



PRIMARY DECOMPOSITION IN NEAR SUBTRACTION SEMIGROUPS

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Abstract: The present research article has to initiate the content of primary decomposition in Near subtraction semigroup. Also investigated the properties of prime ideals of a near subtraction semigroup. The investigator also developed the new concept of Noetherian semigroup and Artinian semigroup and duo Noetherian near subtraction semigroup and some of the results were proved.

Keywords: duo Noetherian near subtraction semigroup, primary decomposition in Near subtraction semigroup, Noetherian semigroup, Artinian semigroup.

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1. Introduction:

The concept of primary ideals in semigroup was studied by A. Anjaneylu in 1980 [1] about the primary ideals in semigroups. Later D.Madhusudana Rao[10] was introduced the same concept to gamma semigroups in 2010. Further A.GangadharRao[16] extended the concept to duo-gamma semigroups. Now in this paper we used the same concept to study the primary ideals in near subtraction semigroups.

2. Preliminaries:

For preliminaries refer the reference [2] and their references

3. PRIMARY DECOMPOSITION IN NEAR SUBTRACTION SEMIGROUPS

DEFINITION 3.1 : Let P be prime ideal in a near subtraction semigroup X . A primary ideal A in X is said to be **P-primary** or P is a prime ideal belonging to A provided $\sqrt{A} = P$.

THEOREM 3.2 : If A_1, A_2, \dots, A_n are **P-primary** ideals in a near subtraction semigroup

X , then $\bigcap_{i=1}^n A_i$ is also a **P-primary** ideal.

Proof : Let $A = \bigcap_{i=1}^n A_i$. Then $\sqrt{A} = \sqrt{\bigcap_{i=1}^n A_i} = \bigcap_{i=1}^n \sqrt{A_i} = \bigcap_{i=1}^n P = P$.

Suppose that $\langle a \rangle \langle b \rangle \subseteq A$ and $b \notin A$. $b \notin A$ implies $b \notin A_i$ for some i . $\langle a \rangle \langle b \rangle \subseteq A_i$ and $b \notin A_i$, since A_i is a primary ideal, we have $a \in \sqrt{A_i} = P = \sqrt{A} \Rightarrow a \in \sqrt{A}$. So A is left primary ideal. Similarly we can show that A is right primary ideal. Thus A is a primary ideal. Hence A is P-primary ideal.

DEFINITION 3.3 : An ideal A in a near subtraction semigroup X is said to have **left primary decomposition** if $A = A_1 \cap A_2 \cap A_3 \cap \dots \cap A_n$ where each A_i is a left primary ideal. If no A_i contains $A_1 \cap A_2 \cap A_3 \cap \dots \cap A_{i-1} \cap A_{i+1} \cap \dots \cap A_n$ and the radicals P_i of the ideals A_i are all distinct, then the left primary decomposition is said to be **reduced**. If P_i is minimal in the set $\{P_1, P_2, P_3, \dots, P_n\}$ then P_i is said to be **beisolated prime**.

DEFINITION 3.4 : An ideal A in a near subtraction semigroup X is said to have **right primary decomposition** if $A = A_1 \cap A_2 \cap A_3 \cap \dots \cap A_n$ where each A_i is a right primary ideal. If no A_i contains $A_1 \cap A_2 \cap A_3 \cap \dots \cap A_{i-1} \cap A_{i+1} \cap \dots \cap A_n$ and the radicals P_i of the ideals A_i are all distinct, then the right primary decomposition is said to be **reduced**. If P_i is minimal in the set $\{P_1, P_2, P_3, \dots, P_n\}$ then P_i is said to be **beisolated prime**.

DEFINITION 3.5 : An ideal A of a near subtraction semigroup X is said to have a **primary decomposition** if $A = A_1 \cap A_2 \cap A_3 \cap \dots \cap A_n$ where each A_i is a right primary ideal. If no A_i contains $A_1 \cap A_2 \cap A_3 \cap \dots \cap A_{i-1} \cap A_{i+1} \cap \dots \cap A_n$ and the radicals P_i of the ideals A_i are all distinct, then the primary decomposition is said to be **reduced**. If P_i is minimal in the set $\{P_1, P_2, P_3, \dots, P_n\}$ then P_i is said to be **beisolated prime**.

