THE FORCING OPEN DETOUR MONOPHONIC NUMBER OF A GRAPH

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Abstract

Let G be a connected graph with atleast two vertices. Let $M \subseteq V$ be an open detour monophonic set of G. A subset $T \subseteq M$ is called a forcing subset for M if M is the unique minimum open detour monophonic set containing T. A forcing open detour monophonic subset for M is the minimum cardinality of a minimum forcing subset of M, denoted by $f_{odm}(M)$, is the cardinality of a minimum forcing subset of M. The forcing open detour monophonic number of G, denoted by $f_{odm}(G)$. $f_{odm}(G) = \min\{f_{odm}(M)\}$, where the minimum is taken over all odm-set M of G.In this paper, we determined the forcing open detour monophonic number of some standard graphs and obtained some results. It is shown that for every pair of integersa and b with $0 \leq a \leq b, b \geq 2$ and $b - a > 3$, there exists a connected graph G such that $f_{odm}(G) = a$and $odm(G) = b$.

Keywords: Detour monophonic number, Open detour monophonic number, Forcing open detour monophonic number.

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1. Introduction

For a graph G consists of a finite non-empty set Vof vertices and a set Eof 2-element subsets of Vcalled edges. For graph theoretic terminologies, we refer reader [1]. If the vertices u and v are joined by the edges e, then the u and v are referred to as neighbors of each other. The neighbors of a vertex v is called the neighborhood of v, is denoted by N(v).Thus $\deg(v) = |N(v)|$. A vertex vis said to be a universal vertex if $\deg(v) = n - 1$. A subgraph Hof Gis called an induced subgraph of Gif whenever u and v are vertices of H and uv is an edge of G, then uv is an edge of H as well as. A vertex v in a graph Gis called a simplicial vertex if the subgraph induced by its neighbourhood is complete. Let Gand Hbe two graphs. The join $G + H$consists of $G \circ H$ and all edges joining a vertices of Gand H. The total graph T(G)of G is the graph with the vwertex set $V \cup E$ and two vertices are adjacent whenever they are either adjacent or incident in G.

The distance between u and v is the shortest length of every $x - y$ path in G, is denoted by $d(u, v)$. A chord of a path Pis an edge which connects two non-adjacent vertices of P. A $x - y$ path is called a monophonic path if it is chord less path. The monophonic distance $d_m(x,y)$from x to y is defined as the length of a longest $x-y$monophonic path in G. A $x-y$ monophonic path with its length $d_m(x,y)$ is called a $x-y$ monophonic. A set Mof vertices of a graph Gis a monophonic set of Glies on a $x-y$monophonic path in Gfor some $x,y \in M$. The monophonic number $m(G)$ is the minimum cardinality of a monophonic set of G. The monophonic number of a graph was studied in [3,4]. A set $M \subseteq V$ is called an open detour monophonic set of G if $J_{dm}(M) = V$. An open detour monophonic number $odm(G)$ is the minimum cardinality of an open detour monophonic set of G. The open detour monophonic number of a graph was studied in [2]. Let $M \subseteq V$ be a detour monophonic set of G. A subset $T \subseteq M$ is called a forcing subset for M if M is the unique minimum detour monophonic set containing T. A forcing subset for M is the minimum cardinality of a minimum forcing subset of M, denoted by $f_{dm}(M)$, is the cardinality of a minimum forcing subset of M. The forcing detour monophonic number of G, denoted by $f_{dm}(G)$. $f_{dm}(G) = \min\{f_{dm}(M)\}$, where the minimum is taken over all odm-sets M of G. The forcing detour monophonic number of a graph was studied in [5]. A vertexof a connected graph Gis said to be a detour monophonic simplicial vertex of Gif v is not an internal vertex of any $x-y$ detour monophonic path for every $x,y \in V$. Each extreme vertex of Gis a detour monophonic simplicial vertex ofG.

