Application of the Preliminary Integration Method for Numerical Simulation of Two-Phase Hydrodynamic Flows

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Abstract: Numerical modeling of heterogeneous (multiphase) mixtures or, in particular, gas suspensions (mixtures of gases with solid particles) is one of the important directions in the study of the dynamics of complex hydrodynamic systems. The study of mathematical models of multiphase system is relevant, primarily from the point of view of applications. Such flows are often encountered in problems of aerodynamics, chemical technology, combustion physics, and others. The study of the influence of suspended particles on the hydrodynamic stability of two-phase flows requires, firstly, a thorough analysis of mathematical models of two-phase media and, secondly, the development of an effective, reliable mathematical apparatus for numerical simulation of hydrodynamic stability equations. In this article, for numerical modeling of the equations of stability of two-phase flows, it is proposed to apply the method of preliminary integration, where Chebyshev polynomials of the first kind is used as basis functions. The essence of the method lies in the fact that before solving the problem, the series in Chebyshev polynomial is pre-integrated the required number of times. Then the integrated series are used in a differential problems and a generalized algebraic eigenvalue problem with complex matrices and vectors is obtained, which is solved by standard methods.

Keywords: two-phase flows, main flow, Reynolds number, wave number, relaxation time and mass concentration of particles.

1. Introduction

Many hydrodynamic systems for both single-phase and two-phase flows exhibit nonlinear behavior with respect to the small disturbances. This phenomenon requires the development of high-precision effective methods for the study of hydrodynamic flows. However, the solution of hydrodynamic equations encounters serious difficulties. They are mainly related to the presence of a small parameter at the highest derivative and, as a consequence, the appearance of areas of strong spatial heterogeneity in the solution. In mathematical modeling of the problem of hydrodynamic stability for two-phase flows, the situation is aggravated, the decrease in the efficiency of numerical methods becomes practically unacceptable. In this case, the difficulties are defined as follows. Firstly, the stability equations contain a small parameter at the highest derivative, so there are significant difficulties in obtaining the solutions with a given degree of accuracy. Secondly, the problem under consideration becomes multiparametric, it is characterized by four parameters and the construction of parametric dependencies leads to a sharp increase in the amount of computational work. Finally, with an increase in the number of equations, since each phase has equations of motion, the order of the generalized algebraic eigenvalue problem solved with complex matrices increases quadratically, which greatly reduces the efficiency of numerical calculations when finding eigenvalues numerically.

In recent years, polynomial approximation has been effectively used to solve problems in the field of fluid mechanics. The Chebyshev collocation method is used for numerical modeling of one-dimensional singular and nonlinear boundary value problems [1]. In this paper, it is shown that the Chebyshev collocation method has an exponential (geometric) convergence rate.

Applications of Chebyshev polynomials for the accuracy of calculating high-order eigenvalues for singular Schrodinger equations are described [2]. In this paper, special attention is paid to problems with almost multiple eigenvalues, as well as problems with a mixed spectrum, the separation of "good" and "bad" eigenvalues.

Many mechanical systems exhibit nonlinear behavior under nonstationary random excitations [3]. However, finding exact solutions to nonlinear problems is very difficult. Therefore, to find approximate solutions to such problems, equivalent linearization methods based on orthogonal functions in time domain are often used for nonlinear systems subject to nonstationary random influences. The results obtained in this work shows that the proposed method has advantage of predicting the response of a nonlinear system with high accuracy and computational efficiency, in addition to that it is applicable to any general term of non-stationary random excitation.

Comparison of computational and experimental studies of hydrodynamics between scaled and initial wing sections in single-phase and two-phase flows and assesses the effectiveness of the proposed method was presented [4]. For computational hydrodynamic modeling, a pressure-based method with a second-order counterwind scheme for spatial discretization and a Spalart-Allmaras turbulence model were used. As a result, it is concluded that using the proposed method, a quick and low-cost study is possible to assess the aerodynamic characteristics of objects with high accuracy.

