$\hat{\rho}$ -SETS WHERE $\rho \in \{ r\omega , r\omega\delta, r^*\omega, r^*\omega\delta \}$

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Abstract: In this paper, the mixed and ordinary operators are characterized in the classes of $\widehat{r\omega}$ -set, $\widehat{r\omega\delta}$ -set, $\widehat{r^*\omega}$-set and $\widehat{r^*\omega\delta}$-set. More over the behavior of $\widehat{r\omega}$ -set, $\widehat{r\omega\delta}$ -set, $\widehat{r^*\omega}$-set and $\widehat{r^*\omega\delta}$-set in spaces are investigated. The Inclusion chains among the mixed and ordinary operators are refined in the domains of $\widehat{r\omega}$ -set, $\widehat{r\omega\delta}$ -set, $\widehat{r^*\omega}$-set and $\widehat{r^*\omega\delta}$-set.

DOI: 10.48047/ecb/2022.11.12.160

1 INTRODUCTION

Throughout this paper (X, τ) (or X) represent topological spaces on which no separation axioms are assumed unless otherwise mentioned. The concept of δ-closure was introduced and studied by Velicko [9] in the year 1968. A point x is in the δ-closure of A if every regular open nbd of x intersects A. $Cl_\delta A$ denotes the δ-closure of A.

A subset A of X is δ-closed [9] if A = $Cl_\delta A$. The complement of a δ-closed set is δ-open. The collection of all δ-open sets is a topology denoted by τ^δ. This τ^δ is called the semi -regularization of τ.

Let $Int_\delta A$ and $Cl_\delta A$ denote the δ-interior and δ-closure of A respectively. Velicko established that the operators $Cl(.)$ and $Cl_\delta(.)$ have the same effect on the class of open sets and the operators $Int(.)$ and $Int_\delta(.)$ coincide on the class of closed sets.

(i) For any open set A, $Cl_\delta A = Cl A$,

(ii) For any closd set B, $Int_\delta B = Int B$.

Let H be a subset of a space (X, τ), a point p in X is called a condensation point of H [1] if for each open set U containing p, U ∩ H is uncountable. A subset H of a space (X, τ) is called ω-closed [1] if it contains all its condensation points. The complement of an ω-closed set is called ω-open. The family of all ω-closed sets is denoted by ωC(X, τ). The family of all ω-open sets is denoted by ωO(X). It is well known that a subset W of a space (X, τ) is ω-open [1] if and only if for each x ∈ W, there exists U ∈ τ such that x ∈ U and U − W is countable. The family of all ω-open sets, denoted by τ_ω, is a topology on X, which is finer than τ. The interior and closure operator in (X, τ_ω) are denoted by Int_ω and Cl_ω respectively.

In this paper, the mixed and ordinary operators are characterized in the classes of $\widehat{r\omega}$ -set, $\widehat{r\omega\delta}$ -set, $\widehat{r^*\omega}$-set and $\widehat{r^*\omega\delta}$-set. More over the behavior of $\widehat{r\omega}$ -set, $\widehat{r\omega\delta}$ -set, $\widehat{r^*\omega}$-set and $\widehat{r^*\omega\delta}$-set in spaces are investigated.

2. PRELIMINARIES

Proposition 2.1 [8]

(i) $Cl_\omega Int_\delta A \subseteq ClInt_\delta A = Cl_\delta Int_\delta A \subseteq Cl_\delta IntA = ClIntA \subseteq ClInt_\omega A \subseteq Cl_\delta Int_\omega A$,

(ii) $Cl_\omega Int_\delta A \subseteq Cl_\omega IntA \subseteq Cl_\omega Int_\omega A \subseteq ClInt_\omega A \subseteq Cl_\delta Int_\omega A$,

(iii) $Int_\delta Cl_\omega A \subseteq IntCl_\omega A \subseteq IntClA = Int_\delta ClA \subseteq Int_\delta Cl_\delta A = IntCl_\delta A \subseteq Int_\omega Cl_\delta A$,

(iv) $Int_\delta Cl_\omega A \subseteq IntCl_\omega A \subseteq Int_\omega Cl_\omega A \subseteq Int_\omega ClA \subseteq Int_\omega Cl_\delta A$.

Proposition 2.2 [8] A subset A of a space (X,τ) is

(i) an rω-set $\Leftrightarrow IntCl_\omega A = Int_\delta ClA$,

(ii) an r*ω-set $\Leftrightarrow ClInt_\omega A = Cl_\delta IntA$.

Corollary 2.3 [8]

(i) The set A is an rω-set $\Leftrightarrow IntCl_\omega A = IntClA = Int_\delta ClA$.

(ii) The set A is an r*ω-set $\Leftrightarrow ClInt_\omega A = ClIntA = Cl_\delta IntA$.

Proposition 2.4 [8] For any subset A of a space (X,τ), the following always hold.

(i) $Cl_\omega IntCl_\omega A \subseteq Cl_\omega IntClA \subseteq Cl_\omega Int_\omega ClA \subseteq ClInt_\omega ClA$,

(ii) $Cl_\omega IntCl_\omega A \subseteq Cl_\omega Int_\omega Cl_\omega A \subseteq Cl_\omega Int_\omega ClA \subseteq ClInt_\omega ClA$,

(iii) $Cl_\omega IntCl_\omega A \subseteq ClIntCl_\omega A \subseteq ClInt_\omega Cl_\omega A \subseteq ClInt_\omega ClA$,

(iv) $Cl_\omega IntCl_\omega A \subseteq ClIntCl_\omega A \subseteq ClIntClA \subseteq ClInt_\omega ClA$.

