Steiner Distance and Steiner Distance Parameters of Double Wheel graph

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Abstract

Let G be a connected graph of order at least 2 and $S \subseteq V(G)$, the Steiner distance $d_G(S)$ among the vertices of S is the minimum size among all the connected subgraphs whose vertex sets contains S. In this paper, we calculate the Steiner distance and Steiner distance parameters such as Steiner Wiener index, Steiner degree distance, Steiner Gutman index, Steiner Harary index and Steiner reciprocal degree distance of a double wheel graph.

Keywords: Steiner distance, Steinerr Wiener index, Steiner degree distance, Steiner Gutman index, Steiner Harary index, Steiner reciprocal degree distance, Double Wheel Graph. 2010 **Mathematics Subject Classification:** 05C12, 05C76

Introduction

All graphs considered in this paper are undirected, finite and simple. For a graph $G(V,E)$ and a set $S \subseteq V(G)$ of at least two vertices, the S-steiner tree or Steiner tree connecting S is a subgraph $T(V,E)$ of G that is a tree with $S \subseteq V$. The Steiner distance $d_G(S)$ among the vertices of S is the minimum size among all connected subgraphs whose vertex set contains S. The Steiner distance of a graph, introduced by Chartrand et al., is a natural generalization of the concept of the distance in graphs[4] .

X.Li et al.[5] generalized the concept of Wiener index in terms of Steiner distance. The *Steiner Wiener index*, $SW_k(G)$, of G is defined as

$$SW_k(G) = \sum_{\substack{S\subseteq V(G),\\ |S|=k}} d_G(S).$$

Recently, Gutman[7] generalized the concept of degree distance by using Steiner distance and termed it as *Steiner degree distance*. The Steiner degree distance, $SDD_k(G)$, of the graph G is defined by

$$SDD_k(G) = \sum_{\substack{S\subseteq V(G),\\ |S|=k}} \left[\sum_{v\in S} deg_G(v)\right] d_G(S).$$

Recently, Y. Mao and K.C. Das[9] introduced the concept of *Steiner Gutman index* by using Steiner distance. The Steiner Gutman index, $SGut_k(G)$, of a graph G is defined by

$$SGut_k(G) = \sum_{\substack{S\subseteq V(G),\\ |S|=k}} \left[\prod_{v\in S} deg_G(v)\right] d_G(S).$$

In 2018, Y.Mao[10] generalized the concept of Harary index in terms of Steiner distance. The *Steiner Harary index* of a graph G is defined as

$$SH_k(G) = \sum_{\substack{S\subseteq V(G),\\ |S|=k}} \frac{1}{d_G(S)}.$$

In 2019, A.Babu et.al[1] introduced the concept of *Steiner reciprocal degree distance* of a graph G and it is defined as

$$SRDD_k(G) = \sum_{\substack{S\subseteq V(G)\\ |S|=k}} \frac{\left[\sum_{v\in S} deg_G(v)\right]}{d_G(S)}.$$

A *Double Wheel graph*, denoted by DW_n, is the graph $2C_n + K_1$ in which there are two copies of cycles of order n whose vertices are all connected to a central vertex. The vertex corresponding to K_1 is known as apex. Clearly, $|V(DW_n)| = 2n + 1$ and $|E(DW_n)| = 4n$. (See FIGURE. 1)

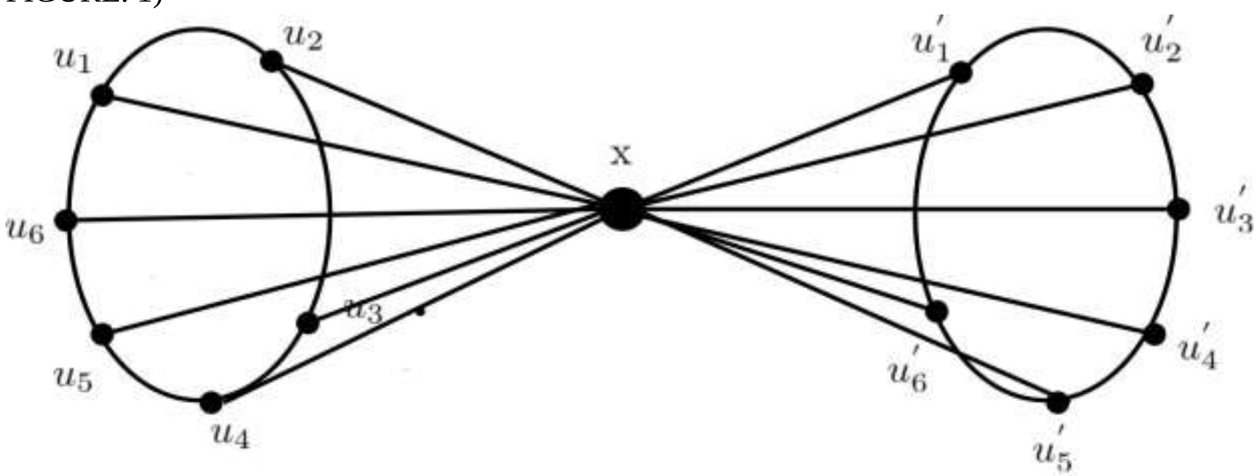

FIGURE 1. DW_6

X.Li et al.[6] computed the Steiner wiener index of wheel graphs. Followed by the results in[6] , V.Sheeba Agnes et al.[12] obtained the Steiner degree distance, Steiner Gutman index, Steiner Harary index and Steiner reciprocal degree distance of the wheel graphs.

In this paper, we obtain the Steiner distance and the Steiner parameters such as Steiner Wiener index, Steiner degree distance, Steiner Gutman index, Steiner Harary index and Steiner reciprocal degree distance of the double Wheel graph.

Steiner Distance of Double Wheel graph

In this section, we obtain the Steiner distance of the double Wheel graph.