The use of the spectral and spectral-grid methods for solving the problem of hydrodynamic stability of two-phase flows is described[5-7]. In these articles, Chebyshev polynomials of the first kind were used as basis functions. Chebyshev polynomials are widely used in the construction of optimal iterative methods, in solving various problems of hydrodynamic stability. Among orthogonal polynomials, only Chebyshev polynomials have a minimax property, which allows more uniform distribution of errors over the entire integration interval. Therefore, all those methods where Chebyshev polynomials are used are optimal or almost optimal. The range of applications of Chebyshev polynomials is increasingly expanding. Let us give a brief overview of numerical methods for the application of Chebyshev polynomials.

A modified Chebyshev orthogonal polynomial and their new derivative and integration operational matrices and modified Chebyshev polynomials of the first kind with explicit formulas was introduced [8]. A direct computational method was proposed for solving a special class of optimal control problems, called the quadratic optimal control problem, using the resulting operational matrices. Modified Chebyshev polynomials of the second kind for solving higher-order differential equations was studied [9]. This method will reduce the original differential equation problem to solving algebraic equations using a simple computational algorithm. In [10], it is devoted to the establishment of new expressions expressing the derivative of any order of orthogonal polynomials, namely, Chebyshev polynomials of the sixth kind in terms of the

Chebyshev polynomials themselves. Ageneral form of polynomial differential equations with fractional delay of variable order was presented [11].

The introduced formulas have variable order polynomials and integral order derivatives, as well as for all terms with a delay or with a normal argument. The polynomial fractional differential equations of variable order were considered as one of the tools for accurately illustrating the behavior of real transient phenomena [12]. The interval scoring for multi-body systems that can provide accurate dynamic prediction and robust reliability design [13]. In order to obtain a reliable numerical model, the uncertainty propagation of multibody systems should take into account the uncertain parameters with several intervals. This article proposes a new method that combines the two-dimensional Chebyshev polynomial and the local expansion of the mean. Subsequently, low-order Chebyshev polynomials can be used to build surrogate models for multivariate amplitude, phase, and residual responses.

In [14], a preliminary integration method was proposed for solving the equations of stability of a single-phase flow. In this article, the pre-integration method is generalized to solve the problem of hydrodynamic stability of two-phase flows.

2. Materials and Methods

The hydrodynamic stability of plane-parallel flows with suspended particles is studied with respect to small two-dimensional perturbations in the form of traveling waves:

$$\Psi(x,\eta,t) = \psi(\eta)\,\mathrm{e}^{ik(x-\lambda t)}$$

$$\Phi(x,\eta,t) = \varphi(\eta)\,\mathrm{e}^{ik(x-\lambda t)}$$

Here Ψ, Φ are stream functions, and $\psi(\eta), \varphi(\eta)$ are the amplitudes of the stream function, for pure gas and particles, respectively, x and η are longitudinal and transverse coordinates, t is time, k is the wave number, $\lambda = \lambda_r + i\lambda_i$ is an unknown constant to be determined. If $\lambda_i > 0$, then the flow under consideration is unstable, if $\lambda_i < 0$ - stable. If $\lambda_i = 0$, then the oscillations are neutrally stable.

For the amplitude of the stream function, the stability equations for two-phase flows have the form:

$$P\psi + M\varphi = 0, \tag{1}$$

$$S\psi + K\varphi = 0 \tag{2}$$

where the differential operators P, M, S, K are defined as:

$$P = \frac{1}{ik\,\mathrm{Re}}\hat{D} - (\mathrm{U}(\eta) - \lambda - \frac{if}{k\tau})D + \frac{d^2U}{d\eta^2},$$

$$M = \frac{f}{ik\tau}D,\ S = \frac{1}{ik\tau}D,$$

$$K = -(\mathrm{U}(\eta) - \lambda - \frac{i}{k\tau})D + \frac{d^2U}{d\eta^2},\ D = \frac{d^2}{d\eta^2} - \mathrm{k}^2,$$

$$\hat{D} = \frac{d^4}{d\eta^4} - 2k^2\frac{d^2}{d\eta^2} + k^4.$$

The dimensionless parameters are defined as follows:

$$\begin{gathered}\mathrm{Re} = U_m\delta/\nu,\ \delta = \sqrt{2\nu L/U_m},\ k\delta = (2\pi/\ell)\delta = G\,\mathrm{Re},\\ G = 2\pi/R_1,\ R_1 = \ell\,\mathrm{U}_m/\nu,\ \tau = T/\mathrm{Re},\\ T = (2/9)\mathrm{s}_1 R_2^2,\ R_2 = \mathrm{aU}_m/\nu,\ s_1 = \rho_2/\rho_1,\end{gathered} \tag{3}$$

where ν is the kinematic viscosity, U_m is the characteristic velocity of the main flow outside the boundary layer, L is the characteristic length along the plate, δ is the thickness of the boundary layer, Re is the Reynolds number, k is the wave number, ℓ is the disturbance wavelength, R_1 is the Reynolds number along wavelength, s_1- density ratio (particles to gas), a- particle radius, T - dimensionless parameter - particle relaxation time, f - mass concentration of particles.