Proposition 2.5 [8]

(i) $Cl_\omega Int_\delta Cl_\omega A \subseteq ClInt_\delta Cl_\omega A \subseteq ClIntCl_\omega A \subseteq ClIntClA = ClInt_\delta ClA \subseteq ClInt_\delta Cl_\delta A$,

(ii) $ClInt_\delta Cl_\delta A = ClIntCl_\delta A \subseteq ClInt_\omega Cl_\delta A \subseteq Cl_\delta A\ Int_\omega Cl_\delta A$,

(iii) $Cl_\omega Int_\delta Cl_\omega A \subseteq ClInt_\delta Cl_\omega A \subseteq Cl_\delta Int_\delta Cl_\omega A \subseteq Cl_\delta IntCl_\omega A \subseteq Cl_\delta IntClA$,

(iv) $Cl_\delta IntClA = Cl_\delta Int_\delta ClA \subseteq Cl_\delta Int_\delta Cl_\delta A = Cl_\delta IntCl_\delta A \subseteq Cl_\delta Int_\omega Cl_\delta A$,

(v) $Cl_\omega Int_\delta Cl_\omega A \subseteq Cl_\omega IntCl_\omega A \subseteq Cl_\omega IntClA = Cl_\omega Int_\delta ClA \subseteq Cl_\omega Int_\delta Cl_\delta A$,

(vi) $Cl_\omega Int_\delta Cl_\delta A = Cl_\omega IntCl_\delta A \subseteq Cl_\omega Int_\omega Cl_\delta A \subseteq ClInt_\omega Cl_\delta A \subseteq Cl_\delta Int_\omega Cl_\delta A$.

Proposition 2.6 [8]

(i) $Cl_\omega Int_\delta Cl_\omega A \subseteq ClInt_\delta Cl_\omega A \subseteq ClIntCl_\omega A \subseteq ClInt_\omega Cl_\omega A$,

(ii) $ClInt_\omega Cl_\omega A \subseteq ClInt_\omega ClA \subseteq ClInt_\omega Cl_\delta A \subseteq Cl_\delta Int_\omega Cl_\delta A$,

(iii) $Cl_\omega Int_\delta Cl_\omega A \subseteq ClInt_\delta Cl_\omega A \subseteq Cl_\delta Int_\delta Cl_\omega A \subseteq Cl_\delta IntCl_\omega A$,

(iv) $Cl_\delta IntCl_\omega A \subseteq Cl_\delta Int_\omega Cl_\omega A \subseteq Cl_\delta Int_\omega ClA \subseteq Cl_\delta Int_\omega Cl_\delta A$,

(v) $Cl_\omega Int_\delta Cl_\omega A \subseteq Cl_\omega IntCl_\omega A \subseteq Cl_\omega Int_\omega Cl_\omega A \subseteq Cl_\omega Int_\omega ClA$,

(vi) $Cl_\omega Int_\omega ClA \subseteq Cl_\omega Int_\omega Cl_\delta A \subseteq ClInt_\omega Cl_\delta A \subseteq Cl_\delta Int_\omega Cl_\delta A$.

Proposition 2.7 [8] For any subset A of a space (X,τ), the following always hold.

(i) $IntCl_\omega IntA \subseteq IntCl_\omega Int_\omega A \subseteq IntCl\, Int_\omega A \subseteq Int_\omega Cl\, Int_\omega A$,

(ii) $IntCl_\omega IntA \subseteq IntClIntA \subseteq IntClInt_\omega A \subseteq Int_\omega ClInt_\omega A$,

(iii) $IntCl_\omega IntA \subseteq Int_\omega Cl_\omega IntA \subseteq Int_\omega ClIntA \subseteq Int_\omega ClInt_\omega A$,

(iv) $IntCl_\omega IntA \subseteq Int_\omega Cl_\omega IntA \subseteq Int_\omega Cl_\omega Int_\omega A \subseteq Int_\omega ClInt_\omega A$.

Proposition 2.8 [8]

(i) $Int_\delta Cl_\omega Int_\delta A \subseteq IntCl_\omega Int_\delta A \subseteq IntClInt_\delta A = IntCl_\delta Int_\delta A \subseteq IntCl_\delta IntA$,

(ii) $IntCl_\delta IntA = IntClIntA \subseteq IntClInt_\omega A \subseteq IntCl_\delta Int_\omega A \subseteq Int_\omega Cl_\delta Int_\omega A$,

(iii) $Int_\delta Cl_\omega Int_\delta A \subseteq Int_\delta ClInt_\delta A = Int_\delta Cl_\delta Int_\delta A \subseteq Int_\delta Cl_\delta IntA = Int_\delta ClIntA$,

(iv) $Int_\delta ClIntA \subseteq Int_\delta ClInt_\omega A \subseteq Int_\delta Cl_\delta Int_\omega A \subseteq IntCl_\delta Int_\omega A \subseteq Int_\omega Cl_\delta Int_\omega A$,

(v) $Int_\delta Cl_\omega Int_\delta A \subseteq IntCl_\omega Int_\delta A \subseteq Int_\omega Cl_\omega Int_\delta A \subseteq Int_\omega ClInt_\delta A = Int_\omega Cl_\delta Int_\delta A$,

(vi) $Int_\omega Cl_\delta Int_\delta A \subseteq Int_\omega Cl_\delta IntA = Int_\omega ClIntA \subseteq Int_\omega ClInt_\omega A \subseteq Int_\omega Cl_\delta Int_\omega A$.

