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TILDE (T) AND MINUS PARTIAL ORDERING ON INTERVAL VALUED INTUITIONISTIC FUZZY MATRICES

M. AROCKIA RANJITHKUMAR[1,2] ,J. BOOBALAN [2]

Department of Mathematics, M.Kumarasamy College of Engineering, Karur - 639 113, Tamilnadu, India.

Department of Mathematics, Annamalai University, Annamalai Nagar - 608 002, Tamilnadu, India

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Abstract: Aim of this article is to characterize the interval valued T and minus partial orderings on Intuitionistic fuzzy matrices (IFM). Also, using the g- inverses, we discuss some Theorems and examples for the interval valued T and minus partial ordering on IFM.

Keywords: Intuitionistic fuzzy matrices (IFM), interval valued T−ordering, interval valued minus ordering, g-inverse, Moore-penrose inverses**.**

I. INTRODUCTION :

Let the IFM P of order m rows and n columns is in the form of $P=[y_{ij},< p_{ij\alpha},\mathrm{p}_{ij\beta} >]$,where $p_{ij\alpha}$ and $p_{ij\beta}$ are called the degree of membership and also the non-membership of y_{ij} in A , it preserving the condition $0 \le p_{ij\alpha} + p_{ij\beta} \le 1$ .In intuitionistic fuzzy matrices, partial ordering is significant. The idea of fuzzy matrix was first presented by Thomosan [2] in 1977 and it has further developments by various researchers. Jian Miao Chen pioneered the partial orderings on fuzzy matrices, which are comparable to the star ordering on complex matrices [3].After that, a lot of works have been done using this notion. A.R. Meenachi [1] characterizes the minus ordering on matrices in terms of their generalized inverses. Another novelty is the way she defines space ordering [6] on fuzzy matrices as a partial order on the set of all idempotent matrices in F_n. Partial ordering is a reflexive, anti-symmetric, transitive crisp binary relation R(X, X) [5]. The properties of this class of relations are denoted by the common symbol ≤ . Therefore, <x ,y> represents <x,y> ∈ R and indicates that x comes before y. The symbol ≥ [9] denotes the inverse partial ordering R^{-1} (X, X). We say that y succeeds x if y ≤ x implying that <x,y> ∈ R^{-1}. The symbols $\le^{P}$, $\le^{Q}$ and $\le^{R}$ are used to denote the various partial orderings P, Q, and R, respectively. In this Section I , As an analogue to the star ordering on complex matrices, we start with the T inverse or reverse ordering on IFM. We explore different ordering on the IFM using a variety of generalized inverses, including g-inverse, group inverse, and Moore-penrose inverses, and we analyses how these ordering relate to T ordering[8]. We derive some equivalent

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conditions for each ordering by using generalized inverses[10]. Boobalan [11] has studied Some Properties of Symmetrical Difference Over Intuitionistic Fuzzy Matrices. Boobalan Sriram [12] have discussed Certain Properties of Intuitionistic Fuzzy Matrices. Additionally, we demonstrate that these orderings are the same for a particular class of IFM. In section II we study the minus ordering for IFM as an analogue of minus ordering for complex matrix studied in [7] and as a generalization of T- ordering for IFM introduced . We show that the minus ordering is only a partial ordering in the set of all regular fuzzy matrices. Finally, we characterize the minus ordering on matrix in terms of their generalized inverses.

II. INTERVAL VALUED T-ORDERING ON INTUITIONISTIC FUZZY MATRICES

Definitions and Theorems

Definition:2.1 Interval-valued intuitionistic fuzzy matrix (IVIFM): An interval valued intuitionistic fuzzy matrix (IVIFM) P of order m×n is defined as $P = [x_{ij}, < p_{ij\mu}, p_{ij\nu} >]_{m\times n}$ where $p_{ij\mu}$ and $p_{ij\nu}$ are both the subsets of [0,1] which are denoted by $p_{ij\mu} = [p_{ij\mu L}, p_{ij\mu U}]$ and $p_{ij\nu} = [p_{ij\nu L}, p_{ij\nu U}]$ which maintaining the condition $0 \le p_{ij\mu U} + p_{ij\nu U} \le 1$, $0 \le p_{ij\mu L} + p_{ij\nu L} \le 1$, $0 \le p_{\mu L} \le p_{\mu U} \le 1$, $0 \le p_{\nu L} \le p_{\nu U} \le 1$ for i = 1,2,⋯,m and j = 1,2,⋯,n.

Example2.1 Let $P = \begin{bmatrix} <[0.2,0.2],[0.3,0.3]> & <[0.2,0.3],[0.3,0.4]> \\ <[0.2,0.3],[0.3,0.4]> & <[0.2,0.2],[0.3,0.3]> \end{bmatrix}$,

$$P_L = \begin{bmatrix} <0.2,0.2> & <0.3,0.3> \\ <0.2,0.2> & <0.3,0.3> \end{bmatrix}, P_U = \begin{bmatrix} <0.2,0.3> & <0.3,0.4> \\ <0.3,0.2> & <0.4,0.3> \end{bmatrix}$$

Definition:2.2 [3] For $P,Q \in (IVIFM)_{m\times n}$ the T−ordering $P \le^T Q$ is well-defined as $P \le^T Q \Leftrightarrow$

(i)$[P_{\mu L}, P_{\nu L}]^T \, [P_{\mu L}, P_{\nu L}] = [P_{\mu L}, P_{\nu L}]^T \, [Q_{\mu L}, Q_{\nu L}]^T$ and $[P_{\mu L}, P_{\nu L}] \, [P_{\mu L}, P_{\nu L}]^T = [Q_{\mu L}, Q_{\nu L}] \, [P_{\mu L}, P_{\nu L}]^T$.

(ii)$[P_{\mu U}, P_{\nu U}]^T \, [P_{\mu U}, P_{\nu U}] = [P_{\mu U}, P_{\nu U}]^T \, [Q_{\mu U}, Q_{\nu U}]^T$ and $[P_{\mu U}, P_{\nu U}] \, [P_{\mu U}, P_{\nu U}]^T = [Q_{\mu U}, Q_{\nu U}] \, [P_{\mu U}, P_{\nu U}]^T$.

Theorem 2.1 . Let $P,Q \in (IF)_{m\times n}$ and Q^+ exists. Then the given conditions are equivalent.

(i) $[P_{\mu L}, P_{\nu L}] \overset{T}{\le} [Q_{\mu L}, Q_{\nu L}], [P_{\mu U}, P_{\nu U}] \overset{T}{\le} [Q_{\mu U}, Q_{\nu U}]$

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(ii) $[P_{\mu L},P_{vL}]^{+}[P_{\mu L},P_{vL}]=[P_{\mu L},P_{vL}]^{+}[Q_{\mu L},Q_{vL}]$, $[P_{\mu L},P_{vL}][P_{\mu L},P_{vL}]^{+}=[Q_{\mu L},Q_{vL}][P_{\mu L},P_{vL}]^{+}$

$\&[P_{\mu U},P_{vU}]^{+}[P_{\mu U},P_{vU}]=[P_{\mu U},P_{vU}]^{+}[Q_{\mu U},Q_{vU}]$,

$[P_{\mu U},P_{vU}][P_{\mu U},P_{vU}]^{+}=[Q_{\mu U},Q_{vU}][P_{\mu U},P_{vU}]^{+}$

(iii) $[P_{\mu L},P_{vL}][P_{\mu L},P_{vL}]^{+}[Q_{\mu L},Q_{vL}]=[P_{\mu L},P_{vL}]=[Q_{\mu L},Q_{vL}][P_{\mu L},P_{vL}]^{+}[Q_{\mu L},Q_{vL}]$,

$[P_{\mu U},P_{vU}][P_{\mu U},P_{vU}]^{+}[Q_{\mu U},Q_{vU}]=[P_{\mu U},P_{vU}]=[Q_{\mu U},Q_{vU}][P_{\mu U},P_{vU}]^{+}[Q_{\mu U},Q_{vU}]$

Proof: (i) $\Rightarrow$ (ii) ,By (i) We have $[P_{\mu L},P_{vL}]^{T}[P_{\mu L},P_{vL}]=[P_{\mu L},P_{vL}]^{T}[Q_{\mu L},Q_{vL}]$,
$[P_{\mu L},P_{vL}][P_{\mu L},P_{vL}]^{T}=[Q_{\mu L},Q_{vL}][P_{\mu L},P_{vL}]^{T}$

Then,

$[P_{\mu L},P_{vL}]^{+}[P_{\mu L},P_{vL}]=[P_{\mu L},P_{vL}]^{+}[P_{\mu L},P_{vL}][P_{\mu L},P_{vL}]^{+}[P_{\mu L},P_{vL}]=[P_{\mu L},P_{vL}]^{+}\left([P_{\mu L},P_{vL}]^{+}\right)^{T}[P_{\mu L},P_{vL}]^{T}[P_{\mu L},P_{vL}]$