The boundary conditions for perturbations in the boundary layer have the form:

$$\psi(\eta)=\frac{d\psi}{d\eta}=0,\ \psi(\eta)=0\ \text{at}\ \eta=0 \tag{4}$$

$$\psi(\eta),\frac{d\psi}{d\eta}\to 0,\ \psi(\eta)=0\ \text{at}\ \eta\to\infty \tag{5}$$

Eqn. (4) expresses the adhesion requirements for pure gas and impermeability for gas and solid particles, and Eqn. (5) defines the conditions for damping perturbations away from the wall.

For numerical modeling of Eqns. (1)-(2) and Eqns. (4)-(5), it is necessary to limit the integration interval to a large but finite value, which is denoted by η_ℓ. Thus, the system of equations given in Eqns. (1)-(2) with boundary conditions in the Eqns. (4)-(5) are considered in the interval $[\eta_0,\eta_\ell]=[0,\eta_\ell]$. To represent approximate solutions ψ,φ as series in Chebyshev polynomials of the first kind, we map the integration interval $[\eta_0,\eta_\ell]$ onto the interval $[-1,+1]$ using the following change of independent variable:

$$\eta=\frac{m}{2}+\frac{\bar{\ell}}{2}y,\ y\in[-1,+1],$$
$$m=\eta_0+\eta_\ell,\ \bar{\ell}=\eta_\ell-\eta_0$$

Thus, we get the following boundary value problem:

$$P\psi+M\varphi=0, \tag{6}$$

$$S\psi+K\varphi=0, \tag{7}$$

$$\psi(\mathrm{y})=\varphi(y)=\frac{d\psi}{dy}=0\ \text{at}\ y=-1, \tag{8}$$

$$\psi(\mathrm{y})=\varphi(y)=\frac{d\psi}{dy}=0\ \text{at}\ y=1 \tag{9}$$

where the operators P,M,S,K take the form:

$$P=\frac{1}{ik\,\mathrm{Re}}\hat{\bar{D}}-\left(U(y)-\lambda-\frac{if}{GT}\right)\bar{D}+\frac{d^2U}{dy^2},$$
$$M=\frac{f}{iGT}\bar{D},\ S=\frac{1}{iGT}\bar{D},$$
$$K=-(\mathrm{U(y)}-\lambda-\frac{i}{GT})\bar{D}+\frac{d^2U}{dy^2},$$
$$\bar{D}=\frac{d^2}{dy^2}-\bar{\mathrm{k}}^2,\ \bar{k}=\frac{\ell}{2}k,\ \bar{\mathrm{R}}\mathrm{e}=\frac{\ell}{2}\mathrm{Re}$$

$$\hat{\bar{D}} = \frac{d^2}{dy^2} - 2\bar{k}\frac{d^2}{dy^2} + \bar{k}^4$$

3. Computational Algorithm

In the method of preliminary integration, the highest derivatives in differential equations given in (6) and (7) are represented as the following series:

$$\psi^{(4)}(y) = \sum_{n=0}^{\bar{N}} a_n T_n(y)$$

$$\varphi^{(2)}(y) = \sum_{n=0}^{\bar{N}} d_n T_n(y) \qquad (10)$$

$$\mathrm{U}(y_p^{(j)}) = \sum_{n=0}^{\bar{N}} b_n T_n(y_p^{(j)})$$

where

$$y_p^{(j)} = \cos(\pi p / \bar{N}),\ p = 0,1,...,\bar{N}$$

nodes of Chebyshev polynomials, $T_n(y)$ Chebyshev polynomials of the first kind, $\bar{N}$ - the number of polynomials used to approximate the solution. The expansion coefficients b_n for the main flow in (10) are determined by the following inverse transformation:

