:

By a ***graph*** we mean a simple, connected, finite and undirected graph G = (V,E) where V is the vertex set whose elements are vertices or nodes and E is the edge set. Unless otherwise stated the graph G with $|V| = n$ and $|E| = q$. Degree of a vertex v is denoted by (v). Let $\Delta(G)$ and $\delta(G)$ denotes the maximum and minimum degree of a graph respectively. We denote a ***complete graph*** on n vertices by K_n. A bipartite graph G = (V, E) with partition $V = (V_1, V_2)$ is said to be a ***complete bipartite graph*** if every vertex in V_1 connected to every vertex of V_2. A gear graph G, is a wheel graph with a graph vertex added between each pair of adjacent graph vertices of the outer cycle. A ***Fan graph*** $F_{m,}$ is defined as the graph join $\bar{K}_m + P_n$, where $\bar{K}_m$ is the empty graph on m nodes and P_n is the path graph on n nodes. The $\boldsymbol{n - Barbell}$ graph is the simple graph obtained by connecting two copies of a complete graph K_n by a bridge. A graph G is connected if any two vertices of G are connected by a path. The complement G of G is the graph with vertex set V in which two vertices are connected if and only if they are not adjacent in G. ***A star graph*** $\boldsymbol{K_{1,n}}$ is a tree on n vertices with one vertex having vertex degree $n - 1$ and the other $n - 1$ having vertex degree one. The ***friendship graph*** $\boldsymbol{F_n}$ can be constructed by joining $'n'$ copies of the cycle graph C_3 with a common vertex which becomes a universal vertex for the graph. A ***wheel graph*** $\boldsymbol{W_n}$ is a graph formed by connecting a single universal vertex to all vertices of a cycle. A graph $C(1)$ is obtained by attaching a path P_2 to any vertex of degree C_m. $C_m + e$ is a graph obtained by adding an edge into a cycle C_m. $C(P_n)$ is a graph obtained by attaching a path P_n to any vertex of C_m. A ***Nordhaus Gaddum*** type result is a lower and upper bound on the sum or product of a parameter of a graph and its complement. A subset S of V is called a ***dominating set*** of G if every vertex In V-S is adjacent to atleast one vertex in S. The domination number (G) is the minimum cardinality of a dominating set. In this paper we introduce the concept of complementary 3-domination number and we present some basic theorems related to this parameter.

Definition:1.1 A dominating set S in a graph G is said to be a ***complementary 3-dominating set*** of G if any vertex in S has atleast three neighbours in V-S. The complentary 3-domination number $\gamma'_3(G)$ of a graph G is the minimum cardinality of a complementary 3-dominating set.

Example: 1.2



$Fig : 1$

For the above example the dominating set is $\{v_1, v_7\}$ and the complementary 3-dominating set is $\{v_1, v_2, \boldsymbol{v_3}\}$ and hence $\gamma'_3(G) = 3$

2. $\boldsymbol{\gamma'_3}$ number for some standard graphs

1, For any complete graph of order $n \geq 4$ $\gamma'_3(K_n) = 1$

2. For any complete bipartite graph $K_{m,n}$ with $m \geq 2, n \geq 3$ $\gamma'_3(K_{m,n}) = \begin{cases} 2 \; if \; m = 2, n = 3 \; and \; m \geq 4, n \geq 4 \\ 3 \; if \; m = n = 3 \; and \; m = 3, m = 4 \end{cases}$

3. For any wheel graph of order $n \geq 4, \gamma'_3(W_n) = 1$

4. For any Friendship graph F_n, $\gamma'_3(G) = 1$ where $n \geq 5$

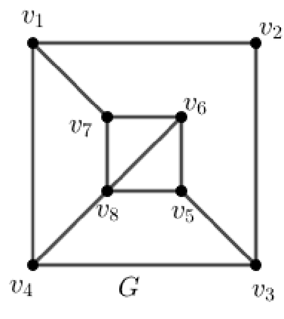
5. For any prism graph CL_p of order $p \geq 6, \gamma'_3(CL_p) = \begin{cases} -1 \; when \; n \; is \; odd \\ n - 2 \; when \; n \; is \; even \end{cases}$ where $p = 2n, n \geq 3$

6. For any gear graph of order $p \geq 7, \gamma'_3(G_p) = n$ where $p = 2n + 1, n \geq 3$

7. For any star graph $K_{1,n}$ of order $n \geq 3, \gamma'_3(K_{1,n}) = 1$

8. For a Petersen graph G, $\gamma_3(G) = 3$

Observation:2.1 Illustrative example for which domination number equals the complementary 3-dominaion number.