DEFINITION 3.6 : A near subtraction semigroup X is said to be a **Noetherian semigroup** provided every ascending chain of ideals becomes stationary.

DEFINITION 3.7 : A near subtraction semigroup X is said to be a **Artinian semigroup** provided every descending chain of ideals becomes stationary.

THEOREM 3.8 : If an ideal A in a near subtraction semigroup X has a primary decomposition, then A has a reduced primary decomposition.

Proof: If $A = A_1 \cap A_2 \cap A_3 \cap \dots \cap A_n$ where each A_i is primary ideal and some A_i contains $A_1 \cap A_2 \cap A_3 \cap \dots \cap A_{i-1} \cap A_{i+1} \cap \dots \cap A_n$,

then $A = A_1 \cap A_2 \cap A_3 \cap \dots \cap A_{i-1} \cap A_{i+1} \cap \dots \cap A_n$ is also a primary decomposition.

By thus eliminating the superfluous A_i reindexing we have $A = A_1 \cap A_2 \cap A_3 \cap \dots \cap A_k$ with no A_i containing the intersection of other A_j . Let P_1, P_2, \dots, P_r be the distinct prime ideals in the set $\sqrt{A_1}, \sqrt{A_2}, \dots, \sqrt{A_k}$. Let $A'_i, 1 \leq i \leq r$ be the intersection of all A_j 's belonging to the prime P_i . By theorem 3.2, each A_i ' is primary for P_i . Clearly no A'_i contains the intersection of all other A'_j . Therefore $A = \bigcap_{i=1}^k A_i = \bigcap_{i=1}^r A'_i$ and hence A has a reduced primary decomposition.

NOTE 3.9 :It is well known that every ideal has a reduced primary decomposition in a commutative Noetherian near subtraction semigroup. But in the case of an arbitrary near subtraction semigroup it is not necessarily true that every ideal has a primary decomposition even if the near subtraction semigroup is finite and pseudo symmetric.

Example 3.10 : Assume that $X = \{x, y, z\}$ in which ‘-‘ and ‘.’ are defined as follows:

•	x	y	z
x	x	x	x
y	x	x	x
z	x	y	z

Then $(X, -, \cdot)$ is a

-	x	y	z
x	x	x	x
y	y	x	y
z	z	z	x

 near subtraction semigroup.

Since $\{b, c\}$ and $\{a, b, c\}$ are the only primary ideals and hence $\{a\}$ has no primary decomposition.

DEFINITION 3.11 *duo near subtraction semigroup* :A near subtraction semigroup X is said to be a **left subtraction semigroup** provided every left ideal of X is a two sided ideal of X .

DEFINITION 3.12 *right duo near subtraction semigroup* :A near subtraction semigroup X is said to be a **right duo near subtraction semigroup** provided every right ideal of X is a two sided ideal of X .

DEFINITION 3.13 : A near subtraction semigroup X is said to be a *duo near subtraction semigroup* provided it is both a left duo near subtraction semigroup and a right duo near subtraction semigroup.

REMARK 3.14: Let A be a semipseudo symmetric ideal of a near subtraction semigroup X . Then the following are equivalent.

- 1) $A_1 =$ The intersection of all completely prime ideals of X containing A .
- 2) $A_1^1 =$ The intersection of all minimal completely prime ideals of X containing A .
- 3) $A_1^{11} =$ The minimal completely semiprime ideal of X relative to containing A .
- 4) $A_2 = \{x \in X : x^n \in A \text{ for some natural number } n\}$
- 5) $A_3 =$ The intersection of all prime ideals of X containing A .
- 6) $A_3^1 =$ The intersection of all minimal prime ideals of X containing A .
- 7) $A_3^{11} =$ The minimal semiprime ideal of X relative to containing A .
- 8) $A_4 = \{x \in X : \langle x \rangle^n \subseteq A \text{ for some natural number } n\}$

REMARK 3.15: Let X be a near subtraction semigroup and A be an ideal in X . Then A is pseudo symmetric if and only if for all $a \in X$ the set $A_r(a) = \{x \in X : ax \in A\}$ is an ideal of X .