Proposition 2.9 [8]

(i) $Int_\delta Cl_\omega Int_\delta A \subseteq IntCl_\omega Int_\delta A \subseteq IntCl_\omega IntA \subseteq IntCl_\omega Int_\omega A$,

(ii) $IntCl_\omega Int_\omega A \subseteq IntClInt_\omega A \subseteq IntCl_\delta Int_\omega A \subseteq IntCl_\delta Int_\omega A$,

(iii) $Int_\delta Cl_\omega Int_\delta A \subseteq Int_\delta Cl_\omega IntA \subseteq Int_\delta Cl_\omega Int_\omega A \subseteq Int_\delta ClInt_\omega A$,

(iv) $Int_\delta ClInt_\omega A \subseteq Int_\delta Cl_\delta Int_\omega A \subseteq IntCl_\delta Int_\omega A \subseteq Int_\omega Cl_\delta Int_\omega A$,

(v) $Int_\delta Cl_\omega Int_\delta A \subseteq IntCl_\omega Int_\delta A \subseteq Int_\omega Cl_\omega Int_\delta A \subseteq Int_\omega Cl_\omega IntA$,

(vi) $Int_\omega Cl_\omega IntA \subseteq Int_\omega Cl_\omega Int_\omega A \subseteq Int_\omega ClInt_\omega A \subseteq Int_\omega Cl_\delta Int_\omega A$.

Definition 2.10 [8] A subset A of a space (X, τ) is

(i) an r-set if $IntCl_\delta A = IntClA$,

(ii) an r*-set if $ClInt_\delta A = ClIntA$ and

(iii) an rr*-set if it is both an r-set and an r*-set.

3. $\hat{\rho}$-sets where $\rho \in \{ r\omega , r^*\omega \}$

Definition 3.1 A subset A of a space (X, τ) is

(i) an $\widehat{r\omega}$ - set if $Int_\omega Cl A = Int Cl A$,

(ii) an $\widehat{r^*\omega}$ - set if $Cl_\omega Int A = Cl Int A$ and

(iii) an $\widehat{rr^*\omega}$ if it is both an $\widehat{r\omega}$ - set and an $\widehat{r^*\omega}$ - set.

Proposition 3.2 A subset A of a space (X, τ) is

(i) an $\widehat{r\omega}$ - set ⇔ X\A is an $\widehat{r^*\omega}$ -set

(ii) an $\widehat{rr^*\omega}$-set ⇔ X\A is an $\widehat{rr^*\omega}$-set.

Proof. The set A is an $\widehat{r\omega}$ -set ⇔ $Int_\omega Cl A = Int Cl A$

⇔ X\ $Int_\omega Cl A$ = X\ $Int Cl A$

⇔ $Cl_\omega Int$ (X\A) = $Cl Int$(X\A)

⇔X\A is an $\widehat{r^*\omega}$-set. This proves (i).

The set A is an $\widehat{rr^*\omega}$-set ⇔ A is an $\widehat{r\omega}$ -set and an $\widehat{r^*\omega}$ -set.

⇔X\A is an $\widehat{r^*\omega}$ -set and an $\widehat{r\omega}$ -set.

⇔ X\A is an $\widehat{r\omega}$ -set and an $\widehat{r^*\omega}$ -set.

⇔ X\A is an $\widehat{rr^*\omega}$-set.

Proposition 3.3 A subset A of a space (X, τ) is

(i) an $\widehat{r\omega}$ - set ⇔ $Int_\omega Cl A = Int Cl A = Int_\delta Cl A$ ⇔ $Int_\omega Cl A = Int_\delta Cl A$,

(ii) an $\widehat{r^*\omega}$ - set ⇔ $Cl_\omega Int A = Cl Int A = Cl_\delta Int A$ ⇔ $Cl_\omega Int A = Cl_\delta Int A$.

Proof. The set A is an $\widehat{r\omega}$ - set ⇔ $Int_\omega Cl A = Int Cl A$. (1)

The set A is an $\widehat{r^*\omega}$ - set ⇔ $Cl_\omega Int A = Cl Int A$. (2)

By using Proposition 2.1 (i) & (iii), we have

$Cl_\delta Int A = Cl Int A$ (3)

$Int Cl A = Int_\delta Cl A$ (4)

Then the assertion (i) follows from (1) and (4). Then the assertion (ii) follows from (2) and (3). This proves the proposition.

Proposition 3.4 If a set A is an $\widehat{r\omega}$ - set then the following chain holds.

$Int_\delta Cl_\omega A \subseteq Int Cl_\omega A \subseteq Int_\omega Cl_\omega A \subseteq Int_\omega Cl A = Int Cl A = Int_\delta Cl A \subseteq Int_\delta Cl_\delta A = Int Cl_\delta A \subseteq Int_\omega Cl_\delta A$.

Proof: Let A be an $\widehat{r\omega}$ - set. Then using Proposition 3.3(i), we have

$Int_\omega Cl A = Int Cl A = Int_\delta Cl A$ (5)

Using Proposition 2.1 (iii) & (iv), we have

$Int_\delta Cl_\omega A \subseteq Int Cl_\omega A \subseteq Int Cl A = Int_\delta Cl A \subseteq Int_\delta Cl_\delta A = Int Cl_\delta A \subseteq Int_\omega Cl_\delta A$.