The Forcing Open Detour Monophonic Number Of A Graph

Definition 2.1

Let M be an open detour monophonic set of G. A subset $T \subseteq M$ is called a forcing subset for M if M is the unique minimumodm-set containing T. A forcing subset for M is the minimum cardinality of a minimum forcing subset of M, denoted by $f_{odm}(M)$, is the cardinality of a minimum forcing subset of M. The forcing open detour monophonic number of G, denoted by $f_{odm}(G)$. $f_{odm}(G) = \min\{f_{odm}(M)\}$, where the minimum is taken over all odm-sets Mof G.

Example 2.2

For the graph G of Figure 2.1, $M_1 = \{v_1, v_4, v_5\}$ and $M_2 = \{v_1, v_4, v_6\}$ are theodm-sets of G such that $f_{odm}(M_1) = 1$ and$f_{odm}(M_2) = 1$ so that $f_{odm}(G) = 1$.

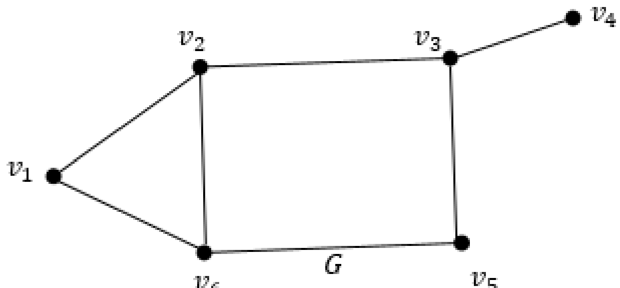

Figure 2.1

The following result follows immediately from the definitions of open detour monophonic number and the forcing open detour monophonic number of a connected graph G.

Theorem 2.3

Let G be a connected graph of order n, Then

1. G has a unique minimum odm-set if and only if $f_{odm}(G) = 0$.
2. $f_{odm}(G) = odm(G)$if and only if no minimum odm-set containing any of its proper subsets.

Theorem 2.4

For the connected graph Gand let S be the set of all detour monophonic simplicial vertices of G.Then $f_{odm}(G) \leq odm(G) - |S|$.

Proof. Let M be any minimum odm-set of G.Then$odm(G) = |M|, S \subseteq M$ and M is the unique odm-set containing $M - S$. Thus $f_{odm}(G) \leq |M - S| = |M| - |S| = odm(G) - |S|$.

Theorem 2.5

For any complete graph $G = K_n$ $(n \geq 2), f_{odm}(G) = 0$.

Proof: Let $G = K_n(n \geq 2)$,then $M = V(G)$is the unique odm-set of G, by Theorem 2.3,$f_{odm}(G) = 0$

Corollary 2.6

(i) For any star graph $G = K_{1,n-1}$ $(n \geq 2), f_{odm}(G) = 0$.

(ii) For any non-trivial tree $T, f_{odm}(G) = 0$.

Proof: This result follows by Theorem 2.5.

Theorem 2.7

LetG be a connected graph f order n, $0 \leq f_{odm}(G) \leq odm(G) \leq n$.

Proof: Since every connected graph have a odm-set, $f_{odm}(G) \geq 0$. Also since forcing subset is a subset of odm-set of G, $f_{odm}(G) \leq odm(G)$. Since V(G) is the unique odm-set of G, $odm(G) \leq n$. Hence $0 \leq f_{odm}(G) \leq odm(G) \leq n$.

Remark 2.8. The bounds in Theorem 2.7 are sharp. For $G = K_n (n \geq 2), f_{odm}(G) = 0$ and $odm(G) = n$. The bounds in strict in Theorem 2.7. For the graph G in Figure 2.1. $odm(G) = 3$ and $f_{odm}(G) = 1$. Thus $0 < f_{odm}(G) < odm(G) < n$.