$=[P_{\mu L},P_{vL}]^{+}\left([P_{\mu L},P_{vL}]^{+}\right)^{T}[P_{\mu L},P_{vL}]^{T}[Q_{\mu L},Q_{vL}]=[P_{\mu L},P_{vL}]^{+}[P_{\mu L},P_{vL}][P_{\mu L},P_{vL}]^{+}[Q_{\mu L},Q_{vL}]$

$[P_{\mu L},P_{vL}]^{+}[P_{\mu L},P_{vL}]=[P_{\mu L},P_{vL}]^{+}[Q_{\mu L},Q_{vL}]$

Similarly, $[P_{\mu L},P_{vL}][P_{\mu L},P_{vL}]^{+}=[Q_{\mu L},Q_{vL}][P_{\mu L},P_{vL}]^{+}$ and

$\&[P_{\mu U},P_{vU}]^{+}[P_{\mu U},P_{vU}]=[P_{\mu U},P_{vU}]^{+}[Q_{\mu U},Q_{vU}]$, $[P_{\mu U},P_{vU}][P_{\mu U},P_{vU}]^{+}=[Q_{\mu U},Q_{vU}][P_{\mu U},P_{vU}]^{+}$

(ii) $\Rightarrow$ (iii) $[P_{\mu L},P_{vL}]^{+}[P_{\mu L},P_{vL}]=[P_{\mu L},P_{vL}]^{+}[Q_{\mu L},Q_{vL}]$ implies
$[P_{\mu L},P_{vL}]=[P_{\mu L},P_{vL}][P_{\mu L},P_{vL}]^{+}[P_{\mu L},P_{vL}]=[P_{\mu L},P_{vL}][P_{\mu L},P_{vL}]^{+}[Q_{\mu L},Q_{vL}]$ and

$[P_{\mu L},P_{vL}][P_{\mu L},P_{vL}]^{+}=[Q_{\mu L},Q_{vL}][P_{\mu L},P_{vL}]^{+}$ implies

$[P_{\mu L},P_{vL}]=[P_{\mu L},P_{vL}][P_{\mu L},P_{vL}]^{+}[P_{\mu L},P_{vL}]=[Q_{\mu L},Q_{vL}][P_{\mu L},P_{vL}]^{+}[P_{\mu L},P_{vL}]$.

Similarly, $[P_{\mu U},P_{vU}][P_{\mu U},P_{vU}]^{+}[Q_{\mu U},Q_{vU}]=[P_{\mu U},P_{vU}]=[Q_{\mu U},Q_{vU}][P_{\mu U},P_{vU}]^{+}[Q_{\mu U},Q_{vU}]$

(iii)$\Rightarrow$ (i) By
$[P_{\mu L},P_{vL}]=[P_{\mu L},P_{vL}][P_{\mu L},P_{vL}]^{+}[Q_{\mu L},Q_{vL}],\left([P_{\mu L},P_{vL}][P_{\mu L},P_{vL}]^{+}\right)^{T}[P_{\mu L},P_{vL}]=\left([P_{\mu L},P_{vL}][P_{\mu L},P_{vL}]^{+}\right)^{T}[Q_{\mu L},Q_{vL}]$

Then $[P_{\mu L},P_{vL}]^{T}\left([P_{\mu L},P_{vL}]^{+}\right)^{T}[P_{\mu L},P_{vL}]^{T}[P_{\mu L},P_{vL}]=[P_{\mu L},P_{vL}]^{T}\left([P_{\mu L},P_{vL}]^{+}\right)^{T}[P_{\mu L},P_{vL}]^{T}[Q_{\mu L},Q_{vL}]$.

Hence $[P_{\mu L},P_{vL}]^{T}[P_{\mu L},P_{vL}]=[P_{\mu L},P_{vL}]^{T}[Q_{\mu L},Q_{vL}]$ and $[P_{\mu L},P_{vL}][P_{\mu L},P_{vL}]^{T}=[Q_{\mu L},Q_{vL}][P_{\mu L},P_{vL}]^{T}$

Similarly,

$$[P_{\mu U},P_{\nu U}]^T[P_{\mu U},P_{\nu U}]=[P_{\mu U},P_{\nu U}]^T[Q_{\mu U},Q_{\nu U}] \text{ and } [P_{\mu U},P_{\nu U}][P_{\mu U},P_{\nu U}]^T=[Q_{\mu U},Q_{\nu U}][P_{\mu U},P_{\nu U}]^T$$

Theorem 2.2 . Let $P,Q\in(IF)_{m,n}$. If P^+ and Q^+ exists., then the given conditions are equivalent.

(i) $[P_{\mu L},P_{\nu L}]\overset{T}{\leq}[Q_{\mu L},Q_{\nu L}],[P_{\mu U},P_{\nu U}]\overset{T}{\leq}[Q_{\mu U},Q_{\nu U}]$

(ii) $[P_{\mu L},P_{\nu L}]^+[P_{\mu L},P_{\nu L}]=[Q_{\mu L},Q_{\nu L}]^+[P_{\mu L},P_{\nu L}]$ and,

$[P_{\mu L},P_{\nu L}][P_{\mu L},P_{\nu L}]^+=[P_{\mu L},P_{\nu L}][Q_{\mu L},Q_{\nu L}]^+$ &

$[P_{\mu U},P_{\nu U}]^+[P_{\mu U},P_{\nu U}]=[Q_{\mu U},Q_{\nu U}]^+[P_{\mu U},P_{\nu U}]$,and

$[P_{\mu U},P_{\nu U}][P_{\mu U},P_{\nu U}]^+=[P_{\mu U},P_{\nu U}][Q_{\mu U},Q_{\nu U}]^+$

(iii) $[Q_{\mu L},Q_{\nu L}]^+[P_{\mu L},P_{\nu L}][P_{\mu L},P_{\nu L}]^+=[P_{\mu L},P_{\nu L}]^+=[P_{\mu L},P_{\nu L}]^+[P_{\mu L},P_{\nu L}][Q_{\mu L},Q_{\nu L}]^+$,

$[Q_{\mu U},Q_{\nu U}]^+[P_{\mu U},P_{\nu U}][P_{\mu U},P_{\nu U}]^+=[P_{\mu U},P_{\nu U}]^+=[P_{\mu U},P_{\nu U}]^+[P_{\mu U},P_{\nu U}][Q_{\mu U},Q_{\nu U}]^+$

(iv) $[P_{\mu L},P_{\nu L}]^T[P_{\mu L},P_{\nu L}][Q_{\mu L},Q_{\nu L}]^T=[P_{\mu L},P_{\nu L}]^T=[Q_{\mu L},Q_{\nu L}]^+[P_{\mu L},P_{\nu L}][P_{\mu L},P_{\nu L}]^T$

$[P_{\mu U},P_{\nu U}]^T[P_{\mu U},P_{\nu U}][Q_{\mu U},Q_{\nu U}]^T=[P_{\mu U},P_{\nu U}]^T=[Q_{\mu U},Q_{\nu U}]^+[P_{\mu U},P_{\nu U}][P_{\mu U},P_{\nu U}]^T$

Proof: (i) $\Rightarrow$ (iv)$[P_{\mu L},P_{\nu L}]^T$ $[P_{\mu L},P_{\nu L}]$ = $[P_{\mu L},P_{\nu L}]^T$ $[Q_{\mu L},Q_{\nu L}]$ implies

$[P_{\mu L},P_{\nu L}]^T[P_{\mu L},P_{\nu L}]=[P_{\mu L},P_{\nu L}]^T[Q_{\mu L},Q_{\nu L}][Q_{\mu L},Q_{\nu L}]^+[Q_{\mu L},Q_{\nu L}]$

Then, (i)$[P_{\mu L},P_{\nu L}]^T[P_{\mu L},P_{\nu L}]=\left([P_{\mu L},P_{\nu L}]^T[P_{\mu L},P_{\nu L}]\right)^T$

$=\left([Q_{\mu L},Q_{\nu L}]^+[Q_{\mu L},Q_{\nu L}]\right)^+\left([P_{\mu L},P_{\nu L}]^T[Q_{\mu L},Q_{\nu L}]\right)^T=[Q_{\mu L},Q_{\nu L}]^+[Q_{\mu L},Q_{\nu L}][P_{\mu L},P_{\nu L}]^T[P_{\mu L},P_{\nu L}]$

Hence,

$[P_{\mu L},P_{\nu L}]^T[P_{\mu L},P_{\nu L}][P_{\mu L},P_{\nu L}]^+=[Q_{\mu L},Q_{\nu L}]^+[Q_{\mu L},Q_{\nu L}][P_{\mu L},P_{\nu L}]^T[P_{\mu L},P_{\nu L}][P_{\mu L},P_{\nu L}]^+$

and $[P_{\mu L},P_{\nu L}]^T\left([P_{\mu L},P_{\nu L}][P_{\mu L},P_{\nu L}]^+\right)^T=[Q_{\mu L},Q_{\nu L}]^+[Q_{\mu L},Q_{\nu L}][P_{\mu L},P_{\nu L}]^T\left([P_{\mu L},P_{\nu L}][P_{\mu L},P_{\nu L}]^+\right)^+$