$Int_\delta Cl_\omega A \subseteq IntCl_\omega A \subseteq Int_\omega Cl_\omega A \subseteq Int_\omega ClA \subseteq Int_\omega Cl_\delta A.$

Using (5) in the above two expressions we have

$Int_\delta Cl_\omega A\subseteq IntCl_\omega A\subseteq IntClA = Int_\omega ClA= Int_\delta ClA\subseteq Int_\delta Cl_\delta A=IntCl_\delta A\subseteq Int_\omega Cl_\delta A.$

$Int_\delta Cl_\omega A \subseteq IntCl_\omega A \subseteq Int_\omega Cl_\omega A \subseteq IntClA = Int_\omega ClA= Int_\delta ClA \subseteq Int_\omega Cl_\delta A.$

Combining the above two expressions we have

$Int_\delta Cl_\omega A\subseteq IntCl_\omega A\subseteq Int_\omega Cl_\omega A\subseteq Int_\omega ClA=IntClA=Int_\delta ClA\subseteq Int_\delta Cl_\delta A=IntCl_\delta A\subseteq Int_\omega Cl_\delta A.$

This proves the proposition**.**

Proposition 3.5 If a set A is an $\widehat{r^*\omega}$ - set then the following chain holds.

$Cl_\omega Int_\delta A\subseteq ClInt_\delta A=Cl_\delta Int_\delta A\subseteq Cl_\delta IntA=Cl_\omega IntA=ClIntA\subseteq Cl_\omega Int_\omega A\subseteq ClInt_\omega A\subseteq Cl_\delta Int_\omega A.$

Proof. Let A be an $\widehat{r^*\omega}$ set . Then $Cl_\omega IntA =ClIntA = Cl_\delta IntA.$ (6)

Using Proposition 2.1(i) & (ii), we have

$Cl_\omega Int_\delta A\subseteq ClInt_\delta A=Cl_\delta Int_\delta A\subseteq Cl_\delta IntA= ClIntA\subseteq ClInt_\omega A\subseteq Cl_\delta Int_\omega A.$

$Cl_\omega Int_\delta A \subseteq Cl_\omega IntA \subseteq Cl_\omega Int_\omega A\subseteq ClInt_\omega A\subseteq Cl_\delta Int_\omega A.$

Using (6) in the above two expressions, we get

$Cl_\omega Int_\delta A\subseteq ClInt_\delta A=Cl_\delta Int_\delta A\subseteq Cl_\delta IntA= Cl_\omega IntA = ClIntA\subseteq ClInt_\omega A\subseteq Cl_\delta Int_\omega A.$

$Cl_\omega Int_\delta A \subseteq Cl_\omega IntA = Cl_\delta IntA=ClIntA \subseteq Cl_\omega Int_\omega A\subseteq ClInt_\omega A \subseteq Cl_\delta Int_\omega A.$

Combining the above two expressions we have

$Cl_\omega Int_\delta A\subseteq ClInt_\delta A=Cl_\delta Int_\delta A\subseteq Cl_\delta IntA= Cl_\omega IntA = ClIntA\subseteq Cl_\omega Int_\omega A\subseteq ClInt_\omega A \subseteq Cl_\delta Int_\omega A.$

This proves the proposition.

Proposition 3.6 If a set A is both an r-set and an $\widehat{r\omega}$ - set then the following chain holds.

$Int_\delta Cl_\omega A\subseteq IntCl_\omega A\subseteq Int_\omega Cl_\omega A\subseteq Int_\omega ClA=IntClA=Int_\delta ClA=Int_\delta Cl_\delta A=IntCl_\delta A\subseteq Int_\omega Cl_\delta A.$

Proof. Let A be an r-set and an $\widehat{r\omega}$ -set .

A is an r-set $\Rightarrow IntCl_\delta A = IntClA.$ (7)

Since A is an $\widehat{r\omega}$ -set using Proposition 3.4, we have

$Int_\delta Cl_\omega A\subseteq IntCl_\omega A\subseteq Int_\omega Cl_\omega A\subseteq Int_\omega ClA=IntClA=Int_\delta ClA\subseteq Int_\delta Cl_\delta A=IntCl_\delta A\subseteq Int_\omega Cl_\delta A.$

Using (7) in the above expression, we have

$Int_\delta Cl_\omega A\subseteq IntCl_\omega A\subseteq Int_\omega Cl_\omega A\subseteq Int_\omega ClA=IntClA=Int_\delta ClA=Int_\delta Cl_\delta A=IntCl_\delta A\subseteq Int_\omega Cl_\delta A.$

This proves the proposition.

Proposition 3.7 If a set A is both an r*-set and an $\widehat{r^*\omega}$ -set then the following chain holds.

$Cl_\omega Int_\delta A\subseteq ClInt_\delta A=Cl_\delta Int_\delta A=Cl_\delta IntA=Cl_\omega IntA=ClIntA\subseteq Cl_\omega Int_\omega A\subseteq ClInt_\omega A\subseteq Cl_\delta Int_\omega A$

Proof. Let A be an r*-set and an $\widehat{r^*\omega}$ -set. Then

A is an r*-set $\Rightarrow ClInt_\delta A = ClIntA$. (8)

Using Proposition 3.5, we have

$Cl_\omega Int_\delta A \subseteq ClInt_\delta A = Cl_\delta Int_\delta A \subseteq Cl_\delta IntA = Cl_\omega IntA = ClIntA \subseteq Cl_\omega Int_\omega A \subseteq ClInt_\omega A \subseteq Cl_\delta Int_\omega A$.