Theorem 2.9. For the cycle $G = C_n (n \geq 3)$,

$$f_{odm}(G) = \begin{cases} 0 & if n = 3,4 \\ 4 & if n = 5 \\ 1 & if n = 6 \\ 3 & if n \geq 7 \end{cases}$$

Proof: Let C_n be $v_1, v_2, \ldots, v_{n-1}, v_n$. For $n = 3,4$, $M = V(G)$ is the unique odm-set of G. By Theorem 2.3, $f_{odm}(G) = 0$. Let $n = 5$. For any $x \in V(G)$, there exists $M = V(G) - \{x\}$, is a odm-set of $G, odm(G) = 4 = f_{odm}(G)$. For $n = 6$, Let x be a vertex of G and y, z be the two antipodal vertices of x and v, w be the antipodal vertex of u. Then $M_1 = \{x, y, z\}$ and $M_2 = \{u, v, w\}$ are the only two odm-sets of G so that $f_{odm}(G) = 1$. For $n \geq 7$, Let $xy \in E(G)$ and $u, v \in V(G)$ such that $d(x, u) = d(y, v) = 2$. Then $M = \{x, y, u, v\}$ is a odm-set of G, odm(G)=4. Since $n \geq 7$, there must be at least 7 odm-sets and so $f_{odm}(M) \geq 2$. Since any two element subset of M is not a forcing subset of M, $f_{odm}(M) \geq 3$. Now $T = \{x, y, u\}$ is a forcing subset of M and so $f_{odm}(M) = 3$. Since this is true for all odm-sets M of G, $f_{odm}(G) = 3$.

Theorem 2.10. For the complete bipartite graph $G = K_{r,s} (2 \leq r \leq s)$,

$$\mathrm{f_{odm}(G)} = \begin{cases} 0 & if r = s = 2 \\ 2 & if\ 2 = r \leq s \\ 4 & if\ 3 \leq r \leq s \end{cases}$$

Proof: If $r = s = 2$, then $G = C_4$, the result follows by Theorem 2.9, $f_{odm}(G) = 0$. If $r = 2$ and $s \geq 3$. Let $X = \{x_1, x_2, \ldots, x_r\}$ and $Y = \{y_1, y_2, \ldots, y_s\}$ be the bipartite sets of G. Then $M_{ij} = \{x_1, x_2, y_i, y_j\}$ is the odm-set of G for some i and $j (1 \leq i, j \leq s)$. $odm(G) = 4$. Since $X = \{x_1, x_2\}$ is a subset of every $odm(G)$-set of G, by Theorem 2.4, $f_{odm}(G) \leq odm(G) - |X| = 4 - 2 = 2$. We prove that $f_{odm}(G) = 2$. Since $s \geq 3$ and y_i lies on more than two odm-sets for some $i (1 \leq i \leq s)$ and y_j lies on more than two odm-sets of G for some $j ((1 \leq j \leq s)$. Since this is true for $f_{odm}(M_{ij}) = 2$ for all i and $j, (1 \leq i, j \leq s)$. Hence it follows that $f_{odm}(G) = 2$. If $r, s \geq 3, M_{ij} = \{x_i, x_j, y_l, y_m\}$ is the odm-set of G for some i and $j (1 \leq i, j \leq r)$ and for some l and $m (1 \leq l, m \leq s), odm(G) = 4$. Since M_{ij} is not the unique odm-set containing any of its proper subsets so that $f_{odm}(G) = 4$.

Theorem 2.11. For the wheel $G = K_n + C_{n-1} (n \geq 4), f_{odm}(G) = \begin{cases} 0 & if n = 4,5 \\ 4 & if n = 6 \\ 1 & if n = 7 \\ 3 & if n \geq 8 \end{cases}$