$$b_n = \frac{2}{\bar{N}c_n}\sum_{p=0}^{\bar{N}}\frac{1}{c_p}\mathrm{U}(y_p^{(j)})T_n(y_p^{(j)}),\ n = 0,1,...,\bar{N}$$

$$c_0 = c_{\bar{N}} = 2,\ c_m = 1\ \text{at}\ m \neq 0, \bar{N}.$$

For Chebyshev polynomials, the following recursive formulas are valid:

$$2T = \frac{c_n}{n+1}T_{n+1}^{(1)} - \frac{d_{n-2}}{n-1}T_{n-1}^{(1)},\ 2yT_n = c_n T_{n+1} + d_{n-1}\mathrm{T}_{n-1}, \qquad (11)$$

where $c_n = d_n = 0, \text{if}\ n < 0, c_0 = 2, d_0 = 1$,
$c_n = d_n = 1, \text{if}\ n > 0$.

The main advantage of the pre-integration method is that before solving the boundary value problem (6)-(9), series (10) are pre-integrated. Thus, all lower derivatives and desired solutions are found:

$$\psi^{(3)}, \psi^{(2)}, \psi^{(1)}, \psi, \varphi^{(1)}, \varphi. \qquad (12)$$

It is known that integration operations improve the smoothness of the series (10). At the same time, in spectral methods, series in Chebyshev polynomials are differentiated.
Using preliminary integration, the functions given in Eqn. (12) are determined as follows:

$$\psi^{(3)} = \sum_{j=0}^{\bar{N}+1}\sum_{i=0}^{\bar{N}} f_{ji}^{(3)} a_i T_j + C_1 T_0,$$

$$\psi^{(2)} = \sum_{j=0}^{\bar{N}+2}\sum_{i=0}^{\bar{N}} f_{ji}^{(2)} a_i T_j + C_1 T_1 + C_2 T_0,$$

$$\psi^{(1)} = \sum_{j=0}^{\bar{N}+3}\sum_{i=0}^{\bar{N}} f_{ji}^{(1)} a_i T_j + \frac{C_1}{4}T_2 + C_1 T_1 + C_2 T_0 \qquad (13)$$

$$\psi=\sum_{j=0}^{\bar{N}+4}\sum_{i=0}^{\bar{N}} f_{ji}^{(0)}a_iT_j+\frac{C_1}{24}T_1+\frac{C_2}{4}T_2+(C_3-\frac{C_1}{8})T_1+C_4T_0,$$
$$\varphi^{(1)}=\sum_{j=0}^{\bar{N}+1}\sum_{i=0}^{\bar{N}}\varphi_{ji}^{(1)}d_iT_j+D_1T_0,$$
$$\varphi=\sum_{j=0}^{\bar{N}+2}\sum_{i=0}^{\bar{N}}\varphi_{ji}^{(0)}d_iT_j+D_1T_1+D_2T_0$$

The constants $C_1, C_2, C_3, C_4, D_1, D_2$ are determined from the satisfaction of boundary conditions (Eqns. (8) and (9)) are:

$$C_1=\sum_{i=0}^{\bar{N}}\frac{3}{2}\Big[\big(\sigma_i^{(0)}-\delta_i^{(0)}\big)-\big(\sigma_i^{(1)}+\delta_i^{(1)}\big)\Big]a_i,$$
$$C_2=\sum_{i=0}^{\bar{N}}\left[-\frac{1}{2}\big(\sigma_i^{(1)}-\delta_i^{(1)}\big)\right]a_i,$$
$$C_3=\sum_{i=0}^{\bar{N}}\left(-\frac{1}{2}\Big[3\big(\sigma_i^{(0)}-\delta_i^{(0)}\big)+\big(\sigma_i^{(3)}+\delta_i^{(3)}\big)\Big]\right)a_i, \tag{14}$$
$$C_4=\sum_{i=0}^{\bar{N}}\left[\frac{1}{2}\left(-\big(\sigma_i^{(0)}+\delta_i^{(0)}\big)+\frac{1}{4}\big(\sigma_i^{(4)}-\delta_i^{(4)}\big)\right)\right]a_i,$$
$$D_1=\sum_{i=0}^{\bar{N}}\frac{3}{2}\Big[\big(\varepsilon_i^{(0)}-\Delta_i^{(0)}\big)-\big(\varepsilon_i^{(1)}+\Delta_i^{(1)}\big)\Big]d_i,$$
$$D_2=\sum_{i=0}^{\bar{N}}\left[-\frac{1}{2}\big(\varepsilon_i^{(1)}-\Delta_i^{(1)}\big)\right]d_i$$