REMARK 3.16: If A is a pseudo symmetric ideal of a near subtraction semigroup X then $A_2 = A_4$.

THEOREM 3.17 : Every left(right) duo near subtraction semigroup X is a pseudo symmetric near subtraction semigroup.

Proof : Let X be a left duo near subtraction semigroup and A be any ideal of X .

Suppose $a \in X$. Let $A_l(a) = \{x \in X : xa \in A\}$.

Let $x, y \in A_l(a) \Rightarrow xa, ya \in A$.

$(x - y)a = xa - ya \in A$, since A is an ideal of X and $xa, ya \in A$.

Therefore $x - y \in A_l(a)$ and hence $A_l(a)$ is a sub algebra of X .

Let $x \in A_l(a) \Rightarrow xa \in A$. $xa \in A$, A is an ideal $\Rightarrow sxa \in A$ for all $s \in X$

$\Rightarrow sx \in A_l(a)$ for all $s \in X$. Therefore $A_l(a)$ is a left ideal of X for all $a \in X$.

Since X is a left duo near subtraction semigroup, $A_l(a)$ is an ideal of X .

So by Remark 3.16, A is a pseudo symmetric ideal of X .

Therefore X is a pseudo symmetric Semigroup.

COROLLARY 3.18: Every duo near subtraction semigroup is a pseudo symmetric near subtraction semigroup.

Proof :By theorem 3.17, we conclude that every duo near subtraction semigroup X is a pseudo symmetric near subtraction semigroup.

THEOREM 3.19 : Every ideal in a (left, right) duo noetherian near subtraction semigroup X has a reduced (left, right) primary decomposition.

Proof :Let Σ be the collection of all ideals in X which has no primary decomposition.

If Σ is nonempty, then since S is Noetherian, Σ contains maximal elements. Let C be a maximal element in Σ . Clearly C is not primary. Suppose C is not left primary, then there exists elements a, b in X such that $\langle a \rangle \langle b \rangle \subseteq C$, $b \notin C$ and $a \notin \sqrt{C}$. Since X is a duo near subtraction semigroup, By theorem 3.18, C is a semi pseudo symmetric ideal, and hence by the remark 3.14, $\sqrt{C} = \{x \in X : x^n \in C \text{ for some natural number } n\}$. Therefore $a^n \notin C$ for any natural number n . For any natural number n , write $B_n = \{x \in X : a^n x \in C\}$. Now $B_1 \subseteq B_2 \subseteq \dots$ is an ascending chain of ideas in X . Since X is Noetherian, there is a natural number k such that $B_k = B_i$ for all $i \geq k$. Since $b \in B_k$, we have B_k contains C properly. Write $D = a^k S \cup C$. Now $a^{k+1} \in D$. Since X is a duo near subtraction semigroup, D is an ideal in X and containing C properly. Now we prove that $C = B_k \cap D$. Clearly $C \subseteq B_k \cap D$. If $x \in B_k \cap D$ and $x \notin C$, then $x = a^k y$ for some $y \in X$. Now $x \in B_k \Rightarrow a^k x \in C$. Therefore $a^{2k} y = a^k \cdot a^k y = a^k x \in C$. Therefore $y \in B_{2k} = B_k$. Thus $y \in B_k \Rightarrow x = a^k y \in C \Rightarrow x \in C$. It is a contradiction. Thus C is left primary. Similarly, we can show that C is right primary. Hence C is primary. Therefore Σ is empty. Thus, every ideal in a duo Noetherian near subtraction semigroup has a primary decomposition and hence by the known result, every ideal has a reduced primary decomposition.

COROLLARY 3.20 : Every ideal in a generalized commutative Noetherian near subtraction semigroup has a right reduced primary decomposition.

Proof :Since every generalized commutative near subtraction semigroup is a left duo near subtraction semigroup, and hence the proof follows from theorem 3.19.

