FIXED POINT THEOREM FROM GENERALIZED $Z_s^{'}$ CONTRACTION IN S- METRIC SPACE

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Abstract

In this paper, we define generalized $Z_s^{'}$ contraction related to γ, where γ is a simulation function in S-metric space and give some examples. Also we derive fixed point of a self mapping by using this definition and support it by some examples.

Key words: Fixed point, contraction mapping, Generalized Z_s Contraction, Simulation function

AMS classification: 54H25,47H10

Notation: R denotes the set of all real numbers and R^+ denotes the set of all non-negative real numbers.

1. Introduction & Preliminaries

Fixed point theorems are very helpful in nonlinear analysis and solving differential equations. Today mathematicians researches around this area for finding fixed points in both complete,non-complete metric spaces. Many of them generalizes the concept and derive new theorems. Khojasteh et al introduced the concept of Z-contractions along with simulation functions and derive some fixed point theorems in complete metric space. Later Erdal Karapinar presented the new concept by defining contractive condition through admissible mapping which is imbedded with simulation function in complete partially ordered metric spaces. S.Sedgi, N.Shobe, A.Aliouche generalized the fixed point theorem in S-Metric Space. Antonio-Francisco, Karpinar established common fixed point in the metric space through simulation function.MuratOlgun, O.Bicerand T.Alyildiz established generalized Z-contraction in relation to the simulation functions and derive the invariant point. Nihal Tas Nihal Ylimaz Ozgur and Nabil Mlaiki researched the invariant point by using collection of simulation mappings in S-metric spaces and then Venkatesh and R.NagaRaju generalized this concept in S-metric space**.**

Definition:1.1[1] A mapping S: $X \times X \times X \to R^+$ where X be any non-empty set is said to be an S-metric on X if S satisfies the following conditions:

(i) $S(a,b,c) > 0 \ \forall\ a,b,c \in X$ with $a \neq b \neq c$

(ii) $S(a,b,c) = 0$ if $a = b = c$

(iii) $S(a,b,c) \leq S(a,a,t) + S(b,b,t) + S(c,c,t)\ \forall a,b,c,t \in X$ If S is a metric on a non-empty set X, then (X,S) is said to be S-metric space.

Example: 1.2[1] Define a mapping $S: X \times X \times X \to R^+$ by $S(a,b,c) = d(a,b) + d(b,c) + d(a,c)$ $\forall\ a,b,c \in X$ where X is the metric space under the metric 'd'. Then (X,S) is S-metric space.

Example: 1.2[3]Take $X = R,\ S: R^3 \to R^+$ defined by $S(a,b,c) = |b + c - 2a| + |b - c|\ \forall\ a,b,c \in X$ is S metric on R & (X,S) is called S-metric space.

Example:1.3[7]Define $S: R^3 \rightarrow R^+$ by $S(a,b,c) = |a-c| + |b-c| \; \forall \; a,b,c \in R$ then (R,S) is S-metric space.

Results: In S-metric space,

(i) $S(a,a,b) = S(b,b,a)$

(ii) $S(a,a,b) \leq 2S(a,a,c) + S(b,b,c)$

(iii) $S(a,a,b) \leq 2S(a,a,c) + S(c,c,b)$

Definition: 1.2 [1]If $S(\eta_n, \eta_n, \eta) \rightarrow 0$ as $n \rightarrow \infty$,then we say that $\{\eta_n\}$ in X converges to $\eta \in X$. (ie) for arbitrary $\varepsilon > 0$,there exists a natural number m such that $S(\eta_n, \eta_n, \eta) < \varepsilon \; \forall \, n \geq m$ .Also we write it by $\lim_{n\rightarrow\infty} \eta_n = \eta$.

Definition: 1.3A sequence $\{\eta_n\}$ in S-metric space (X,S) is said to be Cauchy sequence if for any $\varepsilon > 0$, $m \in N$ such that $S(\eta_n, \eta_n, \eta_m) < \varepsilon \; \forall \, n, m \geq n_0$.

Definition: 1.4A S-metric space (X,S) is said to be complete if every Cauchy sequence in X is converges to a limit in X .

Definition:1.5[1]Let (X,S) be S-metric space. A mapping $h: X \rightarrow X$ is said to be S-contraction if $S(h\eta, h\eta, hv) \leq tS(\eta, \eta, v) \; \forall \, \eta, v \in X \; \& \; 0 \leq t < 1$.

Definition:1.6 [2]A function $\gamma: R^+ \times R^+ \rightarrow R$ is a simulation function if

(i) $\gamma(0,0) = 0$

(ii) $\gamma(s,t) \leq t - s$ for X $s,t > 0$

(iii) If $\{s_n\}$, $\{t_n\}$ are 2 sequences in and $\lim_{n\rightarrow\infty} s_n = \lim_{n\rightarrow\infty} t_n > 0$,then $\lim_{n\rightarrow\infty} \sup \gamma(s_n, t_n) < 0$

The collection of all simulation mappings is indicated by Z .For example, $\gamma(s,t) = kt - s$ for $0 \leq k < 1$ belongs to Z .

Definition: 1.7[5] A self mapping h on a S-metric space (X,S) is said to be Z_s – contraction in relation to γ if $\gamma(S(ha,ha,hb), S(a,a,b)) \geq 0 \; \forall \, a,b \in X \; \& \; \gamma \in Z$

Theorem:[5] Let (X,S) be a S-metric space. Then a self map h on X has a unique fixed point $l \in X$ where l is the limit of the Picard Sequence $\{\eta_n\}$ whenever h is a Z_s – contraction in relation to γ .