where

$$\sigma_i^{(\beta)}=\sum_{j=0}^{\bar{N}+4-\beta} f_{ji}^{(\beta)},\ \delta_i^{(\beta)}=\sum_{j=0}^{\bar{N}+4-\beta}(-1)^i f_{ji}^{(\beta)}, \beta=0,1,2,3$$
$$\varepsilon_i^{(k)}=\sum_{j=0}^{\bar{N}+2-k}\varphi_{ji}^{(k)},\ \Delta_i^{(k)}=\sum_{j=0}^{\bar{N}+2-k}(-1)^j\varphi_{ji}^{(k)}, k=0,1$$

The Coefficients $f_{ji}^{(\beta)}, \beta=0,1,2,3$ and $\varphi_{ji}^{(k)}, k=0,1$ are described by integrating the series in Eqn. (10) and using recurrence formulas in Eqn.(11), one can defined as

$$f_{ji}^{(3)}=\delta_{j,i+1}\beta_i^{(3)}+\delta_{j,i-1}\gamma_i^{(3)}$$
$$f_{ji}^{(2)}=\delta_{j,i+2}\beta_i^{(2)}+\delta_{j,i}\gamma_i^{(2)}+\delta_{j,i-2}\delta_i^{(2)}$$
$$f_{ji}^{(1)}=\delta_{j,i+3}\beta_i^{(1)}+\delta_{j,i+1}\gamma_i^{(1)}+\delta_{j,i-1}\delta_i^{(1)}+\delta_{j,i-3}\varepsilon_i^{(1)},$$
$$f_{ji}^{(0)}=\delta_{j,i+4}\beta_i^{(0)}+\delta_{j,i+2}\gamma_i^{(0)}+\delta_{j,i}\delta_i^{(0)}+\delta_{j,i-2}\varepsilon_i^{(0)}+\delta_{j,i-1}\eta_i^{(0)}, \tag{15}$$
$$\varphi_{ji}^{(1)}=\delta_{j,i+1}\bar{\beta}_i^{(1)}+\delta_{j,i-1}\bar{\gamma}_i^{(1)},$$
$$\varphi_{ji}^{(0)}=\delta_{j,i+1}\bar{\beta}_i^{(0)}+\delta_{j,i}\bar{\gamma}_i^{(0)}+\delta_{j,i-2}\bar{\delta}_i^{(0)}.$$

Here $\delta_{j,i}$ is the Kronecker symbol and non-zero coefficients $\beta_i, \gamma_i, \delta_i, \varepsilon_i, \eta_i$ at $i\geq 0$ are defined as follows:

$$\beta_i^{(3)}=\frac{c_i}{2(i+1)},\ \beta_i^{(\beta-1)}=\frac{\beta_i^{(\beta)}}{2(i+5-\beta)},\ \beta=3,2,1,\ i\geq 0$$

$$\gamma_i^{(3)} = \frac{-1}{2(i-1)},\ i \ge 2,\ \gamma_i^{(\beta-1)} = \frac{-\beta_i^{(\beta)} + \gamma_i^{(\beta)}}{2(i+3-\beta)},\ \ \beta = 3,2,1 \ \ ,\ i \ge \beta - 2$$

$$\delta_i^{(3)} = \frac{-\gamma_i^{(3)}}{2(i-2)},\ i \ge 3,\ \delta_i^{(\beta-1)} = \frac{-\gamma_i^{(\beta)} + \delta_i^{(\beta)}}{2(i+1-\beta)},\ \ \beta = 2,1 \ \ ,\ i \ge \beta$$

$$\varepsilon_i^{(1)} = \frac{-\delta_i^{(2)}}{2(i-3)},\ i \ge 4,\ \varepsilon_i^{(0)} = \frac{-\delta_i^{(1)} + \varepsilon_i^{(1)}}{2(i-2)},\ \ i \ge 3$$