Using (8) in the above expression we have

$Cl_\omega Int_\delta A \subseteq ClInt_\delta A = Cl_\delta Int_\delta A = Cl_\delta IntA = Cl_\omega IntA = ClIntA \subseteq Cl_\omega Int_\omega A \subseteq ClInt_\omega A \subseteq Cl_\delta Int_\omega A$.

This proves the proposition.

Proposition 3.8

(i) The set A is an r-set and an $\widehat{r\omega}$-set $\Leftrightarrow Int_\omega ClA = IntClA = Int_\delta ClA = Int_\delta Cl_\delta A = IntCl_\delta A$.

(ii) The set A is an r*-set and an $\widehat{r^*\omega}$ -set $\Leftrightarrow ClInt_\delta A = Cl_\delta Int_\delta A = Cl_\delta IntA = Cl_\omega IntA = ClIntA$.

(iii) The set A is an rω-set and an $\widehat{r\omega}$-set $\Leftrightarrow IntCl_\omega A = Int_\omega Cl_\omega A = Int_\omega ClA = IntClA = Int_\delta ClA$.

.(iv) The set A is an r*ω-set and an $\widehat{r^*\omega}$ -set $\Leftrightarrow Cl_\delta IntA = Cl_\omega IntA = ClIntA = Cl_\omega Int_\omega A = ClInt_\omega A$.

Proof. The assertions (i) and (ii) follow respectively from Proposition 3.6 and Proposition 3.7.

Let A be an rω-set and an $\widehat{r\omega}$-set. Then using Corollary 2.3(i) and Proposition 3.4 we have $IntCl_\omega A = IntClA = Int_\delta ClA$ and

$Int_\delta Cl_\omega A \subseteq IntCl_\omega A \subseteq Int_\omega Cl_\omega A \subseteq Int_\omega ClA = IntClA = Int_\delta ClA \subseteq Int_\delta Cl_\delta A = IntCl_\delta A \subseteq Int_\omega Cl_\delta A$.

Combining the above two expressions we have

$IntClA = IntCl_\omega A \subseteq Int_\omega Cl_\omega A \subseteq Int_\omega ClA = IntClA = Int_\delta ClA$ that implies

$IntClA = IntCl_\omega A = Int_\omega Cl_\omega A = Int_\omega ClA = IntClA = Int_\delta ClA$. This proves the necessary part of the assertion (iii) . The sufficient part follows easily.

Now let A be an r*ω-set and an $\widehat{r^*\omega}$ -set. Then using Corollary 2.3(ii) and Proposition 3.5 we have $ClInt_\omega A = ClIntA = Cl_\delta IntA$ and

$Cl_\omega Int_\delta A \subseteq ClInt_\delta A = Cl_\delta Int_\delta A \subseteq Cl_\delta IntA = Cl_\omega IntA = ClIntA \subseteq Cl_\omega Int_\omega A \subseteq ClInt_\omega A \subseteq Cl_\delta Int_\omega A$.

Combining the above two expressions we have

$Cl_\delta IntA = Cl_\omega IntA = ClIntA \subseteq Cl_\omega Int_\omega A \subseteq ClInt_\omega A = ClIntA$ that implies

$Cl_\delta IntA = Cl_\omega IntA = ClIntA = Cl_\omega Int_\omega A = ClInt_\omega A = ClIntA$. This proves the necessary part of the assertion (iv) . The sufficient part follows easily. This proves the proposition.

Proposition 3.9 Let A be an $\widehat{r\omega}$ -set. The following results hold.

(i) $ClInt_\omega ClA = ClIntClA = ClInt_\delta ClA$,

(ii) $Cl_\delta Int_\omega ClA = Cl_\delta IntClA = Cl_\delta Int_\delta ClA$,

(iii) $Cl_\omega Int_\omega ClA = Cl_\omega IntClA = Cl_\omega Int_\delta ClA$.

Proof. Let A be an $\widehat{r\omega}$ -set. Then using Proposition 3.3(i) we have

$Int_\omega Cl A = IntCl A = Int_\delta Cl A$ (9)

Applying the operations 'Cl ', 'Cl_δ ', 'Cl_ω ' on (9) we have

$ClInt_\omega Cl A = ClIntCl A = ClInt_\delta Cl A$

$Cl_\delta Int_\omega Cl A = Cl_\delta IntCl A = Cl_\delta Int_\delta Cl A$

$Cl_\omega Int_\omega Cl A = Cl_\omega IntCl A = Cl_\omega Int_\delta Cl A$

This proves the proposition.

Proposition 3.10 Let A be an $\widehat{r^*\omega}$-set. The following results hold.

(i) $IntCl_\omega Int A = IntClInt A = IntCl_\delta Int A$,

(ii) $Int_\delta Cl_\omega Int A = Int_\delta ClInt A = Int_\delta Cl_\delta Int A$,

(iii) $Int_\omega Cl_\omega Int A = Int_\omega ClInt A = Int_\omega Cl_\delta Int A$.

Proof. Let A be an $\widehat{r^*\omega}$-set. Then using Proposition 3.3(ii) we have

$$Cl_\omega Int A = ClInt A = Cl_\delta Int A. \quad (10)$$

Applying the operations 'Cl ', 'Cl_δ ', 'Cl_ω ' on (10) we have

$IntCl_\omega Int A = IntClInt A = IntCl_\delta Int A$.

$Int_\delta Cl_\omega Int A = Int_\delta ClInt A = Int_\delta Cl_\delta Int A$.