2. Main Results

Definition:2.1 Let (X , S) be a complete metric space. A self map h on X is said to be generalized $Z_s^{'}$ contraction related to γ if $\gamma(S(h\zeta, h\zeta, hw), M(\zeta, \zeta, w)) \geq 0 \; \forall \; \zeta, w \in X$ where

$$M(\zeta,\zeta,w) = Max\{S(\zeta,\zeta,w), S(\zeta,h\zeta,h\zeta), S(w,w,hw), \frac{1}{4}[S(\zeta,hw,hw) + S(hw,h\zeta,h\zeta) + S(\zeta,h\zeta,hw) + S(w,h\zeta,hw)]\}$$

Example: 2.1 Let $X = [0,1]$be a complete metric space with the metric S as $S: X \times X \times X \rightarrow R^+$ defined by $S(\zeta, v, w) = |\zeta - w| + |v - w|$ .Define $h: X \rightarrow X$ by

$$h\zeta = \begin{cases} \frac{1}{7} & [0, \frac{4}{7}) \\ \frac{3}{7} & [\frac{4}{7}, 1] \end{cases}$$

Case(i): $\zeta, w \in [0, \frac{4}{7})$

Let $\zeta = 0, w = \frac{1}{2}$,then $h\zeta = hw = \frac{1}{7}$

$$S(h\zeta, h\zeta, hw) = |h\zeta - hw| + |h\zeta - hw| = 2|h\zeta - hw| = 2\left|\frac{1}{7} - \frac{1}{7}\right| = 0$$

$$S(\zeta, \zeta, w) = |\zeta - w| + |\zeta - w| = 2|\xi - w|$$

$$S\left(0, 0, \frac{1}{2}\right) = 2\left|0 - \frac{1}{2}\right| = 2\left|-\frac{1}{2}\right| = 2 \times \frac{1}{2} = 1$$

$$S(\zeta, h\zeta, h\zeta) = |\zeta - h\zeta| + |h\zeta - h\zeta| = |\zeta - h\zeta|$$

$$S\left(0, \frac{1}{7}, \frac{1}{7}\right) = \left|0 - \frac{1}{7}\right| = \frac{1}{7}$$

$$S(w, w, hw) = S\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{7}\right) = \frac{10}{7},\ S(\zeta, hw, hw) = S\left(0, \frac{1}{7}, \frac{1}{7}\right) = \frac{1}{7},$$

$$S(hw, h\zeta, h\zeta) = S\left(\frac{1}{7}, \frac{1}{7}, \frac{1}{7}\right) = 0,\ S(\zeta, h\zeta, hw) = S\left(0, \frac{1}{7}, \frac{1}{7}\right) = \frac{1}{7},$$

$$S(w, h\zeta, hw) = S\left(\frac{1}{2}, \frac{1}{7}, \frac{1}{7}\right) = \frac{5}{14},$$

$$M(\zeta, \zeta, w) = Max\left\{1, \frac{1}{7}, \frac{10}{7}, \frac{1}{4}\left[\frac{1}{7} + 0 + \frac{1}{7} + \frac{5}{14}\right]\right\} = \frac{10}{7}$$

$$Here, S(h\zeta, h\zeta, hw) < M(\zeta, \zeta, w)$$

$$Case(ii): \zeta \in \left[0, \frac{4}{7}\right), w \in \left[\frac{4}{7}, 1\right]$$

(a)Let $\zeta = \frac{1}{7}, w = \frac{5}{7}$,then $h\zeta = \frac{1}{7}, hw = \frac{3}{7},$

$here, \zeta = h\zeta$

$$S(h\zeta, h\zeta, hw) = S\left(\frac{1}{7}, \frac{1}{7}, \frac{3}{7}\right) = \frac{4}{7}$$

$$S(\zeta, \zeta, w) = S\left(\frac{1}{7}, \frac{1}{7}, \frac{5}{7}\right) = \frac{8}{7},\ S(\zeta, h\zeta, h\zeta) = S\left(\frac{1}{7}, \frac{1}{7}, \frac{1}{7}\right) = 0,$$

$$S(w, w, hw) = S\left(\frac{5}{7}, \frac{5}{7}, \frac{3}{7}\right) = \frac{4}{7},\ S(\zeta, hw, hw) = S\left(\frac{1}{7}, \frac{3}{7}, \frac{3}{7}\right) = \frac{2}{7},$$

$$S(hw, h\zeta, h\zeta) = S\left(\frac{3}{7}, \frac{1}{7}, \frac{1}{7}\right) = \frac{2}{7},\ S(\zeta, h\zeta, hw) = S\left(\frac{1}{7}, \frac{1}{7}, \frac{3}{7}\right) = \frac{4}{7},$$

$$S(w, h\zeta, hw) = S\left(\frac{5}{7}, \frac{1}{7}, \frac{3}{7}\right) = \frac{4}{7},$$

$$M(\zeta, \zeta, w) = Max\left\{\frac{8}{7}, 0, \frac{4}{7}, \frac{1}{4}\left[\frac{2}{7} + \frac{2}{7} + \frac{4}{7} + \frac{4}{7}\right]\right\} = \frac{8}{7}$$

$$Here,\ S(h\zeta, h\zeta, hw) < M(\zeta, \zeta, w)$$

(b)Let $\zeta = \frac{2}{7}, w = \frac{5}{7}$,then $h\zeta = \frac{1}{7}, hw = \frac{3}{7},$

$here, \zeta \neq h\zeta\ \&\ w \neq hw$

$S(h\zeta, h\zeta, hw) = S\left(\frac{1}{7}, \frac{1}{7}, \frac{3}{7}\right) = \frac{4}{7}$