NOTATION 3.21 : Let A and B be two ideals in a near subtraction semigroup X . Then we denote $A^l(B) = \{x \in X : \langle x \rangle B \subseteq A\}$ and $A^r(B) = \{x \in X : B \langle x \rangle \subseteq A\}$. Clearly $A^l(B)$ and $A^r(B)$ are ideals of X containing A .

THEOREM 3.22 : Let A and B be two ideals of a near subtraction semigroup X .

- 1) If A is a left primary ideal of X , then $A^l(B)$ is a left primary ideal.
- 2) If A is a right primary ideal of X , then $A^r(B)$ is a right primary ideal.

Proof : (1) If $B \subseteq A$, then clearly $A^l(B) = X$. Suppose $B \not\subseteq A$. Let $b \in B \setminus A$ and $x \in A^l(B)$. $x \in A^l(B)$ implies $\langle x \rangle B \subseteq A$. So $\langle x \rangle \langle b \rangle \subseteq A$.

Since $b \notin A$, we have $x \in \sqrt{A}$, therefore $A^l(B) \subseteq \sqrt{A}$.

Since $A \subseteq A^l(B) \subseteq \sqrt{A} \Rightarrow \sqrt{A} \subseteq \sqrt{A^l(B)} \subseteq \sqrt{\sqrt{A}}$

$\Rightarrow \sqrt{A} \subseteq \sqrt{A^l(B)} \subseteq \sqrt{A} \Rightarrow \sqrt{A} = \sqrt{A^l(B)}$. Let $\langle x \rangle \langle y \rangle \subseteq A^l(B)$ and $y \notin A^l(B)$.

Now $\langle x \rangle \langle y \rangle B \subseteq A$. If $x \notin \sqrt{A^l(B)} = \sqrt{A}$ then $\langle y \rangle B \subseteq A$ and hence $y \in A^l(B)$.

This is a contradiction. So $x \in \sqrt{A^l(B)}$. Therefore $A^l(B)$ is a left primary ideal.

(2) The proof is similar to (1).

THEOREM 3.23 : If Q is a P -primary ideal and if $A \not\subseteq P$, then $Q^l(A) = Q^r(A) = Q$. And also if $A \subseteq P$ and $A \not\subseteq Q$, then $\sqrt{Q^l(A)} = \sqrt{Q^r(A)} = \sqrt{Q}$.

Proof: Clearly $Q \subseteq Q^l(A)$. Let $x \in Q^l(A)$. Then $\langle x \rangle A \subseteq Q$. Since $A \not\subseteq P$, there exists $a \in A \setminus P$. Now $\langle x \rangle \langle a \rangle \subseteq Q$ and $a \notin \sqrt{Q}$. So $x \in Q$. Hence $Q^l(A) \subseteq Q$. Therefore $Q = Q^l(A)$. Similarly we can show that $Q^r(A) = Q$. Therefore $Q^l(A) = Q^r(A) = Q$.

Hence clearly by the theorem 3.22, $\sqrt{Q^l(A)} = \sqrt{Q^r(A)} = \sqrt{Q}$.

THEOREM 3.24 : If A_1, A_2, \dots, A_n, B are ideals of a semigroup X , then

$$[\bigcap_{i=1}^n A_i]^l(B) = \bigcap_{i=1}^n (A_i)^l(B).$$

Proof : Let $x \in [\bigcap_{i=1}^n A_i]^l(B)$. Then $\langle x \rangle B \subseteq \bigcap_{i=1}^n A_i \Rightarrow \langle x \rangle B \subseteq A_i$ for $i = 1, 2, \dots, n$

$$\Rightarrow x \in A_i^l(B) \text{ for } i = 1, 2, \dots, n. \Rightarrow x \in \bigcap_{i=1}^n (A_i)^l(B). \text{ Therefore } [\bigcap_{i=1}^n A_i]^l(B) \subseteq \bigcap_{i=1}^n (A_i)^l(B).$$

Similarly we can show that $\bigcap_{i=1}^n (A_i)^l(B) \subseteq [\bigcap_{i=1}^n A_i]^l(B)$. Hence $[\bigcap_{i=1}^n A_i]^l(B) = \bigcap_{i=1}^n (A_i)^l(B)$.