Theorem 1. Let G be a double wheel graph with $n \geq 3$ vertices. Let $k \geq 2$ be an integer and S be a subset of G with $|S| = k$. Then $k - 1 \leq d_G(S) \leq k$, if $2 \leq k \leq 2n$ and $d_G(S) = k - 1$, if $k = 2n + 1$.

Proof. By definition, G is the join of K_1 and two copies C_1 and C_2 of cycles of order n. Let x be the apex of G and let $U_1 = \{u_1, u_2, \ldots, u_n\}$ be the set of vertices of the cycle C_1 and $U_2 = \{u_1', u_2', \ldots, u_n'\}$ be the set of vertices of the cycle C_2. Then $V(G) = U_1 \cup U_2 \cup \{x\}$.

Let $S \subseteq V(G)$ such that $|S| = k$, where $k \geq 2$ is an integer.

Case (i): $2 \leq k \leq 2n$.

Here we consider two subcases;

Subcase (i)a: $2 \leq k \leq n$

Let $S \subseteq V(G)$ with $|S| = k$. Clearly $x \in S$ or $x \notin S$.

Suppose $x \in S$. Then the remaining $k - 1$ vertices belongs to $U_1 \cup U_2$. Since x is adjacent to all the vertices of U_1 and U_2, we obtain an optimal steiner tree T with x as centre and $k - 1$ vertices as leaves. Hence $G[S]$ has $k - 1$ edges.

Now if $x \notin S$, then S contains the set of vertices of the cycles of G.

Suppose $S \subseteq U_1$ with $|S| = k$.

If S is a set of k consecutive vertices, which belongs to U_1, then the subgraph formed by S is a path with $k - 1$ edges and we obtain a steiner tree on k vertices with $k - 1$ number of edges. Thus $G[S]$ has $k - 1$ edges.

If S is a set of k vertices which contains at least one non-consecutive vertex, then we obtain an optimal steiner tree T which is a star with x as centre and k vertices as leaves. Thus the subgraph $G[S]$ has exactly k edges and hence $d_G(S)$ is either $k - 1$ or k.

If $S \subseteq U_2$, by similar argument as above, we get $d_G(S)$ as either $k - 1$ or k.

Now we consider a set $S = S_1 \cup S_2$ with $|S| = k$, where $S_1 \subseteq U_1$ and $S_2 \subseteq U_2$ with $|S_1| = a$ and $|S_2| = b$ respectively, such that $a + b = k$. By the structure of G, x is adjacent to all the vertices of U_1 and U_2. Hence we obtain a connected subgraph induced by the vertices of S as a star with x as centre and k vertices of S as leaves, which is the optimal steiner tree with k edges. Thus $d_G(S) = k$.

Subcase (i) b: $n + 1 \leq k \leq 2n$

Clearly $x \in S$ or $x \notin S$.

Suppose $x \in S$. Then the remaining $k - 1$ vertices belongs to $U_1 \cup U_2$. Since x is adjacent to all the vertices of U_1 and U_2, we obtain an optimal steiner tree T with x as centre and $k - 1$ vertices as leaves. Hence $G[S]$ has $k - 1$ edges.

Suppose $x \notin S$. We consider a set $S = S_1 \cup S_2$ with $|S| = k$, where $S_1 \subseteq U_1$ and $S_2 \subseteq U_2$ with $|S_1| = a$ and $|S_2| = b$ respectively, such that $a + b = k$. By the structure of G, x is adjacent to all the vertices of U_1 and U_2. We obtain a connected subgraph induced by the vertices of S as a star with x as centre and the k vertices of S as leaves, which is the optimal steiner tree on k edges. Thus $d_G(S) = k$.

Thus $d_G(S)$ is either $k-1$ or k.

Case (ii): $k = 2n+1$

Let $S \subseteq V(G)$ with $k = 2n+1$. By the structure of G, there exists a spanning tree with x as centre and the remaining $2n$ vertices as leaves. This tree has $2n$ edges. Thus $G[S]$ has $2n+1-1 = k-1$ edges. And hence $d_G(S) = k-1$

This completes the proof.

Steiner Wiener index of Double Wheel graph

In this section we obtain the Steiner wiener index of double wheel graph.

Theorem 2. Let $G = DW_n$ be a double wheel graph with $n \geq 3$ vertices. Let k be an integer with $2 \leq k \leq 2n+1$. Let S be a subset of $V(G)$ with $|S| = k$. Then

1. $SW_k(G) = 2n$, if $k = 2n+1$.
2. $SW_k(G) = \sum_{k=n+1}^{2n}\left[\sum_{a=k-1-n}^{n}\binom{n}{a}\binom{n}{k-1-a}(k-1)\right]$ $+\sum_{k=n+1}^{2n}\left[\sum_{a=k-n}^{n}\binom{n}{a}\binom{n}{k-a}(k)\right]$, if $n+1 \leq k \leq 2n$.
3. $SW_k(G) = \binom{2n}{k-1}(k-1) + 2\left[k\binom{n}{k} - n\right] + \sum_{1\leq a\leq k}\binom{n}{a}\binom{n}{k-a}k$, if $2 \leq k \leq n$.

Proof. By definition, G is the join of K_1 and two copies C_1 and C_2 of cycles of order n. Let x be the apex of G and let $U_1 = \{u_1, u_2, \dots, u_n\}$ be the set of vertices of the cycle C_1 and $U_2 = \{u_1', u_2', \dots, u_n'\}$ be the set of vertices of the cycle C_2. Then $V(G) = U_1 \cup U_2 \cup \{x\}$.

Let $S \subseteq V(G)$ such that $|S| = k$, where $k \geq 2$ is an integer.

Case (i): $k = 2n+1$.

In this case S contains all the vertices of G. By Theorem 1, $d_G(S) = k-1 = 2n$. Therefore,

$$SW_k(G) = \sum_{|S|=2n+1} d_G(S) = 2\mathrm{n}.$$

Case (ii): $n+1 \leq k \leq 2n$.

Let S be a set of vertices with $|S| = k$. Clearly $x \in S$ or $x \notin S$.