$S(\zeta, \zeta, w) = S\left(\frac{2}{7}, \frac{2}{7}, \frac{5}{7}\right) = \frac{6}{7}, S(\zeta, h\zeta, h\zeta) = S\left(\frac{2}{7}, \frac{1}{7}, \frac{1}{7}\right) = \frac{1}{7},$

$S(w, w, hw) = S\left(\frac{5}{7}, \frac{5}{7}, \frac{3}{7}\right) = \frac{4}{7}, S(\zeta, hw, hw) = S\left(\frac{2}{7}, \frac{3}{7}, \frac{3}{7}\right) = \frac{1}{7},$

$S(hw, h\zeta, h\zeta) = S\left(\frac{3}{7}, \frac{1}{7}, \frac{1}{7}\right) = \frac{2}{7}, S(\zeta, h\zeta, hw) = S\left(\frac{2}{7}, \frac{1}{7}, \frac{3}{7}\right) = \frac{3}{7},$

$S(w, h\zeta, hw) = S\left(\frac{5}{7}, \frac{1}{7}, \frac{3}{7}\right) = \frac{4}{7},$

$M(\zeta, \zeta, w) = Max\left\{\frac{6}{7}, \frac{1}{7}, \frac{4}{7}, \frac{1}{4}\left[\frac{1}{7} + \frac{2}{7} + \frac{3}{7} + \frac{4}{7}\right]\right\} = \frac{6}{7}$

$\therefore S(h\zeta, h\zeta, hw) < M(\zeta, \zeta, w)$

$(c)\zeta = 0, w = 1$

(lower and upper bounds of the interval in which the function h defined)

Then $h\zeta = \frac{1}{7}, hw = \frac{3}{7},$

$S(h\zeta, h\zeta, hw) = S\left(\frac{1}{7}, \frac{1}{7}, \frac{3}{7}\right) = \frac{4}{7}$

$S(\zeta, \zeta, w) = S(0,0,1) = 2,\ S(\zeta, h\zeta, h\zeta) = S\left(0, \frac{1}{7}, \frac{1}{7}\right) = \frac{1}{7},$

$S(w, w, hw) = S\left(1,1, \frac{3}{7}\right) = \frac{8}{7},\ S(\zeta, hw, hw) = S\left(0, \frac{3}{7}, \frac{3}{7}\right) = \frac{3}{7},$

$S(hw, h\zeta, h\zeta) = S\left(\frac{3}{7}, \frac{1}{7}, \frac{1}{7}\right) = \frac{2}{7},\ S(\zeta, h\zeta, hw) = S\left(0, \frac{1}{7}, \frac{3}{7}\right) = \frac{5}{7},$

$S(w, h\zeta, hw) = S\left(1, \frac{1}{7}, \frac{3}{7}\right) = \frac{6}{7},$

$M(\zeta, \zeta, w) = Max\left\{2, \frac{1}{7}, \frac{8}{7}, \frac{1}{4}\left[\frac{3}{7} + \frac{2}{7} + \frac{5}{7} + \frac{6}{7}\right]\right\} = 2$

$Here, S(h\zeta, h\zeta, hw) < M(\zeta, \zeta, w)$

If ζ & w *are changing their values, we get the same inequality as above.*

$Case(iii): \zeta, w \in \left[\frac{4}{7}, 1\right]$

Let $\zeta = \frac{5}{7}, w = 1$, then $h\zeta = \frac{3}{7}, hw = \frac{3}{7},$

$S(h\zeta, h\zeta, hw) = S\left(\frac{3}{7}, \frac{3}{7}, \frac{3}{7}\right) = 0$

$$S(\zeta,\zeta,w)=S\left(\frac{5}{7},\frac{5}{7},1\right)=\frac{4}{7}, S(\zeta,h\zeta,h\zeta)=S\left(\frac{5}{7},\frac{3}{7},\frac{3}{7}\right)=\frac{2}{7},$$

$$S(w,w,hw)=S\left(1,1,\frac{3}{7}\right)=\frac{8}{7}, S(\zeta,hw,hw)=S\left(\frac{5}{7},\frac{3}{7},\frac{3}{7}\right)=\frac{2}{7},$$

$$S(hw,h\zeta,h\zeta)=S\left(\frac{3}{7},\frac{3}{7},\frac{3}{7}\right)=0, S(\zeta,h\zeta,hw)=S\left(\frac{5}{7},\frac{3}{7},\frac{3}{7}\right)=\frac{2}{7},$$

$$S(w,h\zeta,hw)=S\left(1,\frac{3}{7},\frac{3}{7}\right)=\frac{4}{7},$$

$$M(\zeta,\zeta,w)=Max\left\{\frac{8}{7},0,\frac{4}{7},\frac{1}{4}\left[\frac{2}{7}+\frac{2}{7}+\frac{4}{7}+\frac{4}{7}\right]\right\}=\frac{8}{7}$$

$$Here, S(h\zeta,h\zeta,hw)<M(\zeta,\zeta,w)$$

$$Case(iv): \zeta=w=h\zeta=hw$$

$$Then, S(h\zeta,h\zeta,hw)=M(\zeta,\zeta,w)=0$$

Hence we conclude that

$$S(h\zeta,h\zeta,hw)\le\frac{4}{7} \quad \& \quad S(h\zeta,h\zeta,hw)\le\frac{2}{3}M(\zeta,\zeta,w)$$

h is generalized $Z_s^{'}$ contraction related to γ if γ is defined as $\gamma(p,q)=\frac{2}{3}q-p$

$$Since,\ \gamma(S(h\zeta,h\zeta,hw),M(\zeta,\zeta,w))=\frac{2}{3}\ (M(\zeta,\zeta,w)-S(h\zeta,h\zeta,hw)\ \ge 0$$

Lemma 2.1: Let (X , S) be an S-metric space. Then any contraction mapping h on X is a generalized $Z_s^{'}$ contraction related to γ .