Therefore,$[P_{\mu L},P_{\nu L}]^T=[Q_{\mu L},Q_{\nu L}]^+[Q_{\mu L},Q_{\nu L}][P_{\mu L},P_{\nu L}]^T=[Q_{\mu L},Q_{\nu L}]^+[P_{\mu L},P_{\nu L}][P_{\mu L},P_{\nu L}]^T$

$[P_{\mu L},P_{\nu L}]^T=[P_{\mu L},P_{\nu L}]^T[P_{\mu L},P_{\nu L}][Q_{\mu L},Q_{\nu L}]^+$ by$[P_{\mu L},P_{\nu L}][P_{\mu L},P_{\nu L}]^T=[Q_{\mu L},Q_{\nu L}][P_{\mu L},P_{\nu L}]^T$

Similarly,

$[P_{\mu U},P_{\nu U}]^T[P_{\mu U},P_{\nu U}][Q_{\mu U},Q_{\nu U}]^T=[P_{\mu U},P_{\nu U}]^T=[Q_{\mu U},Q_{\nu U}]^+[P_{\mu U},P_{\nu U}][P_{\mu U},P_{\nu U}]^T$

(iv) $\Rightarrow$ (ii) $By[P_{\mu L},P_{\nu L}]^T=[Q_{\mu U},Q_{\nu U}]^+[P_{\mu L},P_{\nu L}][P_{\mu L},P_{\nu L}]^T,[P_{\mu L},P_{\nu L}]^T\left([P_{\mu L},P_{\nu L}]^+\right)^T$

$=[Q_{\mu U},Q_{\nu U}]^+[P_{\mu L},P_{\nu L}][P_{\mu L},P_{\nu L}]^T\left([P_{\mu L},P_{\nu L}]^+\right)^T$

Then,

$[P_{\mu L},P_{vL}]^{+}[P_{\mu L},P_{vL}]=[Q_{\mu L},Q_{vL}]^{+}[P_{\mu L},P_{vL}][P_{\mu L},P_{vL}]^{+}[P_{\mu L},P_{vL}]=[Q_{\mu L},Q_{vL}]^{+}[P_{\mu L},P_{vL}]$

And

$[P_{\mu L},P_{vL}][P_{\mu L},P_{vL}]^{+}=[P_{\mu L},P_{vL}][Q_{\mu L},Q_{vL}]^{+}$ $by[P_{\mu L},P_{vL}]^{T}=[P_{\mu L},P_{vL}]^{T}[P_{\mu L},P_{vL}][Q_{\mu L},Q_{vL}]^{+}$

Similarly,

$[P_{\mu U},P_{vU}]^{+}[P_{\mu U},P_{vU}]=[Q_{\mu U},Q_{vU}]^{+}[P_{\mu U},P_{vU}]$,and

$[P_{\mu U},P_{vU}][P_{\mu U},P_{vU}]^{+}=[P_{\mu U},P_{vU}][Q_{\mu U},Q_{vU}]^{+}$

(ii) $\Rightarrow$(i)

$[P_{\mu L},P_{vL}]^{+}[P_{\mu L},P_{vL}]=\left([P_{\mu L},P_{vL}]^{+}[P_{\mu L},P_{vL}]\right)^{T}=\left([Q_{\mu L},Q_{vL}]^{+}[P_{\mu L},P_{vL}]\right)^{T}$

$=\left([Q_{\mu L},Q_{vL}]^{+}[Q_{\mu L},Q_{vL}][Q_{\mu L},Q_{vL}]^{+}[P_{\mu L},P_{vL}]\right)^{T}=\left([Q_{\mu L},Q_{vL}]^{+}[P_{\mu L},P_{vL}]\right)^{T}\left([Q_{\mu L},Q_{vL}]^{+}[Q_{\mu L},Q_{vL}]\right)^{T}$

$=\left([P_{\mu L},P_{vL}]^{+}[P_{\mu L},P_{vL}]\right)^{T}[Q_{\mu L},Q_{vL}]^{+}[Q_{\mu L},Q_{vL}]=[P_{\mu L},P_{vL}]^{+}[P_{\mu L},P_{vL}][Q_{\mu L},Q_{vL}]^{+}[Q_{\mu L},Q_{vL}]$

$=[P_{\mu L},P_{vL}]^{+}[P_{\mu L},P_{vL}][P_{\mu L},P_{vL}]^{+}[Q_{\mu L},Q_{vL}]=[P_{\mu L},P_{vL}]^{+}[Q_{\mu L},Q_{vL}]$

And $[P_{\mu L},P_{vL}][P_{\mu L},P_{vL}]^{+}=[Q_{\mu L},Q_{vL}][P_{\mu L},P_{vL}]^{+}$

Similarly, $[P_{\mu U},P_{vU}]^{+}[P_{\mu U},P_{vU}]=[P_{\mu U},P_{vU}]^{+}[Q_{\mu U},Q_{vU}]$

$[P_{\mu U},P_{vU}][P_{\mu U},P_{vU}]^{+}=[Q_{\mu U},Q_{vU}][P_{\mu U},P_{vU}]^{+}$

Thus (i) holds by Theorem 2.1 (ii)

(ii)$\Rightarrow$(iii) *By* $[P_{\mu L},P_{vL}]^{+}[P_{\mu L},P_{vL}]=[Q_{\mu U},Q_{vU}]^{+}[P_{\mu L},P_{vL}],[P_{\mu L},P_{vL}]^{+}=[P_{\mu L},P_{vL}]^{+}[P_{\mu L},P_{vL}],[P_{\mu L},P_{vL}]^{+}$
$=[Q_{\mu U},Q_{vU}]^{+}[P_{\mu L},P_{vL}][P_{\mu L},P_{vL}]^{+}$

$[P_{\mu L},P_{vL}]^{+}[P_{\mu L},P_{vL}]=[P_{\mu L},P_{vL}][Q_{\mu U},Q_{vU}]^{+}\Rightarrow[P_{\mu L},P_{vL}]^{+}=[P_{\mu L},P_{vL}]^{+}[P_{\mu L},P_{vL}][Q_{\mu U},Q_{vU}]^{+}$

Similarly,

$[Q_{\mu U},Q_{vU}]^{+}[P_{\mu U},P_{vU}][P_{\mu U},P_{vU}]^{+}=[P_{\mu U},P_{vU}]^{+}=[P_{\mu U},P_{vU}]^{+}[P_{\mu U},P_{vU}][Q_{\mu U},Q_{vU}]^{+}$

(iii)$\Rightarrow$(ii) *By* $[Q_{\mu U},Q_{vU}]^{+}[P_{\mu L},P_{vL}][P_{\mu L},P_{vL}]^{+}=[P_{\mu L},P_{vL}]^{+}=[P_{\mu L},P_{vL}]^{+}[P_{\mu L},P_{vL}][Q_{\mu U},Q_{vU}]^{+}$

Implies $[P_{\mu L},P_{vL}]^{+}[P_{\mu L},P_{vL}]=[Q_{\mu L},Q_{vL}]^{+}[P_{\mu L},P_{vL}][P_{\mu L},P_{vL}]^{+}[P_{\mu L},P_{vL}]=[Q_{\mu L},Q_{vL}]^{+}[P_{\mu L},P_{vL}]$ and

$[P_{\mu L},P_{vL}]^{+}[P_{\mu L},P_{vL}]=[P_{\mu L},P_{vL}][P_{\mu L},P_{vL}]^{+}[P_{\mu L},P_{vL}][Q_{\mu L},Q_{vL}]^{+}=[P_{\mu L},P_{vL}][Q_{\mu L},Q_{vL}]^{+}$

Similarly, $[P_{\mu U},P_{vU}]^{+}[P_{\mu U},P_{vU}]=[Q_{\mu U},Q_{vU}]^{+}[P_{\mu U},P_{vU}]$, and
$[P_{\mu U},P_{vU}][P_{\mu U},P_{vU}]^{+}=[P_{\mu U},P_{vU}][Q_{\mu U},Q_{vU}]^{+}$

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Theorem 2.3 . If $[P_{\mu L}, P_{\nu L}] \overset{T}{\leq} [Q_{\mu L}, Q_{\nu L}], [P_{\mu U}, P_{\nu U}] \overset{T}{\leq} [Q_{\mu U}, Q_{\nu U}]$ then we have

(i) $[P_{\mu L}, P_{\nu L}]^T \overset{T}{\leq} [Q_{\mu L}, Q_{\nu L}]^T, [P_{\mu U}, P_{\nu U}]^T \overset{T}{\leq} [Q_{\mu U}, Q_{\nu U}]^T$

(ii) $[P_{\mu L}, P_{\nu L}]^+ \overset{T}{\leq} [Q_{\mu L}, Q_{\nu L}]^+, [P_{\mu U}, P_{\nu U}]^+ \overset{T}{\leq} [Q_{\mu U}, Q_{\nu U}]^+$