Proof: Let $V(K_1) = x$ and $V(C_{n-1}) = \{v_1, v_2, \ldots, v_{n-1}\}$. If $n = 4$, then $G = K_4$, by Theorem 2.5, $f_{odm}(G) = 0$. If $n = 5, M = V(C_4)$ is the unique odm-set of G, by Theorem 2.3, $f_{odm}(G) = 0$. Let $n \geq 6$. Since x is a detour monophonic simplicial vertices of $G, \{x\}$ is a subset of every odm-set of G. Let $n = 6$. For any vertex $u \in v(G)$, there exists $M = V(G) - \{u\}$ is a odm-set of G, $odm(G) = 5$. By Theorem 2.4, $f_{odm}(G) \leq odm(G) - 1 = 5 - 1 = 4$. For $n = 7$, Let h be a vertex of G and i, j be the two antipodal vertices of h and v, w be the antipodal vertex of u. Then $M_1 = \{x, h, i, j\}$ and $M_2 = \{x, u, v, w\}$ are the only two odm-sets of G so that $odm(G) = 4$. By Theorem 2.4 $f_{odm}(G) \leq 3$. Since $\{h\}$ is not containing M_2, $f_{odm} = 1$. For $n \geq 8$, Let $yz \in E(G)$ and $u, v \in V(G)$ such that $d(y, u) = d(z, v) = 2$. Then $M = \{x, y, z, u, v\}$ is a odm-set of G, $odm(G)$=5. By Theorem 2.4 $f_{odm}(G) \leq 4$. Since $n \geq 8$, there must be at least 7 odm-sets and so $f_{odm}(M) \geq 2$. Since any two element subset of M is not a forcing subset of M, $f_{odm}(M) \geq 3$. Now $T = \{y, z, u\}$ is a forcing subset of M and so $f_{odm}(M) = 3$. Since this is true for all odm-sets M of G, $f_{odm}(G) = 3$.

Theorem 2.12. For the fan graph $G = F_n = K_1 + P_{n-1}, (n \geq 3), f_{odm}(G) = 0$.

Proof: Let $V(K_1) = \{x\}$ and $V(P_{n-1}) = \{v_1, v_2, \ldots, v_{n-1}\}$.

If $n = 3$, then $G = K_3$, by Theorem 2.5, $f_{odm}(G) = 0$. If $n = 4$, then $M = \{v_1, v_3\}$ is the unique odm-set of G so that $f_{odm}(G) = 0$. Let $n \geq 5$, then $M = \{x, v_1, v_{n-1}\}$ is the unique odm-set of $G, f_{odm}(G) = 0$.

Theorem 2.13. Let $G = \overline{K}_2 + \overline{K}_{n-2} (n \geq 6), f_{odm}(G) = 2$.

Proof: Let $V(K_2) = \{x, y\}$ and $V(K_{n-2}) = \{v_1, v_2, \ldots, v_{n-2}\}$. Since $X = \{x, y\}$ is a subset of every odm-set of G. Now $M_{ij} = \{x, y\} \cup \{v_i, v_j\} i \neq j (1 \leq i, j \leq n - 2)$ is a odm-set of $G, odm(G) = 4$. By Theorem 2.4, $f_{odm}(G) = odm(G) - |X| = 4 - 2 = 2$. Since $\{v_i, v_j\}$ is not containing any odm-set of it proper subsets $f_{odm}(G) = 2$.

Theorem 2.14. For any Ladder graph $G = L_n = P_2 \times P_n (n \geq 2), f_{odm}(G) = 0$.

Proof: Let $V(G) = \{v_1, v_2, \ldots, v_{n-2}\} \cup \{u_1, u_2, \ldots, u_n\}$. Let $M = \{, v_1, u_1, u_n\}$ be unique odm-set of G. By Theorem 2.3, $f_{odm}(G) = 0$

Theorem 2.15. For the total graph of path $G = T(P_n)(n \geq 3), f_{odm}(G) = 0$.
Proof: Let $/V(G) = \{v_1, v_2, \ldots, v_{n-2}\} \cup \{u_1, u_2, \ldots, u_{n-1}\}$. For $n = 3$, then $M = \{v_1, v_2, v_3\}$ is the unique odm-set of $G, f_{odm}(G) = 0$.For $n \geq 4, X = \{v_1, v_n\}$. Since $X = \{v_1, v_n\}$ is the detour monophonic simplicial vertices of G. Let$M = \{v_1, v_n\}$ is the unique odm-set of $G, f_{odm}(G) = 0$.