Proof: Since h is contraction mapping on X .

$$\Rightarrow S(h\zeta,h\zeta,hw)\le k\ S(\zeta,\zeta,w)\le k\ M(\zeta,\zeta,w)\ where\ \ k\in[0,1)$$

$$\Rightarrow k\ M(\zeta,\zeta,w)-S(h\zeta,h\zeta,hw)\ge 0$$

$$\Rightarrow \gamma(S(h\zeta,h\zeta,hw),\ M(\zeta,\zeta,w))\ge 0$$

$\therefore h$ is generalized $Z_s^{'}$ contraction related to γ if γ is defined as $\gamma(p,q)=kq-p$

Definition 2.2: Let (X , S) be an S-metric space. Then a self mapping h on X is said to be asymptotically regular at $\eta\in X$ if $\lim_{k\to\infty}S(h^k\eta,h^k\eta,h^{k+1}\eta)=0$.

Lemma 2.2:If a self mapping h is a generalized $Z_s^{'}$ contraction related to γ . Then h is asymptotically regular at each point $\eta\in X$.

Proof: Let $\eta\in X$

Case (i) Suppose $h^k\eta=h^{k-1}\eta$ for any arbitrary $k\in N$

that means $ha=a$ where $a=h^{k-1}\eta$

Then $h^m\eta=h^{m-1}h\eta=h^{m-1}\eta=...=h\eta=\eta\ \ \forall\ m\in N$

$$\begin{aligned} Now,\ S(h^k\eta,\ h^k\eta,\ h^{k+1}\eta) &= S(h^{k-l+1}h^{l-1}\eta,\ h^{k-l+1}h^{l-1}\eta,\ h^{k-l+2}h^{l-1}\eta) \\ &= S(h^{k-l+1}a,\ h^{k-l+1}a,\ h^{k-l+2}a) \\ &= S(a,a,a) \\ &= 0 \end{aligned}$$

$Hence,\ \lim_{k\to\infty} S(h^k\eta, h^k\eta, h^{k+1}\eta) = 0$

(ie) h is asymptotically regular at η.

Case (ii) Suppose $h^t\eta \neq h^{t+1}\eta \ \ \forall\, t \in N$

Since h is a generalized $Z_s^{'}$ contraction related to γ

$We\ have,\ \gamma(S(h^{k+1}\eta, h^{k+1}\eta, h^k\eta), M(h^k\eta, h^k\eta, h^{k-1}\eta) \geq 0$

$$where,\ M(h^k\eta,h^k\eta,h^{k-1}\eta) = Max\{S(h^k\eta,h^k\eta,h^{k-1}\eta), S(h^k\eta,h^{k+1}\eta,h^{k+1}\eta), S(h^{k-1}\eta,h^{k-1}\eta,h^k\eta),$$

$$\frac{1}{4}[S(h^k\eta,h^k\eta,h^k\eta) + S(h^k\eta,h^{k+1}\eta,h^{k+1}\eta) + S(h^k\eta,h^{k+1}\eta,h^k\eta) + S(h^{k-1}\eta,h^{k+1}\eta,h^k\eta)]\}$$

$$\leq Max\{S(h^k\eta,h^k\eta,h^{k-1}\eta), S(h^k\eta,h^{k+1}\eta,h^{k+1}\eta), \frac{1}{4}[2S(h^k\eta,h^{k+1}\eta,h^{k+1}\eta) + S(h^{k-1}\eta,h^{k-1}\eta,h^k\eta) + S(h^{k+1}\eta,h^{k+1}\eta,h^k\eta)]\}$$

$$= Max\{S(h^k\eta,h^k\eta,h^{k-1}\eta), S(h^k\eta,h^{k+1}\eta,h^{k+1}\eta), \frac{1}{4}[3S(h^k\eta,h^{k+1}\eta,h^{k+1}\eta) + S(h^{k-1}\eta,h^{k-1}\eta,h^k\eta)]\}$$

$$= Max\{S(h^k\eta,h^k\eta,h^{k-1}\eta), S(h^k\eta,h^{k+1}\eta,h^{k+1}\eta)\}$$

$Since,\ \frac{1}{4}[3S(h^k\eta,h^{k+1}\eta,h^{k+1}\eta) + S(h^{k-1}\eta,h^{k-1}\eta,h^k\eta)]\ lies\ between$

$S(h^k\eta,h^k\eta,h^{k-1}\eta)\ \ and\ S(h^k\eta,h^{k+1}\eta,h^{k+1}\eta).$

$\therefore\ M(h^k\eta,h^k\eta,h^{k-1}\eta)\ \leq Max\{S(h^k\eta,h^k\eta,h^{k-1}\eta), S(h^k\eta,h^{k+1}\eta,h^{k+1}\eta)\}.$

$If\ \ S(h^k\eta,h^{k+1}\eta,h^{k+1}\eta) > S(h^k\eta,h^k\eta,h^{k-1}\eta)$,

$then\ M(h^k\eta,h^k\eta,h^{k-1}\eta) = S(h^k\eta,h^{k+1}\eta,h^{k+1}\eta)$

$$\begin{aligned} Now,\ \ 0 &\leq \gamma\ (S(h^{k+1}\eta,\ h^{k+1}\eta,\ h^k\eta),\ S(h^{k+1}\eta,\ h^{k+1}\eta,\ h^k\eta)\\ &< S(h^{k+1}\eta,h^{k+1}\eta,h^k\eta) - S(h^{k+1}\eta,h^{k+1}\eta,h^k\eta)\\ &= 0 \end{aligned}$$

Which is a contradiction to the definition of γ.