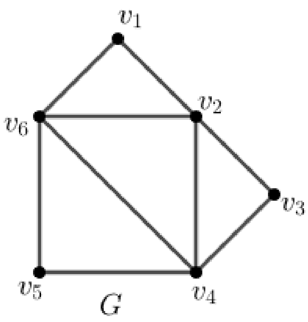


$Fig : 2$

Graph for which $\gamma'_3(G) = (G)$

For the figure:2, $S = \{v_3 v_7\}$ forms $\gamma'_3 - set$ and hence $\gamma'_3(G) = \gamma(G) = 3$

Observation: 2.2 The complement of a complementary 3-dominating set need not be a complementary 3-dominating set.



$Fig:3$

Example for which complement of $\gamma'_3 - set$ need not be a $\gamma'_3 - set$.

In figure:3 the set $S = \{v_4, v_6\}$ forms a $\gamma'_3 - set$ and hence $\gamma'_3(G) = 2$ but not a complementary 3-dominating set.

Observation: 2.3 Every complementary 3-dominating set is a dominating set but the converse need not be true.

Consider $C_4 + e$. Let $\{v_1, v_2, v_3, v_4\}$ be the vertices of $C_4 + e$. Now the set $S = \{v_2\}$ is the complementary 3-dominating set and dominating set.

Consider C_5. Let $\{v_1, v_2, v_3, v_4, v_5\}$ be the vertices of C_5. Now thw set $S = \{v_3, v_5\}$ for the dominating set but not $\gamma'_3 - set$.

Observation: 2.4 Let G be a connected graph and h be a spanning subgraph of G. H has $\gamma'_3 - set$ then $\gamma'_3(G) \le \gamma'_3(H)$ and the bound is sharp.

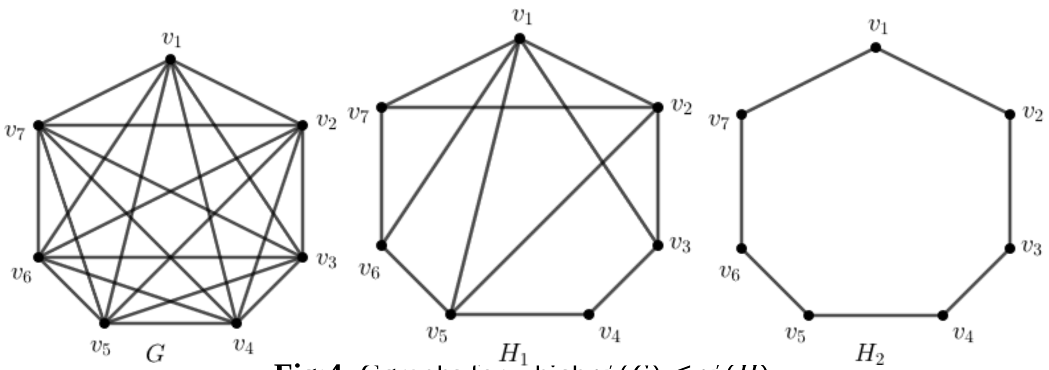


Fig:4 Ggraphs for which $\gamma'_3(G) \le \gamma'_3(H)$

In the above figure, Let the set $S = \{v_1\}$ forms $\gamma'_3 - set$ and hence $\gamma'_3(G) = 1$. Let H_1 *and* H_2 be the spanning subgraphs of C_6. Now $S_1 = \{v_1, v_5\}$ forms $\gamma'_3 - set$ and hence $\gamma'_3(H_1) = 2$. Let $S_2 = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7,\}$ forms $\gamma'_3 - set$ and hence $\gamma'_3(H_2) = 7$. Therefore

Remark: 2.5 For $C_4 + e$ with vertices $\{v_1, v_2, v_3, v_4\}$ with $e \in v_1 v_3$, $\{v_1\}$ forms $\gamma'_3 - set$ and hence $\gamma'_3(G) = 1$. Let $H \cong C_4$ be a spanning subgraph of $C_4 + e$, which implies $\gamma'_3(H) = 4$ and hence $\gamma'_3(G) < \gamma'_3(H)$. Let P_n be the spanning subgraph of C_n which implies $\gamma'_3(C_n) \le \gamma'_3(P_n) = n$.