Suppose $x \in S$. Then by Theorem 1, $d_G(S) = k-1$. By the structure of G, x is adjacent to all the remaining $k-1$ vertices of the cycles C_1 and C_2. The $k-1$ vertices can be chosen as follows; either a number of vertices in C_1 and $(k-1)-a$ vertices in C_2 or a number of vertices in C_2 and $(k-1)-a$ vertices in C_1, $1 \leq a \leq k-1$.

Therefore, $\sum_{|S|=k} d_G(S) = \sum_{k=n+1}^{2n}\left[\sum_{a=k-1-n}^{n}\binom{n}{a}\binom{n}{k-1-a}(k-1)\right]$.

Suppose $x \notin S$. By Theorem 1, We have $d_G(S) = k$. Since $x \notin S$, S contains the vertices of $U_1 \cup U_2$. Hence S is a set with a vertices of cycle C_1 and $k-a$ vertices of the cycle C_2 or S is a set with a vertices of C_2 and $k-a$ vertices of C_1, $1 \leq a \leq k$. Therefore,

$$\sum_{|S|=k} d_G(S) = \sum_{k=n+1}^{2n}\left[\sum_{a=k-n}^{n}\binom{n}{a}\binom{n}{k-a}(k)\right].$$

Hence,

$$SW_k(G) = \sum_{k=n+1}^{2n}\left[\sum_{a=k-1-n}^{n}\binom{n}{a}\binom{n}{k-1-a}(k-1)\right] + \sum_{k=n+1}^{2n}\left[\sum_{a=k-n}^{n}\binom{n}{a}\binom{n}{k-a}(k)\right].$$

Case (iii): $2 \leq k \leq n$.

Let S be a set of vertices with $|S| = k$. Clearly $x \in S$ or $x \notin S$.

Suppose $x \in S$. By Theorem 1, $d_G(S) = k - 1$. By the structure of G, x is adjacent to all the vertices of the two cycles.

Here the remaining $k - 1$ vertices can be chosen from the vertices of cycles in $\binom{2n}{k-1}$ ways.

Therefore, $\sum_{x\in S} d_G(S) = \binom{2n}{k-1}(k-1)$.

Suppose $x \notin S$. Then by Theorem 1, we have $k - 1 \leq d_G(S) \leq k$. Also S contains the vertices of cycles of G. Therefore $S \subseteq U_1$ or $S \subseteq U_2$ or $S \subseteq U_1 \cup U_2$.

Let $S \subseteq U_1$. First we consider S to be set of consecutive vertices of the cycle C_1. Then there are n such sets for $n > k$ and $d_G(S) = k - 1$. Therefore we get, $\sum_{S\subseteq U_1} d_G(S) = n(k-1)$.

If S is a set of k vertices of C_1 which contains at least one non-consecutive vertex, then there are $\left[\binom{n}{k} - n\right]$ such sets with $d_G(S) = k$.

Therefore $\sum_{S\subseteq U_1} d_G(S) = \left[\binom{n}{k} - n\right](k) + n(k-1)$.

Similarly, if $S \subseteq U_2$, we have, $\sum_{S\subseteq U_2} d_G(S) = \left[\binom{n}{k} - n\right](k) + n(k-1)$.

Now we consider a set $S = S_1 \cup S_2$ with $|S| = k$, where $S_1 \subseteq U_1$ and $S_2 \subseteq U_2$ with either $|S_1| = a$ and $|S_2| = k - a$ or $|S_2| = a$ and $|S_1| = k - a, 1 \leq a \leq k$ respectively.

By Theorem 1, we have $d_G(S) = k$. Hence,

$$\sum_{|S|=k} d_G(S). = \sum_{k=2}^{n}\left[\sum_{a=k-n}^{n}\binom{n}{a}\binom{n}{k-a}(k)\right].$$

Therefore,

$$SW_k(G) = \binom{2n}{k-1}(k-1) + 2\left[k\binom{n}{k} - n\right] + \sum_{1\leq a\leq k}\binom{n}{a}\binom{n}{k-a}k, \quad if \quad 2 \leq k \leq n.$$

This completes the proof.

Steiner Degree Distance of Double Wheel Graph

In this section we obtain the Steiner degree distance of double wheel graph.

Theorem 3. Let $G = DW_n$ be a Double wheel graph with $n \geq 3$ vertices. Let k be an integer with $2 \leq k \leq 2n + 1$. Let S be a subset of $V(G)$ with $|S| = k$. Then

1. $SDD_k(G) = 16n^2$, if $k = 2n + 1$.
2. $SDD_k(G) = \sum_{k=n+1}^{2n}\left[(2n+3k-3)\sum_{a=k-1-n}^{n}\binom{n}{a}\binom{n}{k-1-a}(k-1)\right]$

$+\sum_{k=n+1}^{2n}\left[(2n+3k)\sum_{a=k-n}^{n}\binom{n}{a}\binom{n}{k-a}(k)\right]$, if $n + 1 \leq k \leq 2n$.

3. $SDD_k(G) = \left[[2n+3k-3]\binom{2n}{k-1}\right](k-1) + 6k\left[k\binom{n}{k} - n\right] +$

$$+\sum_{k=2}^{n}\left[3k\sum_{a=k-n}^{n}\binom{n}{a}\binom{n}{k-a}(k)\right]. \text{ If } 2 \le k \le n.$$

Proof. By definition, G is the join of K_1 and two copies C_1 and C_2 of cycles of order n. Let x be the apex in G and Let $U_1 = \{u_1, u_2, \ldots, u_n\}$ be the set of vertices of the cycle C_1 and $U_2 = \{u_1', u_2', \ldots, u_n'\}$ be the set of vertices of the cycle C_2. Then $V(G) = U_1 \cup U_2 \cup \{x\}$.
Let $S \subseteq V(G)$ such that $|S| = k$, where $k \ge 2$ is an integer.
Case (i): $k = 2n+1$.
In this case S contains all the vertices of G. By the structure of G, x is adjacent to all the vertices of C_1 and C_2. Hence we have $deg_G(u_i) = 3$, $deg_G(u_i') = 3, 1 \le i \le n$. and $deg_G(x) = 2n$. Also by Theorem 1, we have $d_G(S) = k-1 = 2n$. and $\sum_{v\in S} d\,eg_G(v) = 8n$. Therefore

$$\begin{aligned} SDD_k(G) &= \sum_{|S|=2n+1}\left[\sum_{v\in S} d\,eg_G(v)\right] d_G(S) \\ &= 8\mathrm{n}(2\mathrm{n}) \\ &= 16n^2. \end{aligned}$$