$\therefore\ M(h^k\eta,h^k\eta,h^{k-1}\eta) = S(h^k\eta,h^k\eta,h^{k-1}\eta)$

By using generalized $Z_s^{'}$ contraction related to γ property, we get,

$$\begin{aligned} 0 &\leq \gamma(S(h^{k+1}\eta,h^{k+1}\eta,h^k\eta), M(h^k\eta,h^k\eta,h^{k-1}\eta))\\ &< M(h^k\eta,h^k\eta,h^{k-1}\eta) - S(h^{k+1}\eta,h^{k+1}\eta,h^k\eta)\\ &= S(h^k\eta,h^k\eta,h^{k-1}\eta) - S(h^{k+1}\eta,h^{k+1}\eta,h^k\eta) \end{aligned}$$

$\Rightarrow S(h^{k+1}\eta,h^{k+1}\eta,h^k\eta) < S(h^k\eta,h^k\eta,h^{k-1}\eta)$

This shows that $S(h^k\eta,h^k\eta,h^{k+1}\eta)$ is a strictly monotonic decreasing sequence of non- negative real numbers.

By the property of monotonic, $S(h^k\eta, h^k\eta, h^{k+1}\eta)$ is convergent.

$Let \ \lim_{k\to\infty} S(h^k\eta, h^k\eta, h^{k+1}\eta) = l \ \ where \ \ l \geq 0$

$Suppose \ \ l > 0$

By the definition of generalized $Z_s^{'}$ contraction related to γ property, and simulation function,

$$We \ \ have, \ 0 \leq \limsup_{k\to\infty} \gamma(S(h^{k+1}\eta, h^{k+1}\eta, h^k\eta), M(h^k\eta, h^k\eta, h^{k-1}\eta))$$

$$= \limsup_{k\to\infty} \ \gamma(S(h^{k+1}\eta, h^{k+1}\eta, h^k\eta), S(h^k\eta, h^k\eta, h^{k-1}\eta)) < 0$$

Which is a contradiction.

$\therefore \ l = 0$

$Hence, \ \lim_{k\to\infty} S(h^k\eta, h^k\eta, h^{k-1}\eta) = 0$

Hence h is asymptotically regular at every point $\eta \in X$.

Lemma 2.3: The sequence $\{\eta_n\}$ defined as $h\eta_{n-1} = h\eta_n$ with initial point $\eta_0 \in X$ is bounded if h is a generalized $Z_s^{'}$ contraction related to γ.

[The sequence defined above is known as Picard Sequence]

Proof:

Let $\{\eta_n\}$ be the sequence defined as $h\eta_{n-1} = h\eta_n$ where h is a generalized $Z_s^{'}$ contraction related to γ.

To Prove: $\{\eta_n\}$ is bounded.

Suppose not.

(ie) $\{\eta_n\}$ is not bounded, then there exists a real number $k > 0,$ for this we can find a subsequence $\eta_{n_s} \in X$ of $\{\eta_n\}$ and a smallest positive integer n_{s+1} such that

$$S(\eta_{n_{s+1}}, \eta_{n_{s+1}}, \eta_{n_s}) > k \ \ \& \ \ S(\eta_{n_m}, \eta_{n_m}, \eta_{n_s}) \leq k \ \ where \ \ n_s \leq m \leq n_{s+1} - 1$$

$$Now, k < S(\eta_{n_{s+1}}, \eta_{n_{s+1}}, \eta_{n_s}) \leq 2\, S(\eta_{n_{s+1}}, \eta_{n_{s+1}}, \eta_{n_{s+1}-1}) + S(\eta_{n_s}, \eta_{n_s}, \eta_{n_{s+1}-1})$$

$$\leq 2\, S(\eta_{n_{s+1}}, \eta_{n_{s+1}}, \eta_{n_{s+1}}, \eta_{n_{s+1}-1}) + k$$

$$As \ s \to \infty, \ S(\eta_{n_{s+1}}, \eta_{n_{s+1}}, \eta_{n_s}) = k$$

$$Since \ \{\eta_n\} is \ Picard \ \ Sequence, \ We \ have$$

$$S(h\eta_{n_{s+1}-1}, h\eta_{n_{s+1}-1}, h\eta_{n_s-1}) = S(\eta_{n_{s+1}}, \eta_{n_{s+1}}, \eta_{n_s})$$

$$Also \ h \ is \ generalized \ Z_s^{'} \ contraction \ related \ to \ \gamma \ , hence \ it \ is \ Asymtotically \ regular$$

$$\Rightarrow \lim_{n\to\infty} S(\eta_{n_{s+1}-1}, \eta_{n_{s+1}-1}, \eta_{n_{s+1}}) = 0$$

$$Now, by \ the \ definition \ of \ \gamma$$

$$\gamma(S(h\eta_{n_{s+1}-1}, h\eta_{n_{s+1}-1}, h\eta_{n_s-1}), M(\eta_{n_{s+1}-1}, \eta_{n_{s+1}-1}, \eta_{n_s-1}) \geq 0$$

$$\Rightarrow M(\eta_{n_{s+1}-1}, \eta_{n_{s+1}-1}, \eta_{n_s-1}) - S(h\eta_{n_{s+1}-1}, h\eta_{n_{s+1}-1}, h\eta_{n_s-1}) \geq 0$$

$$\Rightarrow M(\eta_{n_{s+1}-1}, \eta_{n_{s+1}-1}, \eta_{n_s-1}) \geq S(h\eta_{n_{s+1}-1}, h\eta_{n_{s+1}-1}, h\eta_{n_s-1})$$