$Int_\omega Cl_\omega Int A = Int_\omega ClInt A = Int_\omega Cl_\delta Int A$.

This proves the proposition.

Proposition 3.11 Let A be an $\widehat{r\omega}$ -set. The following chains hold.

(i) $Cl_\omega IntCl_\omega A \subseteq Cl_\omega IntCl A \subseteq Cl_\omega Int_\omega Cl A \subseteq ClInt_\omega Cl A = ClIntCl A = ClInt_\delta Cl A$,

(ii) $Cl_\omega IntCl_\omega A \subseteq ClIntCl_\omega A \subseteq ClInt_\omega Cl A = ClIntCl A = ClInt_\delta Cl A$,

(iii) $Cl_\omega Int_\delta Cl_\omega A \subseteq ClInt_\delta Cl_\omega A \subseteq ClIntCl_\omega A \subseteq ClInt_\omega Cl A = ClIntCl A = ClInt_\delta Cl A \subseteq ClInt_\delta Cl_\delta A$,

(iv) $ClInt_\omega Cl_\omega A \subseteq ClInt_\omega Cl A = ClIntCl A = ClInt_\delta Cl A \subseteq ClInt_\omega Cl_\delta A \subseteq Cl_\delta Int_\omega Cl_\delta A$,

(v) $Cl_\delta Int_\omega Cl A = Cl_\delta IntCl A = Cl_\delta Int_\delta Cl A \subseteq Cl_\delta Int_\delta Cl_\delta A = Cl_\delta IntCl_\delta A \subseteq Cl_\delta Int_\omega Cl_\delta A$,

(vi) $Cl_\delta IntCl_\omega A \subseteq Cl_\delta Int_\omega Cl_\omega A \subseteq Cl_\delta Int_\omega Cl A = Cl_\delta IntCl A = Cl_\delta Int_\delta Cl A \subseteq Cl_\delta Int_\omega Cl_\delta A$,

(vii) $Cl_\omega Int_\delta Cl_\omega A \subseteq Cl_\omega IntCl_\omega A \subseteq Cl_\omega Int_\omega Cl A = Cl_\omega IntCl A = Cl_\omega Int_\delta Cl A \subseteq Cl_\omega Int_\delta Cl_\delta A$,

(viii) $Cl_\omega Int_\delta Cl_\omega A \subseteq Cl_\omega IntCl_\omega A \subseteq Cl_\omega Int_\omega Cl_\omega A \subseteq Cl_\omega Int_\omega Cl A = Cl_\omega IntCl A = Cl_\omega Int_\delta Cl A$,

(ix) $Cl_\omega Int_\omega Cl A = Cl_\omega IntCl A = Cl_\omega Int_\delta Cl A \subseteq Cl_\omega Int_\omega Cl_\delta A \subseteq ClInt_\omega Cl_\delta A \subseteq Cl_\delta Int_\omega Cl_\delta A$.

Proof. Let A be an $\widehat{r\omega}$ -set. Then using Proposition 3.9, we have

$ClInt_\omega Cl A = ClIntCl A = ClInt_\delta Cl A$ (11)

$Cl_\delta Int_\omega Cl A = Cl_\delta IntCl A = Cl_\delta Int_\delta Cl A$ (12)

$Cl_\omega Int_\omega Cl A = Cl_\omega IntCl A = Cl_\omega Int_\delta Cl A$ (13)

Using (11) in Proposition 2.4 (i) & (iv) we have

$Cl_{\omega}IntCl_{\omega}A \subseteq Cl_{\omega}IntClA \subseteq Cl_{\omega}Int_{\omega}ClA \subseteq ClInt_{\omega}ClA = ClIntClA = ClInt_{\delta}ClA$ (14)

$Cl_{\omega}IntCl_{\omega}A \subseteq ClIntCl_{\omega}A \subseteq ClInt_{\omega}ClA = ClIntClA = ClInt_{\delta}ClA$ (15)

Using (11) in Proposition 2.5(i), we have

$Cl_{\omega}Int_{\delta}Cl_{\omega}A \subseteq ClInt_{\delta}Cl_{\omega}A \subseteq ClIntCl_{\omega}A \subseteq ClInt_{\omega}ClA = ClIntClA = ClInt_{\delta}ClA \subseteq ClInt_{\delta}Cl_{\delta}A$ (16)

Using (12) in Proposition 2.5(iv), we have

$Cl_{\delta}Int_{\omega}ClA = Cl_{\delta}IntClA = Cl_{\delta}Int_{\delta}ClA \subseteq Cl_{\delta}Int_{\delta}Cl_{\delta}A = Cl_{\delta}IntCl_{\delta}A \subseteq Cl_{\delta}Int_{\omega}Cl_{\delta}A.$ (17)

Using (13) in Proposition 2.5(v), we have

$Cl_{\omega}Int_{\delta}Cl_{\omega}A \subseteq Cl_{\omega}IntCl_{\omega}A \subseteq Cl_{\omega}Int_{\omega}ClA = Cl_{\omega}IntClA = Cl_{\omega}Int_{\delta}ClA \subseteq Cl_{\omega}Int_{\delta}Cl_{\delta}A$ (18)

Using (11) in Proposition 2.6 (ii) , we have

$ClInt_{\omega}Cl_{\omega}A \subseteq ClInt_{\omega}ClA = ClIntClA = ClInt_{\delta}ClA \subseteq ClInt_{\omega}Cl_{\delta}A \subseteq Cl_{\delta}Int_{\omega}Cl_{\delta}A.$ (19)