Case (ii): $n+1 \le k \le 2n$.
Let S be a set of vertices with $|S| = k$. Clearly $x \in S$ or $x \notin S$.
Suppose $x \in S$. By Theorem 1, $d_G(S) = k-1$. By the structure of G, x is adjacent to all the remaining $k-1$ vertices of $C_1 \cup C_2$. The $k-1$ vertices can be chosen as follows; either a number of vertices in C_1 and $(k-1)-a$ vertices in C_2 or a number of vertices in C_2 and $(k-1)-a$ vertices in $C_1, 1 \le a \le k-1$. Also the degree of each vertex of the cycles is 3. Therefore, $\sum deg_G(v) = 2n+3k-3$. Hence,

$$\sum_{\substack{S\subseteq V(G)\\|S|=k}}\left[\sum_{v\in S} d\,eg_G(v)\right] d_G(S) = \sum_{k=n+1}^{2n}\left[(2n+3k-3)\sum_{a=k-1-n}^{n}\binom{n}{a}\binom{n}{k-1-a}(k-1)\right].$$

Suppose $x \notin S$. By Theorem 1, we have $d_G(S) = k$. Here we consider $S = S_1 \cup S_2$ with $|S| = k$, where $S_1 \subseteq U_1$ and $S_2 \subseteq U_2$ with $|S_1| = a$ and $|S_2| = k-a, 1 \le a \le k$ respectively. The degree of each vertex of the cycles is 3. Therefore $\sum deg_G(v) = 2n+3k$. Hence

$$\sum_{\substack{S\subseteq V(G)\\|S|=k}}\left[\sum_{v\in S} d\,eg_G(v)\right] d_G(S) = \sum_{k=n+1}^{2n}\left[(2n+3k)\sum_{a=k-n}^{n}\binom{n}{a}\binom{n}{k-a}(k)\right].$$

Therefore,

$$SDD_k(G) = \sum_{k=n+1}^{2n}\left[(2n+3k-3)\sum_{a=k-1-n}^{n}\binom{n}{a}\binom{n}{k-1-a}(k-1)\right] + \sum_{k=n+1}^{2n}\left[(2n+3k)\sum_{a=k-n}^{n}\binom{n}{a}\binom{n}{k-a}(k)\right].$$

Case (iii): $2 \le k \le n$.
Let S be a set of vertices with $|S| = k$. Clearly $x \in S$ or $x \notin S$.
Suppose $x \in S$. By Theorem 1, $d_G(S) = k - 1$. By the structure of G, x is adjacent to all the vertices of the two cycles.

Here the remaining $k - 1$ vertices can be chosen from the vertices of cycles in $\binom{2n}{k-1}$ ways. Hence, $\sum_{S \subseteq V(G)} d\,eg_G(v) = [2n + 3k - 3]\binom{2n}{k-1}$.
Therefore,

$$\sum_{\substack{S \subseteq V(G) \\ |S|=k}} \left[\sum_{v \in S} d\,eg_G(v)\right] d_G(S) = \left[(2n + 3k - 3)\binom{2n}{k-1}\right](k - 1).$$

Suppose $x \notin S$. By Theorem 1 we have $k - 1 \le d_G(S) \le k$. Also S contains the vertices of cycles of G. Therefore $S \subseteq U_1$ or $S \subseteq U_2$ or $S \subseteq U_1 \cup U_2$.
Let $S \subseteq U_1$. First we consider S to be a set of consecutive vertices of the cycle C_1 of G. Then there are n such sets for $n > k$, and $d_G(S) = k - 1$. The degree of each vertex in S is 3. Then $\sum_{U_1} d\,eg_G(v) = 3kn$. Therefore we get,

$$\sum_{S \subseteq U_1} d\,eg_G(v) = 3kn.$$

If S is a set of k vertices of cycle C_1 which contains at least one non-consecutive vertex, then there are $\left[\binom{n}{k} - n\right]$ such sets with $d_G(S) = k$. The degree of each vertex in S is 3. Then,

$$\sum_{S \subseteq U_1} d\,eg_G(v) = 3k\left[\binom{n}{k} - n\right].$$

Therefore,

$$\sum_{S \subseteq U_1} \left[\sum_{v \in S} d\,eg_G(v)\right] d_G(S) = 3k\left[\binom{n}{k} - n\right](k) + 3kn(k - 1).$$

Similarly, if $S \subseteq U_2$, we have,

$$\sum_{S \subseteq U_2} \left[\sum_{v \in S} d\,eg_G(v)\right] d_G(S) = 3k\left[\binom{n}{k} - n\right](k) + 3kn(k - 1).$$

Now we consider another set $S = S_1 \cup S_2$ with $|S| = k$, where $S_1 \subseteq U_1$ and $S_2 \subseteq U_2$ with $|S_1| = a$ and $|S_2| = k - a, 1 \le a \le k$. Also the degree of each vertex of the cycles is 3. By Theorem 1, we have $d_G(S) = k$. Therefore $\sum_{v \in S} d\,eg_G(v) = 3k\binom{n}{a}\binom{n}{k-a}$. Hence,

$$\sum_{S \subseteq U_1 \cup U2} \left[\sum_{v \in S} d\,eg_G(v)\right] d_G(S) = \sum_{k=2}^{n}\left[3k \sum_{a=k-n}^{n} \binom{n}{a}\binom{n}{k-a}(k)\right].$$

Therefore,

$$SDD_k(G) = \left[[2n+3k-3]\binom{2n}{k-1}\right](k-1) + 6k\left[k\binom{n}{k} - n\right] + \sum_{k=2}^{n}\left[3k\sum_{a=k-n}^{n}\binom{n}{a}\binom{n}{k-a}(k)\right].$$

This completes the proof.