(iii) $[Q_{\mu L}, Q_{\nu L}]^T [P_{\mu L}, P_{\nu L}] \overset{T}{\leq} [Q_{\mu L}, Q_{\nu L}]^T [Q_{\mu L}, Q_{\nu L}], [P_{\mu L}, P_{\nu L}][Q_{\mu L}, Q_{\nu L}]^T \overset{T}{\leq} [Q_{\mu L}, Q_{\nu L}][Q_{\mu L}, Q_L]^T$

$[Q_{\mu U}, Q_{\nu U}]^T [P_{\mu U}, P_{\nu U}] \overset{T}{\leq} [Q_{\mu U}, Q_{\nu U}]^T [Q_{\mu U}, Q_{\nu U}]^T, [P_{\mu U}, P_{\nu U}][Q_{\mu U}, Q_{\nu U}]^T \overset{T}{\leq} [Q_{\mu U}, Q_{\nu U}][Q_{\mu U}, Q_U]^T$

(iv) $[Q_{\mu L}, Q_{\nu L}]^+ [P_{\mu L}, P_{\nu L}] \overset{T}{\leq} [Q_{\mu L}, Q_{\nu L}]^+ [Q_{\mu L}, Q_{\nu L}], [P_{\mu L}, P_{\nu L}][Q_{\mu L}, Q_{\nu L}]^+ \overset{T}{\leq} [Q_{\mu L}, Q_{\nu L}][Q_{\mu L}, Q_L]^+$

$[Q_{\mu U}, Q_{\nu U}]^+ [P_{\mu U}, P_{\nu U}] \overset{T}{\leq} [Q_{\mu U}, Q_{\nu U}]^+ [Q_{\mu U}, Q_{\nu U}], [P_{\mu U}, P_{\nu U}][Q_{\mu U}, Q_{\nu U}]^+ \overset{T}{\leq} [Q_{\mu U}, Q_{\nu U}][Q_{\mu U}, Q_U]^+$

(v) $[P_{\mu L}, P_{\nu L}]^T [P_{\mu L}, P_{\nu L}] \overset{T}{\leq} [Q_{\mu L}, Q_{\nu L}]^T [Q_{\mu L}, Q_{\nu L}], [P_{\mu L}, P_{\nu L}][P_{\mu L}, P_{\nu L}]^T \overset{T}{\leq} [Q_{\mu L}, Q_{\nu L}][Q_{\mu L}, Q_L]^T$

$[P_{\mu U}, P_{\nu U}]^T [P_{\mu U}, P_{\nu U}] \overset{T}{\leq} [Q_{\mu U}, Q_{\nu U}]^T [Q_{\mu U}, Q_{\nu U}], [P_{\mu U}, P_{\nu U}][P_{\mu U}, P_{\nu U}]^T \overset{T}{\leq} [Q_{\mu U}, Q_{\nu U}][Q_{\mu U}, Q_U]^T$

(vi) $[P_{\mu L}, P_{\nu L}]^+ [P_{\mu L}, P_{\nu L}] \overset{T}{\leq} [Q_{\mu L}, Q_{\nu L}]^+ [Q_{\mu L}, Q_{\nu L}], [P_{\mu L}, P_{\nu L}][P_{\mu L}, P_{\nu L}]^+ \overset{T}{\leq} [Q_{\mu L}, Q_{\nu L}][Q_{\mu L}, Q_L]^+$

$[P_{\mu U}, P_{\nu U}]^+ [P_{\mu U}, P_{\nu U}] \overset{T}{\leq} [Q_{\mu U}, Q_{\nu U}]^+ [Q_{\mu U}, Q_{\nu U}], [P_{\mu U}, P_{\nu U}][P_{\mu U}, P_{\nu U}]^+ \overset{T}{\leq} [Q_{\mu U}, Q_{\nu U}][Q_{\mu U}, Q_U]^+$

(vii) if $[Q_{\mu L}, Q_{\nu L}]^T [Q_{\mu L}, Q_{\nu L}]^+ = [Q_{\mu L}, Q_{\nu L}]^+ [Q_{\mu L}, Q_{\nu L}]^T$ *then* $[P_{\mu L}, P_{\nu L}]^T [P_{\mu L}, P_{\nu L}]^+ = [P_{\mu L}, P_{\nu L}]^+ [P_{\mu L}, P_{\nu L}]^T$

if $[Q_{\mu U}, Q_{\nu U}]^T [Q_{\mu U}, Q_{\nu U}]^+ = [Q_{\mu U}, Q_{\nu U}]^+ [Q_{\mu U}, Q_{\nu U}]^T$ *then* $[P_{\mu U}, P_{\nu U}]^T [P_{\mu U}, P_{\nu U}]^+$

$= [P_{\mu U}, P_{\nu U}]^+ [P_{\mu U}, P_{\nu U}]^T$

(viii) if $[Q_{\mu L}, Q_{\nu L}]^+ = [Q_{\mu L}, Q_{\nu L}]^T$ then $[P_{\mu L}, P_{\nu L}]^+ = [P_{\mu L}, P_{\nu L}]^T$

$[Q_{\mu U}, Q_{\nu U}]^+ = [Q_{\mu U}, Q_{\nu U}]^T$ then $[P_{\mu U}, P_{\nu U}]^+ = [P_{\mu U}, P_{\nu U}]^T$

(ix) if $[Q_{\mu L}, Q_{\nu L}]^2 = 0$ then $[P_{\mu L}, P_{\nu L}]^2 = < 0, 0 >$

$[Q_{\mu U}, Q_{\nu U}]^2 = 0$ then $[P_{\mu U}, P_{\nu U}]^2 = < 0, 0 >$

(x) if $[Q_{\mu L}, Q_{\nu L}] = [Q_{\mu L}, Q_{\nu L}]^2$ then $[P_{\mu L}, P_{\nu L}]^2 = [P_{\mu L}, P_{\nu L}]$

$[Q_{\mu U}, Q_{\nu U}] = [Q_{\mu U}, Q_{\nu U}]^2$ then $[P_{\mu U}, P_{\nu U}]^2 = [P_{\mu U}, P_{\nu U}]$

(xi) if $[Q_{\mu L}, Q_{\nu L}] = [Q_{\mu L}, Q_{\nu L}][Q_{\mu L}, Q_{\nu L}]^T$ then $[P_{\mu L}, P_{\nu L}] = [P_{\mu L}, P_{\nu L}][P_{\mu L}, P_{\nu L}]^T$

$[Q_{\mu U}, Q_{\nu U}] = [Q_{\mu U}, Q_{\nu U}][Q_{\mu U}, Q_{\nu U}]^T$ then $[P_{\mu U}, P_{\nu U}] = [P_{\mu U}, P_{\nu U}][P_{\mu U}, P_{\nu U}]^T$

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(xii) if $[Q_{\mu L},Q_{vL}]=[Q_{\mu L},Q_{vL}]^T=[Q_{\mu L},Q_{vL}]^3$ *and* $[P_{\mu L},P_{vL}]=[P_{\mu L},P_{vL}]^T$ then $[P_{\mu L},P_{vL}]=[P_{\mu L},P_{vL}]^3$
$[Q_{\mu U},Q_{vU}]=[Q_{\mu U},Q_{vU}]^T=[Q_{\mu U},Q_{vU}]^3$ *and* $[P_{\mu U},P_{vU}]=[P_{\mu U},P_{vU}]^T$ then $[P_{\mu U},P_{vU}]=[P_{\mu U},P_{vU}]^3$

Proof: (i) and (ii) hold clearly

$(iii)\left([Q_{\mu L},Q_{vL}]^T[P_{\mu L},P_{vL}]\right)^T[Q_{\mu L},Q_{vL}]^T[P_{\mu L},P_{vL}]=[P_{\mu L},P_{vL}]^T[Q_{\mu L},Q_{vL}][Q_{\mu L},Q_{vL}]^T[P_{\mu L},P_{vL}]$

$=[P_{\mu L},P_{vL}]^T[P_{\mu L},P_{vL}][P_{\mu L},P_{vL}]^T[P_{\mu L},P_{vL}]=[P_{\mu L},P_{vL}]^T[P_{\mu L},P_{vL}][P_{\mu L},P_{vL}]^T[Q_{\mu L},Q_{vL}]$

$=[P_{\mu L},P_{vL}]^T[P_{\mu L},P_{vL}][Q_{\mu L},Q_{vL}]^T[Q_{\mu L},Q_{vL}]=[P_{\mu L},P_{vL}]^T[Q_{\mu L},Q_{vL}][Q_{\mu L},Q_{vL}]^T[Q_{\mu L},Q_{vL}].$

Similarly,$[Q_{\mu L},Q_{vL}]^T[P_{\mu L},P_{vL}]\left([P_{\mu L},P_{vL}]^T[P_{\mu L},P_{vL}]\right)^T=[Q_{\mu L},Q_{vL}]^T[Q_{\mu L},Q_{vL}]\left([Q_{\mu L},Q_{vL}]^T[P_{\mu L},P_{vL}]\right)^T$

Thus, $[Q_{\mu L},Q_{vL}]^T[P_{\mu L},P_{vL}]\overset{T}{\leq}[Q_{\mu L},Q_{vL}]^T[Q_{\mu L},Q_{vL}]$