Using (12) in Proposition 2.6 (iv) , we have

$Cl_{\delta}IntCl_{\omega}A \subseteq Cl_{\delta}Int_{\omega}Cl_{\omega}A \subseteq Cl_{\delta}Int_{\omega}ClA = Cl_{\delta}IntClA = Cl_{\delta}Int_{\delta}ClA \subseteq Cl_{\delta}Int_{\omega}Cl_{\delta}A.$ (20)

Using (13) in Proposition 2.6 (v)&(vi), we have

$Cl_{\omega}Int_{\delta}Cl_{\omega}A \subseteq Cl_{\omega}IntCl_{\omega}A \subseteq Cl_{\omega}Int_{\omega}Cl_{\omega}A \subseteq Cl_{\omega}Int_{\omega}ClA = Cl_{\omega}IntClA = Cl_{\omega}Int_{\delta}ClA$ (21)

$Cl_{\omega}Int_{\omega}ClA = Cl_{\omega}IntClA = Cl_{\omega}Int_{\delta}ClA \subseteq Cl_{\omega}Int_{\omega}Cl_{\delta}A \subseteq ClInt_{\omega}Cl_{\delta}A \subseteq Cl_{\delta}Int_{\omega}Cl_{\delta}A.$ (22)

Then the assertions (i) through (ix), follow from (14), (15), (16), (19), (17), (20), (18), (21) and (22) respectively. This proves the proposition.

Proposition 3.12 Let A be an $\widehat{r^*\omega}$-set. The following chains hold.

(i) $IntCl_{\omega}IntA = IntClIntA = IntCl_{\delta}IntA \subseteq IntCl_{\omega}Int_{\omega}A \subseteq IntCl\,Int_{\omega}A \subseteq Int_{\omega}Cl\,Int_{\omega}A$,

(ii) $IntCl_{\omega}IntA = IntClIntA = IntCl_{\delta}IntA \subseteq Int_{\omega}Cl_{\omega}IntA \subseteq Int_{\omega}ClIntA \subseteq Int_{\omega}ClInt_{\omega}A$,

(iii) $IntCl_{\omega}IntA = IntClIntA = IntCl_{\delta}IntA \subseteq Int_{\omega}Cl_{\omega}IntA \subseteq Int_{\omega}Cl_{\omega}Int_{\omega}A \subseteq Int_{\omega}ClInt_{\omega}A$,

(iv) $Int_{\delta}Cl_{\omega}Int_{\delta}A \subseteq IntCl_{\omega}Int_{\delta}A \subseteq IntClInt_{\delta}A = IntCl_{\delta}Int_{\delta}A \subseteq IntCl_{\omega}IntA = IntClIntA = IntCl_{\delta}IntA$,

(v) $IntCl_{\omega}IntA = IntClIntA = IntCl_{\delta}IntA \subseteq IntClInt_{\omega}A \subseteq IntCl_{\delta}Int_{\omega}A \subseteq Int_{\omega}Cl_{\delta}Int_{\omega}A$,

(vi) $Int_{\delta}Cl_{\omega}Int_{\delta}A \subseteq IntCl_{\omega}Int_{\delta}A \subseteq IntCl_{\omega}IntA = IntClIntA = IntCl_{\delta}IntA \subseteq IntCl_{\omega}Int_{\omega}A$,

(vii) $Int_{\delta}Cl_{\omega}Int_{\delta}A \subseteq Int_{\delta}ClInt_{\delta}A = Int_{\delta}Cl_{\delta}Int_{\delta}A \subseteq Int_{\delta}Cl_{\omega}IntA = Int_{\delta}ClIntA = Int_{\delta}Cl_{\delta}IntA$,

(viii) $Int_{\delta}Cl_{\omega}IntA = Int_{\delta}ClIntA = Int_{\delta}Cl_{\delta}IntA \subseteq Int_{\delta}ClInt_{\omega}A \subseteq Int_{\delta}Cl_{\delta}Int_{\omega}A \subseteq IntCl_{\delta}Int_{\omega}A \subseteq Int_{\omega}Cl_{\delta}Int_{\omega}A$,

(ix) $Int_{\delta}Cl_{\omega}Int_{\delta}A \subseteq Int_{\delta}Cl_{\omega}IntA = Int_{\delta}ClIntA = Int_{\delta}Cl_{\delta}IntA \subseteq Int_{\delta}Cl_{\omega}Int_{\omega}A \subseteq Int_{\delta}ClInt_{\omega}A$,

(x) $Int_{\omega}Cl_{\delta}Int_{\delta}A \subseteq Int_{\omega}Cl_{\delta}IntA = Int_{\omega}ClIntA = Int_{\omega}Cl_{\omega}IntA \subseteq Int_{\omega}ClInt_{\omega}A \subseteq Int_{\omega}Cl_{\delta}Int_{\omega}A$,

(xi) $Int_{\omega}Cl_{\delta}IntA = Int_{\omega}ClIntA = Int_{\omega}Cl_{\omega}IntA \subseteq Int_{\omega}Cl_{\omega}Int_{\omega}A \subseteq Int_{\omega}ClInt_{\omega}A \subseteq Int_{\omega}Cl_{\delta}Int_{\omega}A$.