Steiner Gutman index of Double Wheel Graph

In this section we obtain the Steiner Gutman index of double wheel graph.

Theorem 4. Let $G = DW_n$ be a Double wheel graph with $n \geq 3$ vertices. Let k be an integer with $2 \leq k \leq 2n+1$. Let S be a subset of $V(G)$ with $|S| = k$. Then

1. $SGut_k(G) = 36n^4$, if $k = 2n+1$.
2. $SGut_k(G) = \sum_{k=n+1}^{2n}\left[\sum_{a=k-1-n}^{n}[18an(k-1-a)]\binom{n}{a}\binom{n}{k-1-a}(k-1)\right]$ $+ \sum_{k=n+1}^{2n}\left[\sum_{a=k-n}^{n}[18an(k-a)]\binom{n}{a}\binom{n}{k-a}(k)\right],$ if $n+1 \leq k \leq 2n$.
3. $SGut_k(G) = \left[6n(k-1)^2\binom{2n}{k-1}\right] + 2.3^k\left[k\binom{n}{k} - n\right] +$ $+\sum_{k=2}^{n}\left[3k\sum_{a=k-n}^{n}\binom{n}{a}\binom{n}{k-a}(k)\right],$ if $2 \leq k \leq n$.

Proof. By definition, G is the join of K_1 and two copies C_1 and C_2 of cycles of order n. Let x be the apex in G and Let $U_1 = \{u_1, u_2, \ldots, u_n\}$ be the set of vertices of the cycle C_1 and $U_2 = \{u'_1, u'_2, \ldots, u'_n\}$ be the set of vertices of the cycle C_2. Then $V(G) = U_1 \cup U_2 \cup \{x\}$.

Let $S \subseteq V(G)$ such that $|S| = k$, where $k \geq 2$ is an integer.

Case (i): $k = 2n+1$.

In this case S contains all the vertices of G. By the structure of G, x is adjacent to all the vertices of C_1 and C_2. Hence we have $deg_G(u_i) = 3$, $deg_G(u'_i) = 3, 1 \leq i \leq n$. and $deg_G(x) = 2n$. Also by Theorem 1, we have $d_G(S) = k-1 = 2n$. and $\prod_{v\in S} deg_G(v) = 18n^3$.

$$\begin{aligned} SGut_k(G) &= \sum_{|S|=2n+1}\left[\prod_{v\in S} deg_G(v)\right] d_G(S) \\ &= 18n^3(2n) \\ &= 36n^4. \end{aligned}$$

Case (ii): $n+1 \leq k \leq 2n$.

Let S be a set of vertices with $|S| = k$. Clearly $x \in S$ or $x \notin S$.

Suppose $x \in S$. By Theorem 1, $d_G(S) = k-1$. By the structure of G, x is adjacent to all the remaining $k-1$ vertices of $C_1 \cup C_2$. The $k-1$ vertices can be chosen as follows; either a number of vertices in C_1 and $(k-1)-a$ vertices in C_2 or a number of vertices in C_2 and $(k-1)-a$ vertices in C_1, $1 \leq a \leq k-1$. Also the degree of each vertex of the cycles is 3. Therefore, $\prod deg_G(v) = 18an(k-1-a)$. Hence,

$$\sum_{\substack{S\subseteq V(G)\\|S|=k}}\left[\prod_{v\in S} d\,eg_G(v)\right]d_G(S)=\sum_{k=n+1}^{2n}\left[\sum_{a=k-1-n}^{n}[18an(k-1-a)]\binom{n}{a}\binom{n}{k-1-a}(k-1)\right].$$

Suppose $x\notin S$. By Theorem 1, we have $d_G(S)=k$. Here we consider $S=S_1\cup S_2$ with $|S|=k$, where $S_1\subseteq U_1$ and $S_2\subseteq U_2$ with $|S_1|=a$ and $|S_2|=k-a, 1\le a\le k$ respectively. The degree of each vertex of the cycles is 3.

Therefore, $\prod deg_G(v)=18an(k-a)$.

Hence,

$\sum_{\substack{S\subseteq V(G)\\|S|=k}}[\prod_{v\in S} d\,eg_G(v)]\,d_G(S)=\sum_{k=n+1}^{2n}[\sum_{a=k-n}^{n}[18an(k-a)]\binom{n}{a}\binom{n}{k-a}(k)].$

$$SGut_k(G)=\sum_{k=n+1}^{2n}\left[\sum_{a=k-1-n}^{n}[18an(k-1-a)]\binom{n}{a}\binom{n}{k-1-a}(k-1)\right]+\sum_{k=n+1}^{2n}\left[\sum_{a=k-n}^{n}[18an(k-a)]\binom{n}{a}\binom{n}{k-a}(k)\right].$$

Case (iii): $2\le k\le n$.

Let S be a set of vertices with $|S|=k$. Clearly $x\in S$ or $x\notin S$.

Suppose $x\in S$. By Theorem 1, $d_G(S)=k-1$. By the structure of G, x is adjacent to all the vertices of the two cycles.

Here the remaining $k-1$ vertices can be chosen from the vertices of cycles in $\binom{2n}{k-1}$ ways.