$$\eta_i^{(0)} = \frac{-\varepsilon_i^{(1)}}{2(i-4)},\ i \ge 5 \tag{16}$$

$$\bar{\beta}_i^{(1)} = \frac{c_i}{2(i+1)},\quad \bar{\beta}_i^{(0)} = \frac{\bar{\beta}_i^{(1)}}{2(i+2)},\quad i \ge 0$$

$$\bar{\gamma}_i^{(1)} = \frac{-1}{2(i-1)},\ \bar{\gamma}_i^{(0)} = \frac{-\bar{\beta}_i^{(1)} + \bar{\gamma}_i^{(1)}}{2(i+1)}$$

$$\bar{\varepsilon}_i^{(1)} = \frac{-\bar{\gamma}_i^{(1)}}{2i},\ \ i \ge 1,\ \bar{\varepsilon}_i^{(0)} = \frac{-\bar{\gamma}_i^{(0)} + \bar{\delta}_i^{(0)}}{2i}$$

Substituting the values of the constants from the Eqn. (14) into Eqn. (13) we have the following generalizing formulas

$$\psi^{(\beta)} = \sum_{j=0}^{\bar{N}+4-\beta} \sum_{i=0}^{\bar{N}} g_{ji}^{(\beta)} a_i T_j,\ \beta = 0,1,2,3,\ \varphi^{(\beta)} = \sum_{j=0}^{\bar{N}+2-\beta} \sum_{i=0}^{\bar{N}} \bar{g}_{ji}^{(\beta)} d_i T_j,\ \beta = 0,1 \tag{17}$$

In which,

$$g_{ji}^{(3)} = f_{ji}^{(3)} + \delta_{j,0} \frac{3}{2}\Big[\big(\sigma_i^{(0)} - \delta_i^{(0)}\big) - \big(\sigma_i^{(1)} + \delta_i^{(1)}\big)\Big],$$

$$\bar{g}_{ji}^{(1)} = \varphi_{ji}^{(1)} + \delta_{j,0} \frac{3}{2}\Big[\big(\varepsilon_i^{(0)} - \Delta_i^{(0)}\big) - \big(\varepsilon_i^{(1)} + \Delta_i^{(1)}\big)\Big].$$

The coefficients $g_{ji}^{(\beta)}\ (\beta = 0,1,2)$ and $\bar{g}_{ji}^{(0)}$ are also determined.

The coefficients satisfy the boundary conditions and their derivatives. It can be represented the main flow $U(y)$ as the following row

$$U^{(\beta)}(y) = \sum_{n=0}^{N_{b-1+\beta}} b_n^{(\beta)} T_n(y),\ \beta = 0,1,2, \tag{18}$$

and different values of parameter β correspond to the different velocity profiles of the main incoming flow. For example, at $\beta = 0$ the main flow is the Poiseuille flow:

$$U(y) = 1 - \mathrm{y}^2.$$

Thus, substituting Eqns. (17) and (18) into Eqns. (6) and (7) we get a generalized algebraic eigenvalue problem as

$$\sum_{k=0}^{\bar{N}} \left\{ \frac{1}{\bar{i}\bar{k}\bar{Re}} \left(\delta_{ik} - 2\bar{k}^2 g_{ik}^{(2)} + k^4 g_{ik}^{(0)}\right) - \sum_{i=0}^{N_{b+2}} \sum_{j=0}^{\bar{N}+2} \beta_{i\ell_j} b_i^{(0)} g_{ik}^{(2)} + \bar{k}^2 \sum_{i=0}^{N_{b+2}} \sum_{j=0}^{\bar{N}+4} \beta_{i\ell_j} b_i^{(0)} g_{ik}^{(0)} + \sum_{i=0}^{N_b} \sum_{j=0}^{\bar{N}+4} \beta_{i\ell_j} b_i^{(2)} g_{ik}^{(0)} + \frac{\bar{i}\bar{f}}{GT}\left[g_{ik}^{(2)} - \bar{k}^2 g_{ik}^{(0)}\right] \right\} a_k +$$
$$\frac{f}{iGT} \sum_{k=0}^{\bar{N}} \left[\bar{g}_{ik}^{(2)} - \bar{k}^2 \bar{g}_{ik}^{(0)}\right] d_i = -\lambda \sum_{k=0}^{\bar{N}} \left[g_{ik}^{(2)} - \bar{k}^2 g_{ik}^{(0)}\right] a_k,\ i = 0,1,2,\dots,\bar{N} \tag{19}$$