Similarly, we have $[P_{\mu L},P_{vL}][Q_{\mu L},Q_{vL}]^T\overset{T}{\leq}[Q_{\mu L},Q_{vL}][Q_{\mu L},Q_{vL}]^T$

Similarly,

$[Q_{\mu U},Q_{vU}]^T[P_{\mu U},P_{vU}]\overset{T}{\leq}[Q_{\mu U},Q_{vU}]^T[Q_{\mu U},Q_{vU}]^T,[P_{\mu U},P_{vU}][Q_{\mu U},Q_{vU}]^T\overset{T}{\leq}[Q_{\mu U},Q_{vU}][Q_{\mu U},Q_{U}]^T$

(iv)$\left([Q_{\mu L},Q_{vL}]^+[P_{\mu L},P_{vL}]\right)^T[Q_{\mu L},Q_{vL}]^+[P_{\mu L},P_{vL}]=\left([P_{\mu L},P_{vL}]^+[P_{\mu L},P_{vL}]\right)[P_{\mu L},P_{vL}]^+[P_{\mu L},P_{vL}]$

$=[P_{\mu L},P_{vL}]^T\left([P_{\mu L},P_{vL}]^+\right)^T[P_{\mu L},P_{vL}]^+[P_{\mu L},P_{vL}]=[P_{\mu L},P_{vL}]^T\left([P_{\mu L},P_{vL}]^+\right)^T[P_{\mu L},P_{vL}]^+[Q_{\mu L},Q_{vL}]$

$=[P_{\mu L},P_{vL}]^T\left([P_{\mu L},P_{vL}]^+\right)^T[Q_{\mu L},Q_{vL}]^+[Q_{\mu L},Q_{vL}]=[Q_{\mu L},Q_{vL}]^+[P_{\mu L},P_{vL}]\left([Q_{\mu L},Q_{vL}]^+[P_{\mu L},P_{vL}]\right)$

$=[Q_{\mu L},Q_{vL}]^+[Q_{\mu L},Q_{vL}]\left([Q_{\mu L},Q_{vL}]^+[P_{\mu L},P_{vL}]\right).$

Thus,$[Q_{\mu L},Q_{vL}]^+[P_{\mu L},P_{vL}]\overset{T}{\leq}[Q_{\mu L},Q_{vL}]^+[Q_{\mu L},Q_{vL}]$

Similarly, we have $[P_{\mu L},P_{vL}][Q_{\mu L},Q_{vL}]^+\overset{T}{\leq}[Q_{\mu L},Q_{vL}][Q_{\mu L},Q_{vL}]^T$

Similarly,

$[Q_{\mu U},Q_{vU}]^+[P_{\mu U},P_{vU}]\overset{T}{\leq}[Q_{\mu U},Q_{vU}]^+[Q_{\mu U},Q_{vU}],[P_{\mu U},P_{vU}][Q_{\mu U},Q_{vU}]^+\overset{T}{\leq}[Q_{\mu U},Q_{vU}][Q_{\mu U},Q_{U}]^+$

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(v) $\left([P_{\mu L},P_{\nu L}]^T[P_{\mu L},P_{\nu L}]\right)^T[P_{\mu L},P_{\nu L}]^T[P_{\mu L},P_{\nu L}]=[P_{\mu L},P_{\nu L}]^T[P_{\mu L},P_{\nu L}][P_{\mu L},P_{\nu L}]^T[Q_{\mu L},Q_{\nu L}]$

$=[P_{\mu L},P_{\nu L}]^T[P_{\mu L},P_{\nu L}][Q_{\mu L},Q_{\nu L}]^T[Q_{\mu L},Q_{\nu L}]=\left([P_{\mu L},P_{\nu L}]^T[P_{\mu L},P_{\nu L}]\right)^T[Q_{\mu L},Q_{\nu L}]^T[Q_{\mu L},Q_{\nu L}]$

and $[P_{\mu L},P_{\nu L}]^T[P_{\mu L},P_{\nu L}]\left([P_{\mu L},P_{\nu L}]^T[P_{\mu L},P_{\nu L}]\right)^T=[Q_{\mu L},Q_{\nu L}]^T[Q_{\mu L},Q_{\nu L}]\left([P_{\mu L},P_{\nu L}]^T[P_{\mu L},P_{\nu L}]\right)^T$.

Thus, $[P_{\mu L},P_{\nu L}]^T[P_{\mu L},P_{\nu L}]\overset{T}{\leq}[Q_{\mu L},Q_{\nu L}]^T[Q_{\mu L},Q_{\nu L}]$,

Similarly, we have $[P_{\mu L},P_{\nu L}][P_{\mu L},P_{\nu L}]^T\overset{T}{\leq}[Q_{\mu L},Q_{\nu L}][Q_{\mu L},Q_{L}]^T$

Similarly,

$[P_{\mu U},P_{\nu U}]^T[P_{\mu U},P_{\nu U}]\overset{T}{\leq}[Q_{\mu U},Q_{\nu U}]^T[Q_{\mu U},Q_{\nu U}],[P_{\mu U},P_{\nu U}][P_{\mu U},P_{\nu U}]^T\overset{T}{\leq}[Q_{\mu U},Q_{\nu U}][Q_{\mu U},Q_{U}]^T$

(vi) $\left([P_{\mu L},P_{\nu L}]^+[P_{\mu L},P_{\nu L}]\right)^T[Q_{\mu L},Q_{\nu L}]^+[Q_{\mu L},Q_{\nu L}]=[P_{\mu L},P_{\nu L}]^T\left([P_{\mu L},P_{\nu L}]^+\right)^T[Q_{\mu L},Q_{\nu L}]^+[Q_{\mu L},Q_{\nu L}]$

$=[P_{\mu L},P_{\nu L}]^T\left([P_{\mu L},P_{\nu L}]^+\right)[P_{\mu L},P_{\nu L}]^+[P_{\mu L},P_{\nu L}]=\left([P_{\mu L},P_{\nu L}]^+[P_{\mu L},P_{\nu L}]\right)^T[P_{\mu L},P_{\nu L}]^+[P_{\mu L},P_{\nu L}]$

and $[Q_{\mu L},Q_{\nu L}]^+[Q_{\mu L},Q_{\nu L}]\left([P_{\mu L},P_{\nu L}]^+[P_{\mu L},P_{\nu L}]\right)^T=[P_{\mu L},P_{\nu L}]^+[P_{\mu L},P_{\nu L}]\left([P_{\mu L},P_{\nu L}]^+[P_{\mu L},P_{\nu L}]\right)^T$.

Similarly, $[P_{\mu L},P_{\nu L}]^+[P_{\mu L},P_{\nu L}]\overset{T}{\leq}[Q_{\mu L},Q_{\nu L}]^+[Q_{\mu L},Q_{\nu L}],[P_{\mu L},P_{\nu L}][P_{\mu L},P_{\nu L}]^+\overset{T}{\leq}[Q_{\mu L},Q_{\nu L}][Q_{\mu L},Q_{L}]^+$

$[P_{\mu U},P_{\nu U}]^+[P_{\mu U},P_{\nu U}]\overset{T}{\leq}[Q_{\mu U},Q_{\nu U}]^+[Q_{\mu U},Q_{\nu U}],[P_{\mu U},P_{\nu U}][P_{\mu U},P_{\nu U}]^+\overset{T}{\leq}[Q_{\mu U},Q_{\nu U}][Q_{\mu U},Q_{U}]^+$

vii) $[P_{\mu L},P_{\nu L}]^T[P_{\mu L},P_{\nu L}]^+=[P_{\mu L},P_{\nu L}]^+[P_{\mu L},P_{\nu L}][Q_{\mu L},Q_{\nu L}]^T[Q_{\mu L},Q_{\nu L}]^+[P_{\mu L},P_{\nu L}][P_{\mu L},P_{\nu L}]^+$

$=[P_{\mu L},P_{\nu L}]^+[P_{\mu L},P_{\nu L}][Q_{\mu L},Q_{\nu L}]^+[Q_{\mu L},Q_{\nu L}]^T[P_{\mu L},P_{\nu L}][P_{\mu L},P_{\nu L}]^+=[P_{\mu L},P_{\nu L}]^+[P_{\mu L},P_{\nu L}]^T$

Similarly, if $[Q_{\mu U},Q_{\nu U}]^T[Q_{\mu U},Q_{\nu U}]^+=[Q_{\mu U},Q_{\nu U}]^+[Q_{\mu U},Q_{\nu U}]^T$ *then* $[P_{\mu U},P_{\nu U}]^T[P_{\mu U},P_{\nu U}]^+$

$=[P_{\mu U},P_{\nu U}]^+[P_{\mu U},P_{\nu U}]^T$

viii) $[P_{\mu L},P_{\nu L}]^+=[P_{\mu L},P_{\nu L}]^+[P_{\mu L},P_{\nu L}][P_{\mu L},P_{\nu L}]^+=[Q_{\mu L},Q_{\nu L}]^+[P_{\mu L},P_{\nu L}][P_{\mu L},P_{\nu L}]^+=[Q_{\mu L},Q_{\nu L}]^+[P_{\mu L},P_{\nu L}][Q_{\mu L},Q_{\nu L}]^+$