Observation: 2.6 For any connected graph G, $\gamma(G) \le \gamma_c(G) \le \gamma'_3(G)$

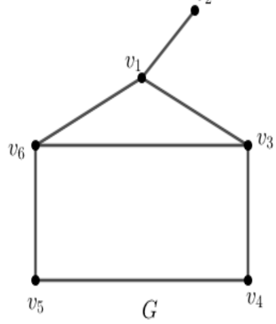


Fig:5 Graph for which $\gamma(G) \le \gamma_c(G) \le \gamma'_3(G)$

In the above figure, $\{v_1, v_5\}$ forms a dominating set, $\{v_1, v_3, v_4,\}$ forms a connected dominating set and $\{v_1, v_2, v_3, v_4, v_5, v_6\}$ forms a cpmplementary 3-dominating set and hence

$$\gamma'_3(G) \le \gamma'_3(H).$$

Theorem:2.7 If G is any connected graph then $1 \le \gamma'_3(G) \le n$

Proof: If G is any non-trivial connected graph containing degree $\Delta(G) = n - 1$ or graphs with $diam = rad = 1$, then $\gamma'_3(G) = 1$, the lower bounds holds. Let $\Delta(G) < n - 1$. For a graph not having two or more vertices of degree two continuously then $1 < \gamma'_3(G) < n$. Let S be the dominating set of G. For some vertex $u \in$ S, $(u) = 1$ *or* 2 then S is not a complementary

Theorem: 2.8 For any connected graph G of order $n \ge 4$, every $\gamma'_3(G) -$ dominating set of G contains its support vertices.

Proof: Let G be a connected graph of order $n \ge 4$ and S be a $\gamma'_3 -$ dominating set of G. Let u be a support vertex of G. Then there exists a pendant vertex v which is adjacent to u in G. Suppose u does not belongs to S. Then v is not dominated by any vertex in S implies $v \in S$. But

$\deg(v) = 1$ shows that S is not a γ'_3- dominating set of G, which is a contradiction. Hence, $u \in S$.

Theorem: 2.9 If G is a graph without isolated vertices $de(G) \geq 3$ and S is a minimal

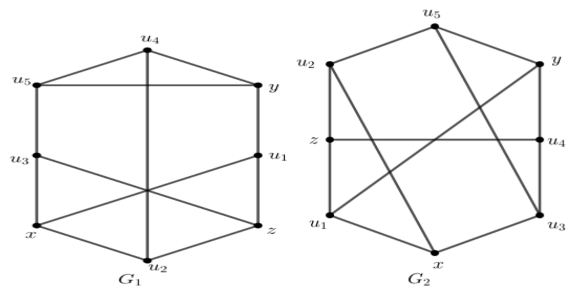
Proof: Let G be a graph without isolated vertices with $\Delta(G) \geq 3$ and S be a minimal

To prove: u is dominated by some vertex in $V(G) - S$

Suppose is not dominated by any vertex in $V(G) - S$. Since G has no isolated vertices and S is a γ'_3- dominating set of G, each vertex in $(G) - S$ has atleast three neighbours in $S - \{u\}$. This contradicts the fact that S is a minimal dominating set of G.

Note: 2.10 If G is connected graph with $de(G) \geq 3$ and S is a γ'_3- dominating set of G then the complement $V - S$ need not be a γ'_3- dominating set.

Theorem: 2.11 For any connected cubic graph of order 8 $\gamma'_3(G) = \chi(G) = 3$ if and only if $G \cong G_1$ *or* G_2.



Proof: If $G \cong G_1$ *or* G_2 then obviously $\gamma'_3(G) = (G) = 3$. Conversely let us assume that $\gamma'_3(G) = (G) = 3$. Let us assume that $S = \{x, y, z\}$ be a minimum complementary 3-dominating set of G and $V - S = \{u_1, u_2, u_3, u_4, u_5\}$. Clearly $< S >$ is not equal to K_3. Therefore we consider three cases.

Case: (1) <S>=$\bar{K_3}$

With no loss of generality, let $(x) = \{u_1, u_2, u_3\}$. Then atleast one of the vertices of $(x) = \{u_1, u_2, u_3\}$ is adjacent to y.

Subcase:(i) One vertex of $N(x)$ adjacent to y, say u_1.