THEOREM 3.25 : (UNIQUENESS THEOREM) Suppose an ideal A in a near subtraction semigroup X has two reduced (one sided) primary decompositions $A = A_1 \cap A_2 \cap \dots \cap A_k = B_1 \cap B_2 \cap \dots \cap B_s$, where A_i is P_i -primary and B_j is Q_j -primary. Then $k = s$ and after reindexing if necessary, $P_i = Q_i$ for $i = 1, 2, 3, \dots, k$. Further if each P_i is an isolated prime, then $A_i = B_i$ for $i = 1, 2, 3, \dots, n$.

Proof : Let P_k be the maximal element in the set $P_1, P_2, \dots, P_k, Q_1, Q_2, \dots, Q_s$. Now we show that P_k occurs among Q_1, Q_2, \dots, Q_s . For this it is enough to show that $P_k \subseteq Q_j$ for some j .

If $A_k \subseteq Q_j$ for some j , then $P_k = \sqrt{A_k} \subseteq Q_j$.

Suppose $A_k \not\subseteq Q_j$ for all j , then by the known result $B_j^l = B_j \forall j$.

$$\text{Now } A^l(A_k) = (B_1 \cap B_2 \cap \dots \cap B_s)^l(A_k) = B_1^l(A_k) \cap B_2^l(A_k) \cap \dots \cap B_s^l(A_k) = A$$

But on the other hand if $1 \leq i < k$, then $P_k \not\subseteq P_i$ and therefore $A_k \not\subseteq P_i$, so that $A_i^l(A_k) = A_i$ and $A_k^l(A_k) = S$. We have

$$A^l(A_k) = (A_1 \cap A_2 \cap \dots \cap A_k)^l(A_k) = A_1^l(A_k) \cap A_2^l(A_k) \cap \dots \cap A_k^l(A_k) = A_1 \cap A_2 \cap \dots \cap A_{k-1}$$

$\therefore A = A_1 \cap A_2 \cap \dots \cap A_{k-1}$. It is a contradiction to the fact that given decomposition is reduced. Thus $A_k \subseteq Q_j$ for some j and hence $P_k \subseteq Q_j$. Therefore $P_k = Q_j$. Without loss of generality we may assume that $P_k = Q_s$. Let $B = A_k \cap B_s$.

By the theorem 3.2, B is a primary ideal and $P_k = Q_s = P$ (say) is a prime ideal belonging to B .

Since $P \not\subseteq P_i$ for all i , $1 \leq i < k$ and $B \subseteq A_k$. We have $A_i^l(B) = A_i$ and $A_k^l(B) = S$.

Therefore $A^l(B) = A_1 \cap A_2 \cap \dots \cap A_{k-1}$.

Similarly, we can show that, $A^l(B) = B_1 \cap B_2 \cap \dots \cap B_{s-1}$.

Therefore $A^l(B) = A_1 \cap A_2 \cap \dots \cap A_{k-1} = B_1 \cap B_2 \cap \dots \cap B_{s-1}$ are two reduced primary decompositions for $A^l(B)$. By continuing the above process, we get $k = s$ and $P_i = Q_i$ for

$i = 1, 2, \dots, k$. Suppose P_i 's are isolated primes. If $A_1 \subseteq B_1$, then since B_1 is primary and $A_1 A_2 \dots A_k \subseteq B_1 \cap B_2 \cap \dots \cap B_k \subseteq B_1$, we have $A_2 A_3 \dots A_k \subseteq \sqrt{B_1} = P_1$.

Now $P_2 P_3 \dots P_k = \sqrt{A_2 A_3 \dots A_k} = P_1$. Since P_1 is a prime ideal $P_i \subseteq P_1$ for some $1 < i \leq k$. This is a contradiction to the fact that P_1 is an isolated prime. So $A_1 \subseteq B_1$. Similarly we can show that $B_1 \subseteq A_1$. By continuing in this way we get $A_i = B_i$ for $i = 1, 2, \dots, k$. This completes the proof of the theorem.

Conclusion: Mainly in this research article, we studied primary decomposition in Near subtraction semigroup.

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