$=[Q_{\mu L},Q_{\nu L}]^+[P_{\mu L},P_{\nu L}][Q_{\mu L},Q_{\nu L}]^T=[Q_{\mu L},Q_{\nu L}]^+[P_{\mu L},P_{\nu L}][P_{\mu L},P_{\nu L}]^T=[P_{\mu L},P_{\nu L}]^T$

Similarly, $[Q_{\mu U},Q_{\nu U}]^+=[Q_{\mu U},Q_{\nu U}]^T$ then $[P_{\mu U},P_{\nu U}]^+=[P_{\mu U},P_{\nu U}]^T$

ix) $[P_{\mu L}, P_{vL}]^2 = [P_{\mu L}, P_{vL}][Q_{\mu L}, Q_{vL}]^+[Q_{\mu L}, Q_{vL}][Q_{\mu L}, Q_{vL}][Q_{\mu L}, Q_{vL}]^+[P_{\mu L}, P_{vL}]$

$= [P_{\mu L}, P_{vL}][Q_{\mu L}, Q_{vL}]^+[Q_{\mu L}, Q_{vL}]^2[Q_{\mu L}, Q_{vL}]^+[P_{\mu L}, P_{vL}] = < 0,0,0 >$

Similarly, $[Q_{\mu U}, Q_{vU}]^2 = 0$ then $[P_{\mu U}, P_{vU}]^2 = < 0,0,0 >$

x) $[P_{\mu L}, P_{vL}]^2 = [P_{\mu L}, P_{vL}][Q_{\mu L}, Q_{vL}]^+[Q_{\mu L}, Q_{vL}][Q_{\mu L}, Q_{vL}][Q_{\mu L}, Q_{vL}]^+[P_{\mu L}, P_{vL}]$

$= [P_{\mu L}, P_{vL}][Q_{\mu L}, Q_{vL}]^+[Q_{\mu L}, Q_{vL}][Q_{\mu L}, Q_{vL}]^+[P_{\mu L}, P_{vL}] = [P_{\mu L}, P_{vL}][Q_{\mu L}, Q_{vL}]^+[P_{\mu L}, P_{vL}]$

$= [P_{\mu L}, P_{vL}][P_{\mu L}, P_{vL}]^+[P_{\mu L}, P_{vL}] = [P_{\mu L}, P_{vL}]$

Similarly, $[Q_{\mu U}, Q_{vU}] = [Q_{\mu U}, Q_{vU}]^2$ then $[P_{\mu U}, P_{vU}]^2 = [P_{\mu U}, P_{vU}]$

xi) By $[Q_{\mu L}, Q_{vL}] = [Q_{\mu L}, Q_{vL}][Q_{\mu L}, Q_{vL}]^T$, *we have* $[Q_{\mu L}, Q_{vL}]^T = [Q_{\mu L}, Q_{vL}]$ and $[Q_{\mu L}, Q_{vL}]^+ = [Q_{\mu L}, Q_{vL}]$.

Then $[P_{\mu L}, P_{vL}][P_{\mu L}, P_{vL}]^T = [Q_{\mu L}, Q_{vL}][P_{\mu L}, P_{vL}]^T = [Q_{\mu L}, Q_{vL}][Q_{\mu L}, Q_{vL}]^T[P_{\mu L}, P_{vL}]$

$= [Q_{\mu L}, Q_{vL}][Q_{\mu L}, Q_{vL}][P_{\mu L}, P_{vL}]^T = [Q_{\mu L}, Q_{vL}][P_{\mu L}, P_{vL}][P_{\mu L}, P_{vL}]^T = [Q_{\mu L}, Q_{vL}]^T[P_{\mu L}, P_{vL}][P_{\mu L}, P_{vL}]^T$

$= [P_{\mu L}, P_{vL}]^T = [P_{\mu L}, P_{vL}]$

Similarly, $[Q_{\mu U}, Q_{vU}] = [Q_{\mu U}, Q_{vU}][Q_{\mu U}, Q_{vU}]^T$ then $[P_{\mu U}, P_{vU}] = [P_{\mu U}, P_{vU}][P_{\mu U}, P_{vU}]^T$

xii) $[P_{\mu L}, P_{vL}]^3 = [P_{\mu L}, P_{vL}][P_{\mu L}, P_{vL}]^T[P_{\mu L}, P_{vL}] = [Q_{\mu L}, Q_{vL}][P_{\mu L}, P_{vL}]^T[P_{\mu L}, P_{vL}][P_{\mu L}, P_{vL}]^+[Q_{\mu L}, Q_{vL}]$

$= [Q_{\mu L}, Q_{vL}][Q_{\mu L}, Q_{vL}]^T[P_{\mu L}, P_{vL}][P_{\mu L}, P_{vL}]^+[Q_{\mu L}, Q_{vL}] = [Q_{\mu L}, Q_{vL}][Q_{\mu L}, Q_{vL}]^T[Q_{\mu L}, Q_{vL}][P_{\mu L}, P_{vL}]^+[Q_{\mu L}, Q_{vL}]$

$= [Q_{\mu L}, Q_{vL}][P_{\mu L}, P_{vL}]^+[Q_{\mu L}, Q_{vL}] = [Q_{\mu L}, Q_{vL}][Q_{\mu L}, Q_{vL}]^+[P_{\mu L}, P_{vL}] = [P_{\mu L}, P_{vL}]$

Similarly,
$[Q_{\mu U}, Q_{vU}] = [Q_{\mu U}, Q_{vU}]^T = [Q_{\mu U}, Q_{vU}]^3$ *and* $[P_{\mu U}, P_{vU}] = [P_{\mu U}, P_{vU}]^T$ then $[P_{\mu U}, P_{vU}] = [P_{\mu U}, P_{vU}]^3$

III. MINUS ORDERING ON INTERVAL VALUED IFM

Definition:3.1 For $P \in (IF)^-_{m,n}$ and $Q \in (IF)_{m\times n}$ the minus ordering as $\bar{<}$ is defined as (i) $[P_{\mu L}, P_{vL}] \bar{<} [Q_{\mu L}, Q_{vL}] \Leftrightarrow [P_{\mu L}, P_{vL}]^-[P_{\mu L}, P_{vL}] = [P_{\mu L}, P_{vL}]^-[Q_{\mu L}, Q_{vL}]$ and $[P_{\mu L}, P_{vL}][P_{\mu L}, P_{vL}]^-$ $= [Q_{\mu L}, Q_{vL}][P_{\mu L}, P_{vL}]^-$ for some $[P_{\mu L}, P_{vL}]^- \in [P_{\mu L}, P_{vL}]\{1\}$

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(ii)$[P_{\mu U},P_{\nu U}] \overset{-}{<} [Q_{\mu U},Q_{\nu U}] \Leftrightarrow [P_{\mu U},P_{\nu U}]^{-}[P_{\mu U},P_{\nu U}] = [P_{\mu U},P_{\nu U}]^{-}[Q_{\mu U},Q_{\nu U}]$ and $[P_{\mu U},P_{\nu U}][P_{\mu U},P_{\nu U}]^{-} = [Q_{\mu U},Q_{\nu U}][P_{\mu U},P_{\nu U}]^{-}$ for some $[P_{\mu U},P_{\nu U}]^{-} \in [P_{\mu U},P_{\nu U}]\{1\}$

Theorem:3.1 For $P \in (IF)^{-}_{m,n}$ and $Q \in (IF)_{m\times n}$ the given conditions are equivalent

(i) $P \overset{-}{<} Q$

(ii) $[P_{\mu L},P_{\nu L}] = [P_{\mu L},P_{\nu L}][P_{\mu L},P_{\nu L}]^{-}[Q_{\mu L},Q_{\nu L}] = [Q_{\mu L},Q_{\nu L}][P_{\mu L},P_{\nu L}]^{-}[P_{\mu L},P_{\nu L}] = [Q_{\mu L},Q_{\nu L}][P_{\mu L},P_{\nu L}]^{-}[Q_{\mu L},Q_{\nu L}]$,

$[P_{\mu U},P_{\nu U}] = [P_{\mu U},P_{\nu U}][P_{\mu U},P_{\nu U}]^{-}[Q_{\mu U},Q_{\nu U}] = [Q_{\mu U},Q_{\nu U}][P_{\mu U},P_{\nu U}]^{-}[P_{\mu U},P_{\nu U}] = [Q_{\mu U},Q_{\nu U}][P_{\mu U},P_{\nu U}]^{-}[Q_{\mu U},Q_{\nu U}]$.