For this case u_4 and u_5 are adjacent to y. Suppose now z is adjacent or non adjacent to u_1.

Suppose if z is adjacent to u_1 then z is adjacent to u_2 and u_3 (or equivalently $u_4$and u_5) or u_2 (or equivalently u_3) and u_4 (or equivalently u_5). If z is adjacent to u_2 and u_3, then u_2 is non adjacent to u_3. Therefore $u_2$must be adjacent to u_4 (or equivalently u_5) and then u_5 is adjacent to u_3 and u_4 which implies $G \cong G_1$. Suppose if z is adjacent to u_2 and u_4 then u_2 is non adjacent to $u_4$and so u_2 is adjacent to u_3 or u_5. If u_2 is adjacent to u_3, then $\{y, u_2\}$ is not a complementary 3-dominating set. Suppose if u is adjacent to u_2, u_3 and u_5 and since G is γ'_3- dominating set of G then $V(G) -$ cubic, u_3 is adjacent to $u_4$and $u_5$which implies $G \cong G_2$. γ'_3-dominating set of G. Let $u \in (G)$.

Suppose z is non adjacent to $u_1$then with no loss of generality, let $(z) = \{u_2, u_3, u_4\}$. Then $u_2$is adjacent to $u_1$or $u_3$or $u_5$or u_4. If $u_2$is adjacent to $u_3$then $\{u_2, y\}$ is not a complementary 3-dominating set. Suppose if $u_2$is adjacent to $u_1$and since G is cubic, $u_5$is adjacent to $u_4$and $u_3$and so $\{x, u_4\}$ is not a complementary 3-dominating set. If $u_2$is adjacent to u_4, then $u_5$adjacent to $u_1$and $u_3$and so $\{z, u_1\}$ is not a complementary 3-dominating set. Suppose if $u_2$adjacent to u_5, then $u_4$is adjacent to $u_5$then $\{x, u_4\}$ is not a complementary 3-dominating set. If $u_4$adjacent to $u_3$then $u_5$adjacent to $u_1$and hence $\{u_3, u_5\}$ is not a complementary 3-dominating set. If $u_1$adjacent to $u_4$then $_3$ adjacent to $u_5$which implies $G \cong G_1$.

Subcase:(ii) Two vertices if $N(z)$ is adjacent to y, say $u_1$and u_2

For this case, let us assume that y is adjacent to u_4. Now z ix adjacent to u_1 (or equivalently u_2) or non adjacent to u_1 (or equivalently u_2).

If z is adjacent to u_1, then $u_2$is adjacent to u_3 (or equivalently u_5) or z or u_4. If $u_2$is adjacent to $u_3$then z is non adjacent to $u_3$and u_4. Also z is non adjacent to $u_3$and u_5. If z adjacent to $u_4$and u_5 then $u_5$adjacent to $u_4$and $u_3$and so $S = \{x, u_4\}$ is not a complementary 3-dominating set. If $u_2$adjacent to z and since G is cubic, z is non adjacent to u_4 (or equivqlently u_3). Therefore z must be adjacent to u_5. Now $u_5$adjacent to $u_3$and $u_4$and then $u_3$adjacent to $u_4$which implies $G \cong G_2$.

Suppose if $u_2$adjacent to u_5, then z adjacent to $u_3$and $u_4$or $u_5$and u_3 (or equivalently $u_5$and u_4). If z adjacent to $u_4$and u_3 then $u_5$adjacent to $u_4$and $u_3$and so $\{x, u_4\}$ is not a complementary 3-dominating set. If z adjacent to $u_3$and $u_5$then $u_4$adjacent to $u_3$and $u_5$which implies $G \cong G_2$.

Suppose if z is non adjacent to $u_1$then let us assume that z be adjacent to $u_4$or $u_3$and u_5. Now $u_5$adjacent to $u_3$and $u_4$or $u_1$and $u_2$or $u_1$and u_4 (or equivalently $u_2$and u_3). If $u_5$adjacent to $u_3$and $u_4$then $u_1$adjacent to $u_2$and so $\{u_2, x\}$ is not a complementary 3-dominating set. If $u_5$adjacent to $u_1$and $u_4$then $u_2$adjacent to u_3, if $u_5$adjacent to $u_1$and $u_2$then $u_3$adjacent to $u_4$which implies $G \cong G_1$.