Theorem 2.16
For every pair of integers$a$ and b with $0 \leq a \leq b, b \geq 2$ and $b - a > 3$, there exists a connected graph G such that $f_{odm}(G) = a$and $odm(G) = b$.
Proof: For $a = 0$, let $G = K_b$. Then by Theorem 2.5, $f_{odm}(G) = 0$and$odm(G) = b$. Thus we assume $0 \leq a \leq b, b \geq 2$.
Let $P: x, y$be a path on two vertices and $P_i: x_i, y_i, z_i (1 \leq i \leq a)$ be a copy of a path on 3 vertices. Let Hbe the graph obtained from $P$and $P_i (1 \leq i \leq a)$ and introduce a vertex w and introduce the edges $xx_i (1 \leq i \leq a)$, $yz_i (1 \leq i \leq a)$ and $wx_i (1 \leq i \leq a)$, $wz_i (1 \leq i \leq a)$. Let G be the graph obtained from H by adding the new vertices$v_1, v_2, u_j (1 \leq j \leq b - a - 2)$and introducing the edges $yu_j (1 \leq j \leq b - a - 3)$, xu_{b-a-2}, wv_1 and wv_2.The graph $G$is shown in Figure 2.2.

First we prove that $odm(G) = b$.Let$X = \{u_1, u_2, \ldots, u_{b-a-2}, v_1, v_2\}$be the set of all extreme vertices of G. By Theorem1.1,X contains every minimum odm-set of $G, odm(G) \geq b - a - 2 + 2 = b - a$. We observe that every minimum odm-set contains exactly one vertex from $H_i = \{x_i, z_i\} (1 \leq i \leq a)$. Thus $odm(G) \geq b - a + a = b$. Let $M = X \cup \{x_1, x_2, \ldots, x_a\}$. Then $M$is an odm-set of G so that $odm(G) = b$. Next we prove that $f_{odm}(G)$=a.Since every minimum odm-set contains X, by Theorem 2.4, $f_{odm}(G) \leq \text{odm(G)} - |X| = b - (b - a) = a$. We prove that $f_{odm}(G)$=a. On the contrary, suppose that $f_{odm}(G) < a$. Now since $odm(G) = b$ and every odm-set contains X and every odm-set of G contains at least one vertex from each$H_i (1 \leq i \leq a)$. It is easily seen that every odm-set M is of the $X \cup \{e_1, e_2, \ldots, e_a\}$, where $e_i \in H_i (1 \leq i \leq a)$.Let Gbe any proper subset of M with $|T| < a$. Then there exists $e_j \in H_j (1 \leq j \leq a)$ such that $e_j \in T$. Let f_j be the vertex of $H_i (1 \leq i \leq a)$distinct from e_j.Then $M' = (M - \{e_j\}) \cup \{f_j\}$ is a odm-set of G properly containing T. Thus M is not the unique odm-set of G containing T so that T is not a forcing subset of M. This is true for all odm-sets containing G so that $f_{odm}(M) = a$.

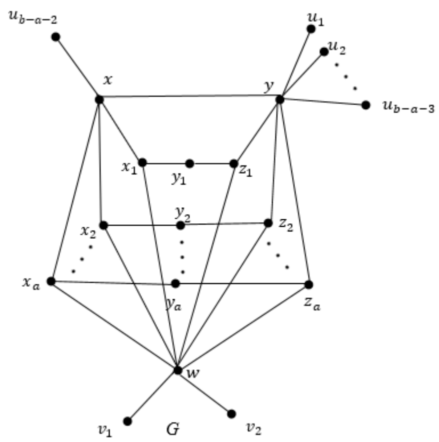

Figure 2.2

2. Conclusion

This paper exhibits the forcing open detour monophonic number of some standard graphs.

3. References

1. G. Chartrand and P.Zhang, Introduction to Graph Theory, Tata McGraw Hill (2006).
2. K.KrishnaKumari, S.Kavitha and D.Nidha, On the upper detour monophonic number of a graph, Malaya Journal of Matematik, 9(1), (2021), 765-769.
3. A.P. Santhakumaran and P. Titus, Monophonic distance in graphs, Discrete Mathematics, Algorithms and Applications, 03 (2), (2011), 159 – 169
4. A.P. Santhakumaran, P. Titus and K. Ganesamoorthy, On the monophonic number of a graph, J.Appl. Math. Informatics, 32 (1-2),(2014), 255 - 266.
5. P.Titus and K.Ganeshmoorthy, Forcing detour monophonic number of a graph, FactaUniversitatis,Ser math Inform. 28, No 2 (2013),211-220.