Proof (i) implies (ii)

$[P_{\mu L},P_{\nu L}] \overset{-}{<} [Q_{\mu L},Q_{\nu L}] \Leftrightarrow [P_{\mu L},P_{\nu L}]^{-}[P_{\mu L},P_{\nu L}] = [P_{\mu L},P_{\nu L}]^{-}[Q_{\mu L},Q_{\nu L}]$ and $[P_{\mu L},P_{\nu L}][P_{\mu L},P_{\nu L}]^{-} = [Q_{\mu L},Q_{\nu L}][P_{\mu L},P_{\nu L}]^{-}$ for some $[Q_{\mu L},Q_{\nu L}]^{-} \in [Q_{\mu L},Q_{\nu L}]\{1\}$

Now, $[P_{\mu L},P_{\nu L}] = [P_{\mu L},P_{\nu L}]\left([P_{\mu L},P_{\nu L}]^{-}[P_{\mu L},P_{\nu L}]\right) = [P_{\mu L},P_{\nu L}][P_{\mu L},P_{\nu L}]^{-}[Q_{\mu L},Q_{\nu L}]$

$[P_{\mu L},P_{\nu L}] = \left([P_{\mu L},P_{\nu L}][P_{\mu L},P_{\nu L}]^{-}\right)[P_{\mu L},P_{\nu L}] = [Q_{\mu L},Q_{\nu L}]\left([P_{\mu L},P_{\nu L}]^{-}[P_{\mu L},P_{\nu L}]\right)$

$[P_{\mu L},P_{\nu L}] = [Q_{\mu L},Q_{\nu L}]\left([P_{\mu L},P_{\nu L}]^{-}[P_{\mu L},P_{\nu L}]\right) = [Q_{\mu L},Q_{\nu L}][P_{\mu L},P_{\nu L}]^{-}[Q_{\mu L},Q_{\nu L}]$

Similarly,
$[P_{\mu U},P_{\nu U}] = [P_{\mu U},P_{\nu U}][P_{\mu U},P_{\nu U}]^{-}[Q_{\mu U},Q_{\nu U}] = [Q_{\mu U},Q_{\nu U}][P_{\mu U},P_{\nu U}]^{-}[P_{\mu U},P_{\nu U}] = [Q_{\mu U},Q_{\nu U}][P_{\mu U},P_{\nu U}]^{-}[Q_{\mu U},Q_{\nu U}]$.

(ii) implies (i) Let

Let $X = [P_{\mu L},P_{\nu L}]^{-}[P_{\mu L},P_{\nu L}][P_{\mu L},P_{\nu L}]^{-}$

$[P_{\mu L},P_{\nu L}]X[P_{\mu L},P_{\nu L}] = [P_{\mu L},P_{\nu L}]\left([P_{\mu L},P_{\nu L}]^{-}[P_{\mu L},P_{\nu L}][P_{\mu L},P_{\nu L}]^{-}\right)[P_{\mu L},P_{\nu L}]$

$= \left([P_{\mu L},P_{\nu L}][P_{\mu L},P_{\nu L}]^{-}[P_{\mu L},P_{\nu L}]\right)[P_{\mu L},P_{\nu L}]^{-}[P_{\mu L},P_{\nu L}] = [P_{\mu L},P_{\nu L}] \Rightarrow X \in [P_{\mu L},P_{\nu L}]\{1\}$

$Now, X[P_{\mu L},P_{\nu L}] = \left([P_{\mu L},P_{\nu L}]^{-}[P_{\mu L},P_{\nu L}][P_{\mu L},P_{\nu L}]^{-}\right)[P_{\mu L},P_{\nu L}][P_{\mu L},P_{\nu L}]^{-}[Q_{\mu L},Q_{\nu L}]$

$= [P_{\mu L},P_{\nu L}]^{-}\left([P_{\mu L},P_{\nu L}]^{-}[P_{\mu L},P_{\nu L}][P_{\mu L},P_{\nu L}]^{-}\right)[P_{\mu L},P_{\nu L}]^{-}[Q_{\mu L},Q_{\nu L}]$

$=\left([P_{\mu L},P_{vL}]^{-}[P_{\mu L},P_{vL}][P_{\mu L},P_{vL}]^{-}\right)[Q_{\mu L},Q_{vL}]$

$=X[Q_{\mu L},Q_{vL}]$

Similarly, $[P_{\mu L},P_{vL}]X=[Q_{\mu L},Q_{vL}]X$

Similarly, $X[P_{\mu U},P_{vU}]=X[Q_{\mu U},Q_{vU}]$ and $[P_{\mu U},P_{vU}]X=[Q_{\mu U},Q_{vU}]X$

Hence $P\,\overline{<}\,Q$ with respect to $X\in[P_{\mu U},P_{vU}]\{1\}$

Theorem:3.2 Let $P,Q\in(IF)^{-}_{m,n}$. If $P\,\overline{<}\,Q$, then $Q\{1\}\subseteq P\{1\}$.

Proof:

$P\,\overline{<}\,Q\Rightarrow[P_{\mu L},P_{vL}]=[P_{\mu L},P_{vL}][P_{\mu L},P_{vL}]^{-}[Q_{\mu L},Q_{vL}]=[Q_{\mu L},Q_{vL}][P_{\mu L},P_{vL}]^{-}[P_{\mu L},P_{vL}]$

For $[Q_{\mu L},Q_{vL}]^{-}\in[Q_{\mu L},Q_{vL}]\{1\}$

$[P_{\mu L},P_{vL}][Q_{\mu L},Q_{vL}]^{-}[P_{\mu L},P_{vL}]=\left([P_{\mu L},P_{vL}][P_{\mu L},P_{vL}]^{-}[Q_{\mu L},Q_{vL}]\right)[Q_{\mu L},Q_{vL}$

$=[P_{\mu L},P_{vL}][P_{\mu L},P_{vL}]^{-}\left([Q_{\mu L},Q_{vL}][Q_{\mu L},Q_{vL}]^{-}[Q_{\mu L},Q_{vL}]\right)[P_{\mu L},P_{vL}]^{-}[P_{\mu L},P_{vL}]$

$=\left([P_{\mu L},P_{vL}][P_{\mu L},P_{vL}]^{-}[Q_{\mu L},Q_{vL}]\right)[P_{\mu L},P_{vL}]^{-}[P_{\mu L},P_{vL}]=[P_{\mu L},P_{vL}][P_{\mu L},P_{vL}]^{-}[P_{\mu L},P_{vL}]=[P_{\mu L},P_{vL}]$

Hence, $[P_{\mu L},P_{vL}][Q_{\mu L},Q_{vL}]^{-}[P_{\mu L},P_{vL}]=[P_{\mu L},P_{vL}]$ for each $[Q_{\mu L},Q_{vL}]^{-}\in[Q_{\mu L},Q_{vL}]\{1\}$

Similarly, $[P_{\mu U},P_{vU}][Q_{\mu U},Q_{vU}]^{-}[P_{\mu U},P_{vU}]=[P_{\mu U},P_{vU}]$ for each $[Q_{\mu U},Q_{vU}]^{-}\in[Q_{\mu U},Q_{vU}]\{1\}$

Therefore, $Q\{1\}\subseteq P\{1\}$.

Theorem3.3 For $P,Q\in(IF)^{-}_{m,n}$ then the given conditions are equivalent

$(i)P\,\overline{<}\,Q$

(ii) $[P_{\mu L},P_{vL}]=[P_{\mu L},P_{vL}][Q_{\mu L},Q_{vL}]^{-}[Q_{\mu L},Q_{vL}]=[Q_{\mu L},Q_{vL}][Q_{\mu L},Q_{vL}]^{-}[P_{\mu L},P_{vL}]$

$=[P_{\mu L},P_{vL}][Q_{\mu L},Q_{vL}]^{-}[P_{\mu L},P_{vL}]$ for all $[Q_{\mu L},Q_{vL}]^{-}\in[Q_{\mu L},Q_{vL}]\{1\}$

$[P_{\mu U},P_{vU}]=[P_{\mu U},P_{vU}][Q_{\mu U},Q_{vU}]^{-}[Q_{\mu U},Q_{vU}]=[Q_{\mu U},Q_{vU}][Q_{\mu U},Q_{vU}]^{-}[P_{\mu U},P_{vU}]$

$=[P_{\mu U},P_{vU}][Q_{\mu U},Q_{vU}]^{-}[P_{\mu U},P_{vU}]$ for all $[Q_{\mu U},Q_{vU}]^{-}\in[Q_{\mu U},Q_{vU}]\{1\}$

(ii) $R\left([P_{\mu L},P_{vL}]\right)\subseteq R\left([Q_{\mu L},Q_{vL}]\right), C\left([P_{\mu L},P_{vL}]\right)\subseteq C\left([Q_{\mu L},Q_{vL}]\right)$ and $[P_{\mu L},P_{vL}][Q_{\mu L},Q_{vL}]^{-}[P_{\mu L},P_{vL}]=[P_{\mu L},P_{vL}]$.

$R\left([P_{\mu U},P_{vU}]\right)\subseteq R\left([Q_{\mu U},Q_{vU}]\right), C\left([P_{\mu U},P_{vU}]\right)\subseteq C\left([Q_{\mu U},Q_{vU}]\right)$ and $[P_{\mu U},P_{vU}][Q_{\mu U},Q_{vU}]^{-}[P_{\mu U},P_{vU}]=[P_{\mu U},P_{vU}]$.