Subcase:(iii) All the vertices of $N(x)$ are adjacent to y, say u_1, u_2, u_3

For this case, $u_2$adjacent to u_1 (or equivalently u_3) or z (or equivalently $u_4$or u_5). Since G is cubic, $u_2$will not be adjacent to u_1 (or equivalently u_3). Therefore $u_2$must be adjacent to z, then z adjacent to $u_1$and $u_2$or $u_4$and u_5. Since G is cubic, z will not be adjacent to $u_1$and also z will not be adjacent to $u_1$and u_5. Therefore z must adjacent to $u_4$and u_5. Suppose if z adjacent to $u_4$and u_5, then $u_1$will not be adjacent to u_3. Therefore $u_1$must be adjacent to u_4 (or equivalently u_5) and so $u_5$adjacent to $u_3$and $u_4$which implies $G \cong G_1$.

Case:(2) $< S >= P_1 \cup P_2$

Let xy be an edge. With no loss of generality, let us assume that x be adjacent to $u_1$and u_3. Now z adjacent to $u_1$and $u_2$and adjacent to $u_3$or $u_4$or z adjacent to u_1 (or equivalently u_2) and adjacent to any two of $\{u_3, u_4, u_5\}$.

Suppose if z adjacent to u_1, u_2, u_3, then $u_4$adjacent to u_1 (or equivalently u_2) or not adjacent to u_1 (or equivalently to u_2). If $u_4$not adjacent to u_1 (or equivalent to u_2u$_2$) then $u_4$adjacent to u_3, z and $u_5$and so $S = \{x, z, u_4\}$ such that $< S >= \overline{K_3}$ which will fall under the case (1). If $u_4$adjacent to $u_1$then $u_5$is adjacent or non adjacent to to u_4, If $u_5$non adjacent to u_4, then u_5 adjacent to $u_2, u_3$and y and so $S = \{u_1, u_5\}$ which is not a complementary 3-dominating set. If $u_5$adjacent to $u_4$then $S = \{x, z, u_5\}$ and so $< S >= \overline{K_3}$which will fall under the case (1).

If z is adjacent to u_1, $u_2$and $u_3$then $u_5$is adjacent or non adjacent to u_1. If $u_5$adjacent to $u_1$then for $S = \{x, z, u_5\}$ and so <S>=$\overline{K_3}$ which will come under the case (1). If $u_5$adjacent to u_1 then $u_5$adjacent to any three of $\{y, u_2, u_3, u_4\}$. Let $u_5$adjacent to $u_3, u_4$and y. Hence for $S = \{x, z, u_5\}$ and so <S>=$\overline{K_3}$ which will fall under the case (1).

Case:(3) $< S >= P_3$

Let us assume that y adjacent to x and z. Then with no loss of generality, let y adjacent to u_1. Now $u_2$adjacent to x, and one of $\{u_3, u_4, u_5\}$ or z and two of $\{u_3, u_4, u_5\}$. Suppos if $u_2$adjacent to x, z and u_3, then $u_4$adjacent or non adjacent to z (or equivalently x). If $u_4$non adjacent to z, then $u_4$adjacent to u_1, u_5 and $u_3$and so S={y, u_2, u_4} such that $< S >= \overline{K_3}$ which will fall under the case (1). If $u_4$adjacent to z, then $u_5$adjacent or non adjacent to u_4. If u_5 non adjacent to $u_4$then $u_5$adjacent to x, $u_1$and $u_3$and hence $S = \{z, u_5\}$ is not a complementary 3-dominating set. If u_5 adjacent to $u_4$then for $S = \{y, u_2, u_5\}$ and so $< S >= \overline{K_3}$ which will fall under the case (1).

Suppose if $u_2$adjacent to z, $u_3$and $u_4$then $u_5$is adjacent or non adjacent to z. If $u_5$adjacent to z, then for $S = \{y, u_2, u_5\}$, $< S >= \overline{K_3}$ which will come under the case (1). If u_5 non adjacent to z, then $u_5$adjacent to three of $\{x, u_1, u_3, u_4\}$. Let us assume that $u_5$adjacent to x, $u_1$and u_3. Hence $S = \{y, u_2, u_5\}$ and so $< S >= \overline{K_3}$ which will come under the case (1). Therefore $\gamma_3'(G) = (G) = 3$ if and only if $G \cong G_1$ *or* G_2.

Conclusion: In this paper we successfully described the complementary 3-domination number of some graphs, bounds and γ_3' number for cubic gaphs.

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