Hence, $\prod_{S\subseteq V(G)} d\,eg_G(v)=[6n(k-1)]\binom{2n}{k-1}$. Therefore,

$$\sum_{\substack{S\subseteq V(G)\\|S|=k}}\left[\prod_{v\in S} d\,eg_G(v)\right]d_G(S)\quad=\quad\left[6n(k-1)^2\binom{2n}{k-1}\right].$$

Suppose $x\notin S$. By Theorem 1, we have $k-1\le d_G(S)\le k$. Also S contains the vertices of cycles of G. Therefore $S\subseteq U_1$ or $S\subseteq U_2$ or $S\subseteq U_1\cup U_2$

Let $S\subseteq U_1$. First we consider S to be a set of consecutive vertices in the cycle C_1 of G. Then there are n such sets for $n>k$, and $d_G(S)=k-1$. The degree of each vertex in S is 3. Then $\prod_{U_1} d\,eg_G(v)=3^k n$ Therefore we get

Also S is a set of k vertices of C_1 which contains at least one non-consecutive vertex, then there are $\left[\binom{n}{k}-n\right]$ such sets with $d_G(S)=k$. The degree of each vertex in S is 3. Then,

$$\prod_{S\subseteq U_1} d\,eg_G(v)\quad=\quad 3^k\left[\binom{n}{k}-n\right].$$

Therefore,

$$\sum_{S\subseteq U_1}\left[\prod_{v\in S} d\,eg_G(v)\right]d_G(S)\quad=\quad 3^k\left[\binom{n}{k}-n\right](k)+3^k n(k-1).$$

Similarly, if $S \subseteq U_2$, we have,

$$\sum_{S\subseteq U_2}\left[\prod_{v\in S} d\,eg_G(v)\right] d_G(S) \quad = \quad 3^k\left[\binom{n}{k}-n\right](k)+3^k n(k-1).$$

Now we consider another set $S = S_1 \cup S_2$ with $|S| = k$, where $S_1 \subseteq U_1$ and $S_2 \subseteq U_2$ with $|S_1| = a$ and $|S_2| = k - a, 1 \leq a \leq k$ respectively. Also the degree of each vertex of the cycles is 3. By Theorem 1, we have $d_G(S) = k$. Therefore $\prod_{v\in S} d\,eg_G(v) = 3a.3(k-a)\binom{n}{a}\binom{n}{k-a}$.

Hence,

$$\sum_{S\subseteq U_1\cup U2}\left[\prod_{v\in S} d\,eg_G(v)\right] d_G(S) \quad = \quad \sum_{k=2}^{n}\left[9a(k-a)\sum_{a=k-n}^{n}\binom{n}{a}\binom{n}{k-a}(k)\right].$$

Therefore,

$$SGut_k(G) = \left[6n(k-1)^2\binom{2n}{k-1}\right] + 2.3^k\left[k\binom{n}{k}-n\right] + \sum_{k=2}^{n}\left[3k\sum_{a=k-n}^{n}\binom{n}{a}\binom{n}{k-a}(k)\right].$$

This completes the proof.

Steiner Harary index of Double Wheel Graph

In this section we obtain the Steiner Harary index of double wheel graph.

Theorem 5. Let $G = DW_n$ be a Double wheel graph with $n \geq 3$ vertices. Let k be an integer with $2 \leq k \leq 2n+1$. Let S be a subset of $V(G)$ with $|S| = k$. Then

1. $SH_k(G) = \frac{1}{2n}$, if $k = 2n+1$.
2. $SH_k(G) = \sum_{k=n+1}^{2n}\frac{\left[\sum_{a=k-1-n}^{n}\binom{n}{a}\binom{n}{k-1-a}\right]}{k-1} + \sum_{k=n+1}^{2n}\frac{\left[\sum_{a=k-n}^{n}\binom{n}{a}\binom{n}{k-a}\right]}{k}$, if $n+1 \leq k \leq 2n$.
3. $SH_k(G) = \binom{2n}{k-1}\frac{1}{(k-1)} + \frac{2}{k}\left[\binom{n}{k}-n\right] + \frac{2n}{k-1} + \sum_{k=2}^{n}\frac{\left[\sum_{a=k-n}^{n}\binom{n}{a}\binom{n}{k-a}\right]}{k}$, if $2 \leq k \leq n$.

Proof. By definition, G is the join of K_1 and two copies C_1 and C_2 of cycles of order n. Let x be the apex in G and Let $U_1 = \{u_1, u_2, \ldots, u_n\}$ be the set of vertices of the cycle C_1 and $U_2 = \{u'_1, u'_2, \ldots, u'_n\}$ be the set of vertices of the cycle C_2. Then $V(G) = U_1 \cup U_2 \cup \{x\}$.

Let $S \subseteq V(G)$ such that $|S| = k$, where $k \geq 2$ is an integer.

Case (i): $k = 2n+1$.

In this case S contains all the vertices of G. By Theorem 1, $d_G(S) = 2n$. Therefore,

$$
\begin{aligned}
SH_k(G) &= \sum_{|S|=2n+1} \frac{1}{d_G(S)} \\
&= \frac{1}{k-1} \\
&= \frac{1}{2n}.
\end{aligned}
$$

Case (ii): $n+1 \le k \le 2n$.

Let S be a set of vertices with $|S| = k$. Clearly $x \in S$ or $x \notin S$.

Suppose $x \in S$, By Theorem 1, $d_G(S) = k-1$. By the structure of G, x is adjacent to all the vertices of the cycles C_1 and C_2. The $k-1$ vertices can be chosen as follows; either a number of vertices in C_1 and $(k-1)-a$ verices in C_2 or a number of vertices in C_2 and $(k-1)-a$ vertices in $C_1, 1 \le a \le k-1$.

Therefore, $\sum_{|S|=k} d_G\,(S) = \sum_{k=n+1}^{2n} \frac{\left[\sum_{a=k-1-n}^{n} \binom{n}{a}\binom{n}{k-1-a}\right]}{k-1}$.

Suppose $x \notin S$. By Theorem 1 we have $d_G(S) = k$. Since $x \notin S$, S contains the vertices of $U_1 \cup U_2$. Hence S is a set with a vertices of cycle C_1 and $(k-a)$ vertices of the cycle C_2. or S is a set with a vertices from C_2 and $(k-a)$ vertices from $C_1, 1 \le a \le k$. And hence

$$\sum_{|S|=k} d_G\,(S) = \sum_{k=n+1}^{2n} \frac{\left[\sum_{a=k-n}^{n} \binom{n}{a}\binom{n}{k-a}\right]}{k}.$$

Therefore,

$$SH_k(G) = \sum_{k=n+1}^{2n} \frac{\left[\sum_{a=k-1-n}^{n} \binom{n}{a}\binom{n}{k-1-a}\right]}{k-1} + \sum_{k=n+1}^{2n} \frac{\left[\sum_{a=k-n}^{n} \binom{n}{a}\binom{n}{k-a}\right]}{k}$$

Case (iii): $2 \le k \le n$.