$$\sum_{k=0}^{\bar{N}} \left[\frac{1}{\bar{i}GT} \left(\bar{g}_{ik}^{(2)} - \bar{k}^2 g_{ik}^{(0)}\right) a_k \right] + \frac{\bar{i}}{GT} \sum_{k=0}^{\bar{N}} \left\{ \left[\bar{g}_{ik}^{(2)} - \bar{k}^2 g_{ik}^{(0)}\right] d_i - \sum_{i=0}^{N_{b+2}} \sum_{j=0}^{\bar{N}+2} \beta_{i\ell_j} b_i^{(0)} \bar{g}_{ik}^{(2)} + \bar{k}^2 \sum_{i=0}^{N_{b+2}} \sum_{j=0}^{\bar{N}+4} \beta_{i\ell_j} b_i^{(0)} \bar{g}_{ik}^{(0)} + \sum_{i=0}^{N_b} \sum_{j=0}^{\bar{N}+4} \beta_{i\ell_j} \bar{g}_{ik}^{(2)} \right\} d_k =$$

$$= -\lambda \sum_{k=0}^{\bar{N}} \left[\bar{g}_{ik}^{(2)} - \bar{k}^2 \bar{g}_{ik}^{(0)} \right] d_k \,,\, i = 0,1,2,...,\bar{N} \tag{20}$$

where *i* denotes the imaginary unit,

$$\beta_{i\ell_j} = \frac{2}{\pi c_{i-1}} \sum_{-1}^{+1} \left(1 - y^2\right)^{-1/2} T_i T_\ell T_j dy = \frac{1}{2} \left[\delta_{i,\ell+j} + \frac{1}{c_\ell} \left(\delta_{i,\ell-j} + \delta_{i,j-\ell} \right) \right]$$

Here the following identity is used

$$2T_j T_\ell = T_{j+\ell} + T_{|j-\ell|} \ .$$

The system of equations given in (19) and (20) forms an algebraic eigenvalue problem of the following form

$$\left(A - \lambda B\right) X = 0 \tag{21}$$

where A and B are known as complex square matrices, λ is the eigenvalue of the problem and X is the eigenvector, moreover λ and X are complex values:

$$\lambda = \left(\lambda_0, \lambda_1, ..., \lambda_{2\bar{N}} \right)$$

$$X = \begin{bmatrix} x_{00} & x_{10} & \cdots & x_{2\bar{N}+1,0} \\ x_{01} & x_{11} & \cdots & x_{2\bar{N}+1,1} \\ \vdots & \vdots & \vdots & \vdots \\ x_{0,2\bar{N}+1} & x_{1,2\bar{N}+1} & \cdots & x_{2\bar{N}+1,2\bar{N}+1} \end{bmatrix}$$

Eigenvalues and eigenvectors of system (21) can be found using the standard numerical methods. In this paper, they can be determined using the following algorithm.

Thus, the algorithm of the preliminary integration method for numerical simulation of the problem of hydrodynamic stability of two-phase flows is performed in the following sequence:

1) constants (16) are calculated;
2) functions (15) are calculated;
3) characteristic parameters $\bar{k}, \bar{R}e, G, T, f$;
4) an algebraic system (21) is formed;
5) using the QR algorithm, the eigenvalues and eigenvectors of the system (21) are determined;
6) in plane $\left(R_j, R_\ell\right)$, such a sequence of characteristic parameters is chosen for which $\lambda_i = 0$, i.e. point of neutral stability;
7) drawing a line through all points of neutral stability in plane $\left(\bar{\mathrm{k}}, \bar{\mathrm{R}}\mathrm{e}\right)$, a neutral stability curve is constructed, which separates the flow stability region from the instability region.

4. Conclusion

1. A preliminary integration method is proposed for numerically solving the problem of hydrodynamic stability of two-phase flows.
2. Using the method of preliminary integration, a differential eigenvalue problem for a system of nonlinear ordinary differential equations with a small parameter at the highest derivative to a generalized algebraic eigenvalue problem.
3. Development of an algorithm for the numerical solution of the problem of hydrodynamic stability by the method of preliminary integration.

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