$$= S(\eta_{n_{s+1}}, \eta_{n_{s+1}}, \eta_{n_s})$$

$$= k$$

$$Thus,\ k \le M(\eta_{n_{s+1}-1}, \eta_{n_{s+1}-1}, \eta_{n_s-1})$$
$$= Max\{S(\eta_{n_{s+1}-1}, \eta_{n_{s+1}-1}, \eta_{n_s-1}), S(\eta_{n_{s+1}-1}, h\eta_{n_{s+1}-1}, h\eta_{n_{s+1}-1}), S(\eta_{n_s-1}, \eta_{n_s-1}, h\eta_{n_s-1}),$$
$$\frac{1}{4}[S(\eta_{n_{s+1}-1}, h\eta_{n_s-1}, h\eta_{n_s-1}) + S(h\eta_{n_s-1}, h\eta_{n_{s+1}-1}, h\eta_{n_{s+1}-1})$$
$$+ S(\eta_{n_{s+1}-1}, h\eta_{n_{s+1}-1}, h\eta_{n_s-1}) + S(\eta_{n_s-1}, h\eta_{n_{s+1}-1}, h\eta_{n_s-1})]\}$$
$$= Max\{S(\eta_{n_{s+1}-1}, \eta_{n_{s+1}-1}, \eta_{n_s-1}), S(\eta_{n_{s+1}-1}, \eta_{n_{s+1}}, \eta_{n_{s+1}}), S(\eta_{n_s-1}, \eta_{n_s-1}, \eta_{n_s}),$$
$$\frac{1}{4}[S(\eta_{n_{s+1}-1}, \eta_{n_s}, \eta_{n_s}) + S(\eta_{n_s}, \eta_{n_{s+1}}, \eta_{n_{s+1}})$$
$$+ S(\eta_{n_{s+1}-1}, \eta_{n_{s+1}}, \eta_{n_s}) + S(\eta_{n_s-1}, \eta_{n_{s+1}}, \eta_{n_s})]\}$$
$$\le Max\{S(\eta_{n_s-1}, \eta_{n_s-1}, \eta_{n_{s+1}-1}), S(\eta_{n_{s+1}-1}, \eta_{n_{s+1}-1}, \eta_{n_{s+1}}), S(\eta_{n_s-1}, \eta_{n_s-1}, \eta_{n_s}),$$
$$\frac{1}{4}[S(\eta_{n_{s+1}-1}, \eta_{n_{s+1}-1}, \eta_{n_s}) + S(\eta_{n_s}, \eta_{n_s}, \eta_{n_{s+1}}) + S(\eta_{n_{s+1}-1}, \eta_{n_{s+1}-1}, \eta_{n_s}) +$$
$$S(\eta_{n_{s+1}}, \eta_{n_{s+1}}, \eta_{n_s}) + S(\eta_{n_s-1}, \eta_{n_s-1}, \eta_{n_s}) + S(\eta_{n_{s+1}}, \eta_{n_{s+1}}, \eta_{n_s})]\}$$
$$\le Max\{2S(\eta_{n_s-1}, \eta_{n_s-1}, \eta_{n_s}) + S(\eta_{n_{s+1}-1}, \eta_{n_{s+1}-1}, \eta_{n_s}), S(\eta_{n_s-1}, \eta_{n_s-1}, \eta_{n_s})$$
$$\frac{1}{4}[2S(\eta_{n_{s+1}-1}, \eta_{n_{s+1}-1}, \eta_{n_s}) + 3S(\eta_{n_s}, \eta_{n_s}, \eta_{n_{s+1}}) + S(\eta_{n_s-1}, \eta_{n_s-1}, \eta_{n_s})]\}$$
$$\le Max\{2S(\eta_{n_s-1}, \eta_{n_s-1}, \eta_{n_s}) + k, S(\eta_{n_s-1}, \eta_{n_s-1}, \eta_{n_s}),$$
$$\frac{1}{4}[2k + 3S(\eta_{n_s}, \eta_{n_s}, \eta_{n_{s+1}}) + S(\eta_{n_s-1}, \eta_{n_s-1}, \eta_{n_s})]\}$$

$$As\ s \to \infty,\ k \le \lim_{s\to\infty} M(\eta_{n_{s+1}-1}, \eta_{n_{s+1}-1}, \eta_{n_s-1}) \le k$$
$$\Rightarrow \lim_{s\to\infty} M(\eta_{n_{s+1}-1}, \eta_{n_{s+1}-1}, \eta_{n_s-1}) = k$$
$$= S(\eta_{n_{s+1}}, \eta_{n_{s+1}}, \eta_{n_s})$$

By the definition of γ and generalized $Z_s^{'}$ contraction related to γ property,

$$0 \le \lim_{s\to\infty} Sup\ \gamma(S(h\eta_{n_{s+1}-1}, h\eta_{n_{s+1}-1}, h\eta_{n_s-1}), M(\eta_{n_{s+1}-1}, \eta_{n_{s+1}-1}, \eta_{n_s-1}))$$
$$= \lim_{s\to\infty} Sup\ \gamma(S(\eta_{n_{s+1}}, \eta_{n_{s+1}}, \eta_{n_s}), S(\eta_{n_{s+1}}, \eta_{n_{s+1}}, \eta_{n_s}))$$
$$= 0$$

Which is a contradiction to the definition of γ, since the limit supremum is strictly less than 0
Hence our assumption is wrong.
Therefore, $\{\eta_n\}$ is bounded.