Let S be a set of vertices with $|S| = k$. Clearly $x \in S$ or $x \notin S$

Suppose $x \in S$, By Theorem 1, $d_G(S) = k-1$. By the structure of G, x is adjacent to all the vertices of the two cycles.

Here the remaining $k-1$ vertices can be chosen from the vertices of cycles in $\binom{2n}{k-1}$ ways.

Therefore, $\sum_{x \in S} d_G\,(S) = \binom{2n}{k-1}\frac{1}{(k-1)}$.

Suppose $x \notin S$, Then by Theorem 1 we have $k-1 \le d_G(S) \le k$. Also S contains the vertices of cycles of G. Therefore $S \subseteq U_1$ or $S \subseteq U_2$ or $S \subseteq U_1 \cup U_2$.

Let $S \subseteq U_1$. First we consider S to be set of consecutive vertices of the cycle C_1. Then there are n such sets for $n > k$ and $d_G(S) = k-1$. Therefore we get $\sum_{S \subseteq U_1} \frac{1}{d_G(S)} = \frac{n}{(k-1)}$.

Also S is a set of k vertices of C_1 which contains at least one non-consecutive vertex, then there are $\left[\binom{n}{k} - n\right]$ such sets with $d_G(S) = k$.

Therefore $\sum_{S\subseteq U_1} \frac{1}{d_G(S)} = \left[\binom{n}{k} - n\right]\frac{1}{(k)} + \frac{n}{(k-1)}$.
Similarly, if $S \subseteq U_2$,
we have, $\sum_{S\subseteq U_2} \frac{1}{d_G(S)} = \left[\binom{n}{k} - n\right]\frac{1}{(k)} + \frac{n}{(k-1)}$.
we consider a set $S = S_1 \cup S_2$ with $|S| = k$, where $S_1 \subseteq U_1$ and $S_2 \subseteq U_2$ with $|S_1| = a$ and $|S_2| = k - a, 1 \leq a \leq k$ respectively. By Theorem 1, we have $d_G(S) = k$.

$$\sum_{|S|=k} \frac{1}{d_G(S)}. = \sum_{k=2}^{n} \frac{\left[\sum_{a=k-n}^{n}\binom{n}{a}\binom{n}{k-a}\right]}{k}.$$

$$SH_k(G) = \binom{2n}{k-1}\frac{1}{(k-1)} + \frac{2}{k}\left[\binom{n}{k} - n\right] + \frac{2n}{k-1} + \sum_{k=2}^{n} \frac{\left[\sum_{a=k-n}^{n}\binom{n}{a}\binom{n}{k-a}\right]}{k}.$$

This completes the proof.

Steiner Reciprocal Degree Distance of Double Wheel Graph

In this section we obtain the Steiner degree distance of double wheel graph.

Theorem 6. Let $G = DW_n$ be a Double wheel graph with $n \geq 3$ vertices. Let k be an integer with $2 \leq k \leq 2n + 1$. Let S be a subset of $V(G)$ with $|S| = k$. Then

1. $SRDD_k(G) = 4$, if $k = 2n + 1$.
2. $SRDD_k(G) = \sum_{k=n+1}^{2n} \frac{\left[(2n+3k-3)\sum_{a=k-1-n}^{n}\binom{n}{a}\binom{n}{k-1-a}\right]}{k-1} + \sum_{k=n+1}^{2n} \frac{\left[(2n+3k)\sum_{a=k-n}^{n}\binom{n}{a}\binom{n}{k-a}\right]}{k}$, if $n + 1 \leq k \leq 2n$.
3. $SRDD_k(G) = \left[[2n + 3k - 3]\binom{2n}{k-1}\right]\frac{1}{k-1} + 6\left[\binom{n}{k} - n + \frac{kn}{k-1}\right] + \sum_{k=2}^{n} \frac{\left[3k\sum_{a=k-n}^{n}\binom{n}{a}\binom{n}{k-a}\right]}{k}$, if $2 \leq k \leq n$.

Proof. By definition, G is the join of K_1 and two copies C_1 and C_2 of cycles of order n. Let x be the apex in G and Let $U_1 = \{u_1, u_2, \ldots, u_n\}$ be the set of vertices of the cycle C_1 and $U_2 = \{u_1', u_2', \ldots, u_n'\}$ be the set of vertices of the cycle C_2. Then $V(G) = U_1 \cup U_2 \cup \{x\}$.
Let $S \subseteq V(G)$ such that $|S| = k$, where $k \geq 2$ is an integer.

Case (i): $k = 2n + 1$.

In this case S contains all the vertices of G. By the structure of G, x is adjacent to all the vertices of C_1 and C_2. Hence we have $deg_G(u_i) = 3$, $deg_G(u_i') = 3, 1 \leq i \leq n$. and $deg_G(x) = 2n$. Also by Theorem 1, we have $d_G(S) = k - 1 = 2n$. and $\sum_{v\in S} deg_G(v) = 8n$. Therefore,

$$\begin{aligned} SRDD_k(G) &= \sum_{|S|=2n+1} \frac{\left[\sum_{v\in S} deg_G(v)\right]}{d_G(S)} \\ &= 4 \end{aligned}$$

Case (ii): $n + 1 \leq k \leq 2n$.

Let S be a set of vertices with $|S| = k$. Clearly $x \in S$ or $x \notin S$.