Proof. Let A be an $\widehat{r^*\omega}$-set. Then using Proposition 3.10, we have

$IntCl_\omega IntA = IntClIntA = IntCl_\delta IntA.$ (23)

$Int_\delta Cl_\omega IntA = Int_\delta ClIntA = Int_\delta Cl_\delta IntA.$ (24)

$Int_\omega Cl_\omega IntA = Int_\omega ClIntA = Int_\omega Cl_\delta IntA.$ (25)

Using Proposition 2.7, we have

$IntCl_\omega IntA \subseteq IntCl_\omega Int_\omega A \subseteq IntCl\ Int_\omega A \subseteq Int_\omega Cl\ Int_\omega A.$

$IntCl_\omega IntA \subseteq IntClIntA \subseteq IntClInt_\omega A \subseteq Int_\omega ClInt_\omega A$

$IntCl_\omega IntA \subseteq Int_\omega Cl_\omega IntA \subseteq Int_\omega ClIntA \subseteq Int_\omega ClInt_\omega A$

$IntCl_\omega IntA \subseteq Int_\omega Cl_\omega IntA \subseteq Int_\omega Cl_\omega Int_\omega A \subseteq Int_\omega ClInt_\omega A$

Using (23) in the above four expressions, we have

$IntCl_\omega IntA = IntClIntA = IntCl_\delta IntA \subseteq IntCl_\omega Int_\omega A \subseteq IntCl\ Int_\omega A \subseteq Int_\omega Cl\ Int_\omega A,$

$IntCl_\omega IntA = IntClIntA = IntCl_\delta IntA \subseteq IntClInt_\omega A \subseteq Int_\omega ClInt_\omega A.$

$IntCl_\omega IntA = IntClIntA = IntCl_\delta IntA \subseteq Int_\omega Cl_\omega IntA \subseteq Int_\omega ClIntA \subseteq Int_\omega ClInt_\omega A$

$IntCl_\omega IntA = IntClIntA = IntCl_\delta IntA \subseteq Int_\omega Cl_\omega IntA \subseteq Int_\omega Cl_\omega Int_\omega A \subseteq Int_\omega ClInt_\omega A$

Combining the above four expressions we have

$IntCl_\omega IntA = IntClIntA = IntCl_\delta IntA \subseteq IntCl_\omega Int_\omega A \subseteq IntCl\ Int_\omega A \subseteq Int_\omega Cl\ Int_\omega A$ (26)

$IntCl_\omega IntA = IntClIntA = IntCl_\delta IntA \subseteq Int_\omega Cl_\omega IntA \subseteq Int_\omega ClIntA \subseteq Int_\omega ClInt_\omega A$ (27)

$IntCl_\omega IntA = IntClIntA = IntCl_\delta IntA \subseteq Int_\omega Cl_\omega IntA \subseteq Int_\omega Cl_\omega Int_\omega A \subseteq Int_\omega ClInt_\omega A$ (28)

Using (23) in Proposition 2.8 (i) & (ii), we have

$Int_\delta Cl_\omega Int_\delta A \subseteq IntCl_\omega Int_\delta A \subseteq IntClInt_\delta A = IntCl_\delta Int_\delta A \subseteq IntCl_\omega IntA = IntClIntA$

$= IntCl_\delta IntA$ (29)

$IntCl_\omega IntA = IntClIntA = IntCl_\delta IntA \subseteq IntClInt_\omega A \subseteq IntCl_\delta Int_\omega A \subseteq Int_\omega Cl_\delta Int_\omega A.$ (30)

Using (24) in Proposition 2.8 (iii) & (iv), we have

$Int_\delta Cl_\omega Int_\delta A \subseteq Int_\delta ClInt_\delta A = Int_\delta Cl_\delta Int_\delta A \subseteq Int_\delta Cl_\omega IntA = Int_\delta ClIntA = Int_\delta Cl_\delta IntA.$ (31)

$Int_\delta Cl_\omega IntA = Int_\delta ClIntA = Int_\delta Cl_\delta IntA \subseteq Int_\delta ClInt_\omega A \subseteq Int_\delta Cl_\delta Int_\omega A \subseteq IntCl_\delta Int_\omega A$

$\subseteq Int_\omega Cl_\delta Int_\omega A.$ (32)

Using (25) in Proposition 2.8 (vi) , we have

$Int_\omega Cl_\delta Int_\delta A \subseteq Int_\omega Cl_\delta IntA = Int_\omega ClIntA = Int_\omega Cl_\omega IntA \subseteq Int_\omega ClInt_\omega A \subseteq Int_\omega Cl_\delta Int_\omega A.$ (33)

Using (23) in Proposition 2.9 (i), , we have

$Int_\delta Cl_\omega Int_\delta A \subseteq IntCl_\omega Int_\delta A \subseteq IntCl_\omega IntA = IntClIntA = IntCl_\delta IntA \subseteq IntCl_\omega Int_\omega A$ (34)

Using (24) in Proposition 2.9 (iii),, we have

$Int_\delta Cl_\omega Int_\delta A \subseteq Int_\delta Cl_\omega IntA = Int_\delta ClIntA = Int_\delta Cl_\delta IntA \subseteq Int_\delta Cl_\omega Int_\omega A \subseteq Int_\delta ClInt_\omega A$ (35)

Using (25) in Proposition 2.9 (vi), we have

$Int_\omega Cl_\delta Int A = Int_\omega Cl Int A = Int_\omega Cl_\omega Int A \subseteq Int_\omega Cl_\omega Int_\omega A \subseteq Int_\omega Cl Int_\omega A \subseteq Int_\omega Cl_\delta Int_\omega A$. (36)

Then the assertions (i) through (xi), follow from (26), (27), (28), (29), (30), (34), (31), (32), (35), (33), and (36) respectively. This proves the proposition.

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