Theorem:2.1 Let (X,S) be a complete S-metric space and if $h: X \to X$ is generalized $Z_s^{'}$ contraction related to γ. Then there exists a unique fixed point for h and the Picard Sequence converges to the fixed point of h.

Proof: Let $\{\eta_n\}$ be the Picard Sequence with initial point $\eta_0 \in X$.
First we have to prove that $\{\eta_n\}$ is a Cauchy sequence.
Consider, $A_n = Sup\ \{S(\eta_t, \eta_t, \eta_s) / t, s \ge n\}$
Clearly $\{A_n\}$ is strictly monotonic decreasing sequence of non-negative real numbers.
Hence it is convergent.Let $\lim_{n\to\infty} A_n = a$

To prove $a=0$

Suppose not, then $a>0$ .From the definition of $\{A_n\}$ We can find s_i, t_i such that $i \le s_i < t_i$ and

$$A_i - \frac{1}{i} < S(\eta_{s_i}, \eta_{s_i}, \eta_{t_i}) \le A_i$$

$$\lim_{i\to\infty} S(\eta_{s_i}, \eta_{s_i}, \eta_{t_i}) = a$$

$$\begin{aligned} Now,\ S(\eta_{s_i}, \eta_{s_i}, \eta_{t_i}) &\le S(\eta_{s_i-1}, \eta_{s_i-1}, \eta_{t_i-1}) \\ &\le 2S(\eta_{s_i-1}, \eta_{s_i-1}, \eta_{s_i}) + S(\eta_{t_i-1}, \eta_{t_i-1}, \eta_{s_i}) \\ &\le 2S(\eta_{s_i-1}, \eta_{s_i-1}, \eta_{s_i}) + 2S(\eta_{t_i-1}, \eta_{t_i-1}, \eta_{t_i}) + S(\eta_{s_i}, \eta_{s_i} \eta_{t_i}) \end{aligned}$$

$$As\ s \to \infty, \lim_{s\to\infty} S(\eta_{s_i}, \eta_{s_i}, \eta_{t_i}) = a$$

$$(\because by\ lemma\ 2.2, \lim_{s\to\infty} S(\eta_{s_i-1}, \eta_{s_i-1}, \eta_{s_i}) = \lim_{t\to\infty} S(\eta_{t_i-1}, \eta_{t_i-1}, \eta_{t_i}) = 0)$$

Since h is generalized $Z_s^{'}$ contraction related to γ .

$$\gamma(S(h\eta_{s_i-1}, h\eta_{s_i-1}, h\eta_{t_i-1}), M(\eta_{s_i-1}, \eta_{s_i-1}, \eta_{t_i-1})) \ge 0$$

$$\gamma(S(\eta_{s_i}, \eta_{s_i}, \eta_{t_i}), M(\eta_{s_i-1}, \eta_{s_i-1}, \eta_{t_i-1})) \ge 0$$

$$\Rightarrow M(\eta_{s_i-1}, \eta_{s_i-1}, \eta_{t-1_i}) - S(\eta_{s_i}, \eta_{s_i}, \eta_{t_i}) \ge 0$$

$$\Rightarrow S(\eta_{s_i}, \eta_{s_i}, \eta_{t_i}) \le M(\eta_{s_i-1}, \eta_{s_i-1}, \eta_{t-1_i})$$

$$\Rightarrow a \le M(\eta_{s_i-1}, \eta_{s_i-1}, \eta_{t-1_i})$$

$$\begin{aligned} = Max\{ &S(\eta_{s_i-1}, \eta_{s_i-1}, \eta_{t_i-1}), S(\eta_{s_i-1}, h\eta_{s_i-1}, h\eta_{s_i-1}), S(\eta_{t_i-1}, \eta_{t_i-1}, h\eta_{t_i-1}), \\ &\frac{1}{4}[S(\eta_{s_i-1}, h\eta_{t_i-1}, h\eta_{t_i-1}) + S(h\eta_{t_i-1}, h\eta_{s_i-1}, h\eta_{s_i-1}) + \\ &S(\eta_{s_i-1}, h\eta_{s_i-1}, h\eta_{t_i-1}) + S(\eta_{t_i-1}, h\eta_{s_i-1}, h\eta_{t_i-1})]\} \end{aligned}$$

$$\begin{aligned} = Max\{ &S(\eta_{s_i-1}, \eta_{s_i-1}, \eta_{t_i-1}), S(\eta_{s_i-1}, \eta_{s_i}, \eta_{s_i}), S(\eta_{t_i-1}, \eta_{t_i-1}, \eta_{t_i}), \\ &\frac{1}{4}[S(\eta_{s_i-1}, \eta_{t_i}, \eta_{t_i}) + S(\eta_{t_i}, \eta_{s_i}, \eta_{s_i}) + S(\eta_{s_i-1}, \eta_{s_i}, \eta_{t_i}) + S(\eta_{t_i-1}, \eta_{s_i}, \eta_{t_i})]\} \end{aligned}$$

$$\begin{aligned} \le Max\{ &S(\eta_{s_i-1}, \eta_{s_i-1}, \eta_{t_i-1}), S(\eta_{s_i-1}, \eta_{s_i-1}, \eta_{s_i}), S(\eta_{t_i-1}, \eta_{t_i}, \eta_{t_i}), \\ &\frac{1}{4}[S(\eta_{s_i-1}, \eta_{s_i-1}, \eta_{t_i}) + S(\eta_{t_i}, \eta_{s_i}, \eta_{s_i}) + S(\eta_{s_i-1}, \eta_{s_i-1}, \eta_{s_i}) + S(\eta_{s_i}, \eta_{s_i}, \eta_{s_i}) + \\ &S(\eta_{s_i} \eta_{s_i}, \eta_{t_i}) + S(\eta_{t_i-1}, \eta_{t_i-1}, \eta_{t_i}) + S(\eta_{s_i}, \eta_{s_i}, \eta_{t_i}) + S(\eta_{t_i}, \eta_{t_i}, \eta_{t_i})]\} \end{aligned}$$