Suppose $x \in S$, By Theorem 1, $d_G(S) = k - 1$. By the structure of G, x is adjacent to all the remaining $k - 1$ vertices of $C_1 \cup C_2$. The $k - 1$ vertices can be chosen as follows; either a number of vertices in C_1 and $(k - 1) - a$ verices in C_2 or a number of vertices in C_2 and $(k - 1) - a$ vertices in $C_1, 1 \leq a \leq k - 1$. Also the degree of each vertex of the cycles is 3. Therefore, $\sum deg_G(v) = 2n + 3k - 3$. Hence,

$$\sum_{\substack{S \subseteq V(G) \\ |S|=k}} \frac{\left[\sum_{v \in S} d\, eg_G(v)\right]}{d_G(S)} = \sum_{k=n+1}^{2n} \frac{\left[(2n + 3k - 3) \sum_{a=k-n}^{n} \binom{n}{a} \binom{n}{k-1-a}\right]}{k - 1}.$$

Suppose $x \notin S$. By Theorem 1, we have $d_G(S) = k$. Here we consider $S = S_1 \cup S_2$ with $|S| = k$, where $S_1 \subseteq U_1$ and $S_2 \subseteq U_2$ with $|S_1| = a$ and $|S_2| = k - a, 1 \leq a \leq k$ respectively. The degree of each vertex of the cycles is 3. Therefore, $\sum deg_G(v) = 2n + 3k$. Hence,

$$\sum_{\substack{S \subseteq V(G) \\ |S|=k}} \frac{\left[\sum_{v \in S} d\, eg_G(v)\right]}{d_G(S)} = \sum_{k=n+1}^{2n} \frac{\left[(2n + 3k) \sum_{a=k-n}^{n} \binom{n}{a} \binom{n}{k-a}\right]}{k}.$$

$$SRDD_k(G) = \sum_{k=n+1}^{2n} \frac{\left[(2n + 3k - 3) \sum_{a=k-1-n}^{n} \binom{n}{a} \binom{n}{k-1-a}\right]}{k - 1} + \sum_{k=n+1}^{2n} \frac{\left[(2n + 3k) \sum_{a=k-n}^{n} \binom{n}{a} \binom{n}{k-a}\right]}{k}.$$

Case (iii): $2 \leq k \leq n$.

Let S be a set of vertices with $|S| = k$. Clearly $x \in S$ or $x \notin S$.

Suppose $x \in S$. By Theorem 1, $d_G(S) = k - 1$. By the structure of G, x is adjacent to all the vertices of the two cycles.

Here the remaining $k - 1$ vertices can be chosen from the vertices of cycles in $\binom{2n}{k-1}$ ways. Hence, $\sum_{S \subseteq V(G)} d\, eg_G(v) = [2n + 3k - 3]\binom{2n}{k-1}$. Therefore,

$$\sum_{\substack{S \subseteq V(G) \\ |S|=k}} \frac{\left[\sum_{v \in S} d\, eg_G(v)\right]}{d_G(S)} = \left[(2n + 3k - 3)\binom{2n}{k - 1}\right] \frac{1}{(k - 1)}.$$

Suppose $x \notin S$. Then by Theorem 1, we have $k - 1 \leq d_G(S) \leq k$. And S contains the vertices in cycles of G. Therefore $S \subseteq U_1$ or $S \subseteq U_2$ or $S \subseteq U_1 \cup U_2$.

Let $S \subseteq U_1$. First we consider S to be a set of consecutive vertices of the cycle C_1 of G. Then there are n such sets for $n > k$, and $d_G(S) = k - 1$. The degree of each vertex in S is 3. Then $\sum_{U_1} d\, eg_G(v) = 3kn$. Therefore we get,

$$\sum_{S \subseteq U_1} d\, eg_G(v) = 3kn$$

If S is a set of k vertices of C_1 which contains at least one non-consecutive vertex, then there are $\left[\binom{n}{k} - n\right]$ such sets with $d_G(S) = k$. The degree of each vertex in S is 3. Then,

$$\sum_{S\subseteq U_1} deg_G(v) \quad = \quad 3k\left[\binom{n}{k}-n\right]$$

Therefore,

$$\sum_{S\subseteq U_1} \frac{[\sum_{v\in S} deg_G(v)]}{d_G(S)} \quad = \quad 3k\left[\binom{n}{k}-n\right]\frac{1}{k}+3kn\frac{1}{k-1}.$$

Similarly, if $S \subseteq U_2$. we have,

$$\sum_{S\subseteq U_2} \frac{[\sum_{v\in S} deg_G(v)]}{d_G(S)} \quad = \quad 3k\left[\binom{n}{k}-n\right]\frac{1}{k}+3kn\frac{1}{k-1}.$$

Now we consider another set $S = S_1 \cup S_2$ with $|S| = k,$ where $S_1 \subseteq U_1$ and $S_2 \subseteq U_2$ with $|S_1| = a$ and $|S_2| = k-a, 1 \le a \le k$ respectively. Also the degree of each vertex of the cycles is 3. By Theorem 1, we have $d_G(S) = k$. Therefore $\sum_{v\in S} deg_G(v) = 3k\binom{n}{a}\binom{n}{k-a}$. Hence,

$$\sum_{S\subseteq U_1\cup U2} \frac{[\sum_{v\in S} deg_G(v)]}{d_G(S)} \quad = \quad \sum_{k=2}^{n}\frac{\left[3k\sum_{a=k-n}^{n}\binom{n}{a}\binom{n}{k-a}\right]}{k}.$$

Therefore,

$$SRDD_k(G) = \left[[2n+3k-3]\binom{2n}{k-1}\right]\frac{1}{k-1}+6\left[\binom{n}{k}-n+\frac{kn}{k-1}\right] + \sum_{k=2}^{n}\frac{\left[3k\sum_{a=k-n}^{n}\binom{n}{a}\binom{n}{k-a}\right]}{k}.$$

This completes the proof.

Conclusion

In this paper, we obtained the Steiner distance of double wheel graph. By using the Steiner distance, we obtained the Steiner Wiener index, Steiner degree distance, Steiner Gutman index, Seiner Harary index and Steiner reciprocal degree distance of double wheel graphs.

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