Proof:

(i) $\Rightarrow$ (ii) $[P_{\mu L},P_{vL}]=[Q_{\mu L},Q_{vL}][P_{\mu L},P_{vL}]^{-}[Q_{\mu L},Q_{vL}]$ (By Theorem 3.1)

$=[Q_{\mu L},Q_{vL}][P_{\mu L},P_{vL}]^{-}\left([Q_{\mu L},Q_{vL}][Q_{\mu L},Q_{vL}]^{-}[Q_{\mu L},Q_{vL}]\right)$

$=\left([Q_{\mu L},Q_{vL}][P_{\mu L},P_{vL}]^{-}[Q_{\mu L},Q_{vL}]\right)[Q_{\mu L},Q_{vL}]^{-}[Q_{\mu L},Q_{vL}]$

$=[P_{\mu L},P_{vL}][Q_{\mu L},Q_{vL}]^{-}[Q_{\mu L},Q_{vL}]$ (By Theorem 3.1)

Therefore, $[P_{\mu L},P_{vL}]=[P_{\mu L},P_{vL}][Q_{\mu L},Q_{vL}]^{-}[Q_{\mu L},Q_{vL}]$ for each $[Q_{\mu U},Q_{vU}]^{-}\in[Q_{\mu U},Q_{vU}]\{1\}$

Similarly, we have $[P_{\mu L},P_{vL}]=[Q_{\mu L},Q_{vL}][Q_{\mu L},Q_{vL}]^{-}[P_{\mu L},P_{vL}]$ for each $[Q_{\mu U},Q_{vU}]^{-}\in[Q_{\mu U},Q_{vU}]\{1\}$

Also $[P_{\mu L},P_{vL}]=[P_{\mu L},P_{vL}][Q_{\mu L},Q_{vL}]^{-}[P_{\mu L},P_{vL}]$ (By Theorem 3.2)

Similarly, $[P_{\mu U},P_{vU}]=[P_{\mu U},P_{vU}][Q_{\mu U},Q_{vU}]^{-}[Q_{\mu U},Q_{vU}]=[Q_{\mu U},Q_{vU}][Q_{\mu U},Q_{vU}]^{-}[P_{\mu U},P_{vU}]$

(ii) $\Rightarrow$ (iii): Immidiate consequence of Theorem 3.1

(iii) $\Rightarrow$ (i) Let $X=[Q_{\mu L},Q_{vL}]^{-}[P_{\mu L},P_{vL}][Q_{\mu L},Q_{vL}]^{-}$

$[P_{\mu L},P_{vL}]X[P_{\mu L},P_{vL}]=[P_{\mu L},P_{vL}]\left([Q_{\mu L},Q_{vL}]^{-}[P_{\mu L},P_{vL}][Q_{\mu L},Q_{vL}]^{-}\right)[P_{\mu L},P_{vL}]$

$=\left([P_{\mu L},P_{vL}][Q_{\mu L},Q_{vL}]^{-}[P_{\mu L},P_{vL}]\right)[Q_{\mu L},Q_{vL}]^{-}[P_{\mu L},P_{vL}]=[P_{\mu L},P_{vL}][Q_{\mu L},Q_{vL}]^{-}[P_{\mu L},P_{vL}]$

$=[P_{\mu L},P_{vL}]\Rightarrow X\in[P_{\mu L},P_{vL}]\{1\}$

Now, $[P_{\mu L},P_{vL}]X=[P_{\mu L},P_{vL}]\left([Q_{\mu L},Q_{vL}]^{-}[P_{\mu L},P_{vL}][Q_{\mu L},Q_{vL}]^{-}\right)$

$=[Q_{\mu L},Q_{vL}]^{-}[Q_{\mu L},Q_{vL}][P_{\mu L},P_{vL}]\left([Q_{\mu L},Q_{vL}]^{-}[P_{\mu L},P_{vL}][Q_{\mu L},Q_{vL}]^{-}\right)$ (By Theorem 3.1)

$=[Q_{\mu L},Q_{vL}][Q_{\mu L},Q_{vL}]^{-}\left([P_{\mu L},P_{vL}][Q_{\mu L},Q_{vL}]^{-}[P_{\mu L},P_{vL}]\right) \quad =[Q_{\mu L},Q_{vL}]X$

Similarly, $X[P_{\mu L},P_{vL}]=X[Q_{\mu L},Q_{vL}]$

Similarly, $[P_{\mu U},P_{vU}]X= \;=[Q_{\mu U},Q_{vU}]X$

Similarly, $X[P_{\mu U}, P_{vU}] = = X[Q_{\mu U}, Q_{vU}]$ follow by Theorem (3.1) and

$[P_{\mu U}, P_{vU}][Q_{\mu U}, Q_{vU}]^{-}[P_{\mu U}, P_{vU}] = [P_{\mu U}, P_{vU}]$.

Hence, $P \bar{<} Q$ with respect to $X \in P\{1\}$

Theorem3.4 For $(IF)^{-}_{m,n}$ the minus ordering $\bar{<}$ is a partial ordering.

Theorem 3.5 For $P \in (IF)^{-}_{m,n}$ and $Q \in (IF)_{m \times n}$ with $P \bar{<} Q$

(i) If $[P_{\mu L}, P_{vL}] = [P_{\mu L}, P_{vL}]^2$, then $[Q_{\mu L}, Q_{vL}] = [Q_{\mu L}, Q_{vL}]^2$

$[P_{\mu U}, P_{vU}] = [P_{\mu U}, P_{vU}]^2$, then $[Q_{\mu U}, Q_{vU}] = [Q_{\mu U}, Q_{vU}]^2$

(ii) If $[P_{\mu L}, P_{vL}]^2 = 0$, then $[Q_{\mu L}, Q_{vL}]^2 = 0$

$[P_{\mu U}, P_{vU}]^2 = 0$, then $[Q_{\mu U}, Q_{vU}]^2 = 0$

Proof: $[P_{\mu L}, P_{vL}]^2 = [P_{\mu L}, P_{vL}][P_{\mu L}, P_{vL}]$

$= \left([P_{\mu L}, P_{vL}][P_{\mu L}, P_{vL}]^{-}[Q_{\mu L}, Q_{vL}]\right)\left([Q_{\mu L}, Q_{vL}][P_{\mu L}, P_{vL}]^{-}[P_{\mu L}, P_{vL}]\right)$ (By Theorem 3.1)

$= [P_{\mu L}, P_{vL}][P_{\mu L}, P_{vL}]^{-}[Q_{\mu L}, Q_{vL}]^2[P_{\mu L}, P_{vL}]^{-}[P_{\mu L}, P_{vL}]$

$= \left([P_{\mu L}, P_{vL}][P_{\mu L}, P_{vL}]^{-}[Q_{\mu L}, Q_{vL}]\right)[P_{\mu L}, P_{vL}]^{-}[P_{\mu L}, P_{vL}]$ (By $[Q_{\mu L}, Q_{vL}] = [Q_{\mu L}, Q_{vL}]^2$)

$= [P_{\mu L}, P_{vL}][P_{\mu L}, P_{vL}]^{-}[P_{\mu L}, P_{vL}] = [P_{\mu L}, P_{vL}]$

$[P_{\mu L}, P_{vL}]^2 = [P_{\mu L}, P_{vL}]$

Similarly, $[P_{\mu U}, P_{vU}]^2 = [P_{\mu U}, P_{vU}]$

(ii) $[P_{\mu L}, P_{vL}]^2 = [P_{\mu L}, P_{vL}][P_{\mu L}, P_{vL}] = \left([P_{\mu L}, P_{vL}][P_{\mu L}, P_{vL}]^{-}[Q_{\mu L}, Q_{vL}]\right)\left([Q_{\mu L}, Q_{vL}][P_{\mu L}, P_{vL}]^{-}[P_{\mu L}, P_{vL}]\right)$

$= [P_{\mu L}, P_{vL}][P_{\mu L}, P_{vL}]^{-}[Q_{\mu L}, Q_{vL}]^2[P_{\mu L}, P_{vL}]^{-}[P_{\mu L}, P_{vL}] = 0$

$[P_{\mu U}, P_{vU}]^2 = 0$

Conclusion:

We derive some equivalent conditions for each ordering by using generalized inverses. Additionally, we demonstrate that these orderings are the same for a particular class of IVIFM. We study the minus ordering for IFM as an analogue of minus ordering for complex matrix studied and as a generalization of T- ordering for IVIFM introduced . We prove that under certain conditions Tilde reduces to the T-ordering on IFM. We establish a set of necessary condition for IVIFM with specified row and column spaces to be under sharp order. The concept of Tilde orderings for IVIFM as an analogue of complex matrices. We show that these ordering preserve its Moore-penrose inverse property. By using various generalized inverses the new type of minus orderings are discussed. Finally, we show that these ordering are identical for certain class of IVIFM. In future, we shall prove some related properties of g-inverse of Tilde and Minus Partial Ordering on Intuitionistic Fuzzy Matrices.

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