$$\begin{aligned} As\ i \to \infty, a &\le M(\eta_{s_i-1}, \eta_{s_i-1}, \eta_{t_i-1}) \\ &\le Max\{a, 0, 0, \frac{1}{4}[a + a + 0 + 0 + a + 0 + a + 0]\} \\ &= a \end{aligned}$$

$$\therefore M(\eta_{s_i-1}, \eta_{s_i-1}, \eta_{t_i-1}) = a$$

Hence we get,

$$0 \le \lim_{i\to\infty}\ Sup\ \gamma(S(\eta_{s_i}, \eta_{s_i}, \eta_{t_i}), M(\eta_{s_i-1}, \eta_{s_i-1}, \eta_{t_i-1}) = 0$$

Which is a contradiction to the definition of γ
our assumption that $a > 0$ is wrong.
Hence $a = 0$
$\therefore$ $\{\eta_n\}$ is a Cauchy sequence in complete metric space and hence convergent.
Let $\lim_{n\to\infty} \eta_n = l$
To prove l is the fixed point of h.
(ie) To prove $hl = l$
Suppose not. Then $hl \neq l$

$$Now, M(\eta_n,\eta_n,l) = Max\{S(\eta_n,\eta_n,l)), S(\eta_n,h\eta_n,h\eta_n), S(l,l,hl), \frac{1}{4}[S(\eta_n,hl,hl)+S(hl,h\eta_n,h\eta_n)+S(\eta_n,h\eta_n,hl)+S(l,h\eta_n,hl)]\}$$

$$\lim_{n\to\infty} M(\eta_n,\eta_n,l) = Max\{S(l,l,l), S(l,hl,hl), S(l,l,hl), \frac{1}{4}[S(l,hl,hl)+S(hl,hl,hl)+S(l,hl,hl)+S(l,l,hl)]\}$$

$$= S(l,l,hl)$$

$$Now, 0 \le \lim_{n\to\infty} Sup\ \gamma(S(h\eta_n, h\eta_n, hl), M(\eta_n,\eta_n,l))$$
$$= Sup\ \gamma(S(hl,hl,hl), S(l,l,hl))$$
$$= Sup\ \gamma(0, S(l,l,hl))$$
$$= Sup\ (S(l,l,hl)-0)$$
$$= Sup\ (S(l,l,hl))$$

Which contradict the condition that $\lim_{n\to\infty} Sup\ \gamma(S(h\eta_n,h\eta_n,hl), M(\eta_n,\eta_n,l)) < 0$

$Hence\ \ hl = l$

(ie) l is the fixed point for h.
To prove the uniqueness
Suppose there exists two elements l_1, l_2 in X such that $h\,l_1 = l_1, h\,l_2 = l_2$ and $l_1 \neq l_2$

$$Consider, M(l_1,l_1,l_2) = Max\{S(l_1,l_1,l_2), S(l_1,hl_1,hl_1), S(l_2,l_2,hl_2), \frac{1}{4}[S(l_1,hl_2,hl_2)+S(hl_2,hl_1,hl_1)+S(l_1,hl_1,hl_2)+S(l_2,hl_1,hl_2)]\}$$

$$= S(l_1,l_1,l_2)$$

$$Now, 0 \le \gamma(S(hl_1,hl_1,hl_2), M(l_1,l_1,l_2))$$
$$= \gamma(S(l_1,l_1,l_2), M(l_1,l_1,l_2))$$
$$= \gamma(S(l_1,l_1,l_2), S(l_1,l_1,l_2))$$
$$= 0$$

Which is a contradiction
$\therefore l_1 = l_2$
Hence proved.

Example:2.2 If $X = \left[0, \frac{1}{4}\right]$ &S : $X \times X \times X \to R^+$ defined by $S(a,b,c) = |a+c-2b| + |a-c|$

Then (X , S) is a complete S metric space. Define a self map h on X by $ha = \frac{a}{1+a}$

Then h is a Z -contraction related to γ where $\gamma(u,v) = \frac{v}{v+\frac{1}{4}} - u \ \forall\, u,v \in [0,\infty]$ [2]

Thus for all $a, b \in X$ we get,

$$0 \le \gamma(S(ha,ha,hb),S(a,a,b))$$
$$= \frac{S(a,a,b)}{S(a,a,b)+\frac{1}{4}} - S(ha,ha,hb)$$
$$\le \frac{M(a,a,b)}{M(a,a,b)+\frac{1}{4}} - S(ha,ha,hb)$$
$$= \gamma(S(ha,ha,hb),M(a,a,b))$$

(ie) h is a generalized $Z_s^{'}$ contraction related to γ.
Hence by the above theorem, h has a unique fixed point & the fixed point is 0.

Example: 2.3 If $X = [0,1]$&S : $X \times X \times X \to R^{+}$
$$S(a,b,c) = |a-c| + |b-c|$$
Then (X , S) is a complete S- metric space.

Define h a == $\begin{cases} \frac{1}{5} & [0,\frac{4}{7}) \\ \frac{2}{3} & [\frac{4}{7},1] \end{cases}$

Then h is not a generalized $Z_s^{'}$ contraction related to γ where $\gamma(u,v) = \tau v - u \ \forall \tau \in [0,1)$

Hence by theorem , h has no unique fixed point but have 2 fixed points as $\frac{1}{5}$ & $\frac{2}{3}$.

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