Edge Product in Cluster Hypergraphs

[1]Befija Minnie. J and [2]Mary Christal Flower. C

[1] Assistant Professor, Department of Mathematics
Holy Cross College (Autonomous), Nagercoil-4, India.
[2] Research Scholar, Department of Mathematics
Holy Cross College (Autonomous), Nagercoil-4, India.
[1]befija@gmail.com, [2]marychristal01@gmail.com
Affliated to Manonmaniam Sundaranar University,
Abishekapatti, Tirunelveli-627012.

Abstract

The cluster hypergraph $H = (V_x, E)$ is said to be an Edge Product Cluster Hypergraph if there exists an edge function $f : E \rightarrow I$ such that the edge function f and the corresponding edge product function F of f on $V_x(H)$ have the following two conditions:

(i) $F(v) \in I$ for every $v \in V_x(H)$

(ii) if $f(e_1) \times f(e_2) \times \ldots \times f(e_i) \in I$ for some edges $e_1, e_2 \ldots e_i \in E(H)$, then the edges $e_1, e_2 \ldots e_i$ are all adjacent to some vertex $v \in V_x(H)$. The edge product cluster hypergraph, unit edge product cluster hypergraph and some theorems related to this concept have been discussed in this paper.

Keywords : Edge function, Edge product function, Edge product cluster hyergraph, Unit edge product cluster hypergraph, Cluster hypergraph.

AMS Subject Classification : 05C65.

DOI: 10.48047/ecb/2023.12.si8.655

1 Introduction

Harary F introduced the notation of sum graph [6]. A graph $G(V, E)$ is said to be a sum graph if there exists a bijection labeling f from the vertex set V to a set S of positive integers such that $xy \in E$ if and only if $f(x) + f(y) \in S$. The product analogue of sum graphs was first introduced by Thavamani in 2011 [8]. He introduced the edge product graphs and the edge product number of a graph [6], [13]. A graph G is said to be an edge product graph if the edges of G can be labelled with distinct positive integers such that the product of all label of the edges incident on a vertex is again an edge label of G [8] . The hypergraph is a generalization of a graph in which edges can connect any number of vertices. Hypergraph was introduced by Berge in 1973 [1]-[2]. In 2020 S. Samantha introduced the concept of cluster hypergraphs [8]. The concept cluster hypergraph was introduced to generalize the concept of hypergraph in which cluster nodes are allowed. Jadhar and Pawar introduced the notation of an edge function and using this edge function an edge product hypergraph is defined [9]. In this paper the concept of edge product cluster hypergraph, unit edge product cluster hypergraph and non-unit edge product cluster hypergraph have been discussed.

2 Preliminaries

Definition 2.1. Let $H = (V_x, E)$ be a simple cluster hypergraph with vertex set $V_x(H)$ and edge set $E(H)$. Let I be the set of positive integers such that $|E| = |I|$. Then any bijection $f\colon E \to I$ is called an **Edge Function** of the cluster hyergraph .

Definition 2.2. The function $F(v) = \prod\{f(e)$; edge e incident to the vertex $v\}$ on $V_x(H)$ is called an **Edge Product Function** of the edge function f.

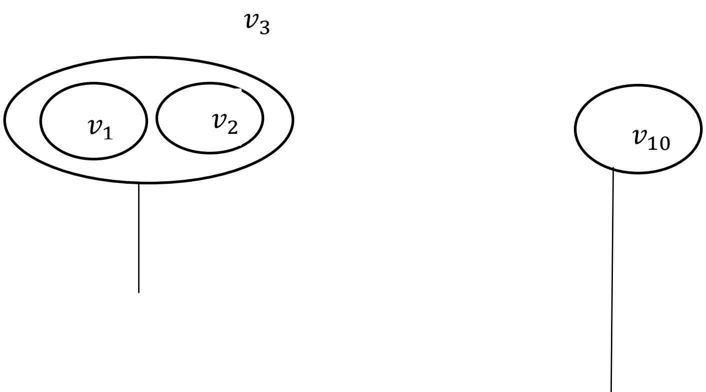
Definition 2.3. The cluster hypergraph $H = (V_x, E)$ is said to be an Edge Product Cluster Hypergraph if there exists an edge function $f : E \to I$ such that the edge function f and the corresponding edge product function F of f on $V_x(H)$ have the following two conditions:

(i) $F(v) \in I$ for every $v \in V_x(H)$

(ii) if $f(e_1) \times f(e_2) \times \ldots \times f(e_i) \in I$ for some edges $e_1, e_2 \ldots e_i \in E(H)$, then the edges $e_1, e_2 \ldots e_i$ are all adjacent to some vertex $v \in V_x(H)$.

Example 2.3. Let $H = (V_x, E)$ be a cluster hypergraph, with vertex set $V_x(H) = \{v_1, v_2, v_3 = \{v_1, v_2\}, v_4, v_5, v_6 = \{v_4, v_5\}, v_7, v_8, v_9, v_{10}, v_{11}, v_{12}\}$ and edge set $E(H) = \{e_1, e_2, \ldots, e_8\}$. Here the edges of H are defined as follows $e_1 = \{v_1, v_3\}$, $e_2 = \{v_2, v_3\}$, $e_3 = \{v_3, v_6\}$, $e_4 = \{v_4, v_6\}$, $e_5 = \{v_5, v_6\}$, $e_6 = \{v_6, v_7, v_8\}$, $e_7 = \{v_6, v_7, v_9\}$, $e_8 = \{v_{10}, v_{11}, v_{12}\}$. Now we define the edge function $f : E \to I$ by $f(e_1) = 15$, $f(e_2) = 8$, $f(e_3) = 20$, $f(e_4) = 2$, $f(e_5) = 3$, $f(e_6) = 4$, $f(e_7) = 5$, $f(e_8) = 2400$. The edge product function F of f is defined by $F(v_1) = 15$, $F(v_2) = 8$, $F(v_3) = 2400$, $F(v_4) = 2$, $F(v_5) = 3$, $F(v_6) = 2400$, $F(v_7) = 20$, $F(v_8) = 4$, $F(v_9) = 5$, $F(v_{10}) = 2400$, $F(v_{11}) = 2400$, $F(v_{12}) = 2400$.

Thus the given cluster hypergraph in figure 1 is an edge product cluster hypergraph.



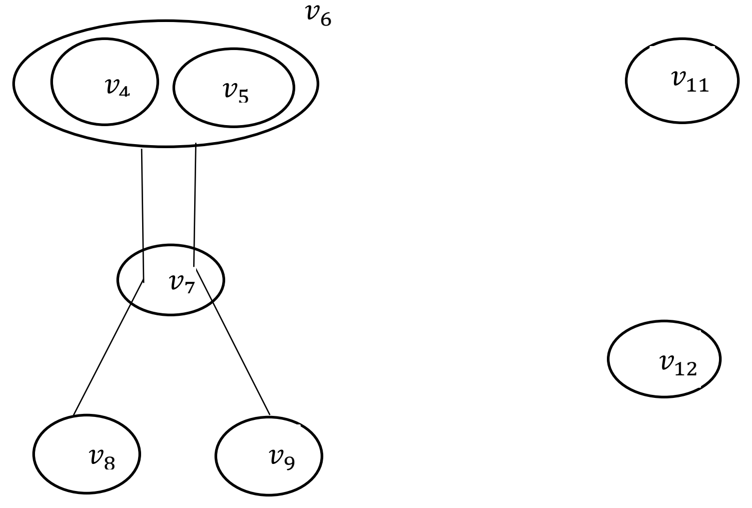


Figure 1, Cluster hypegraph

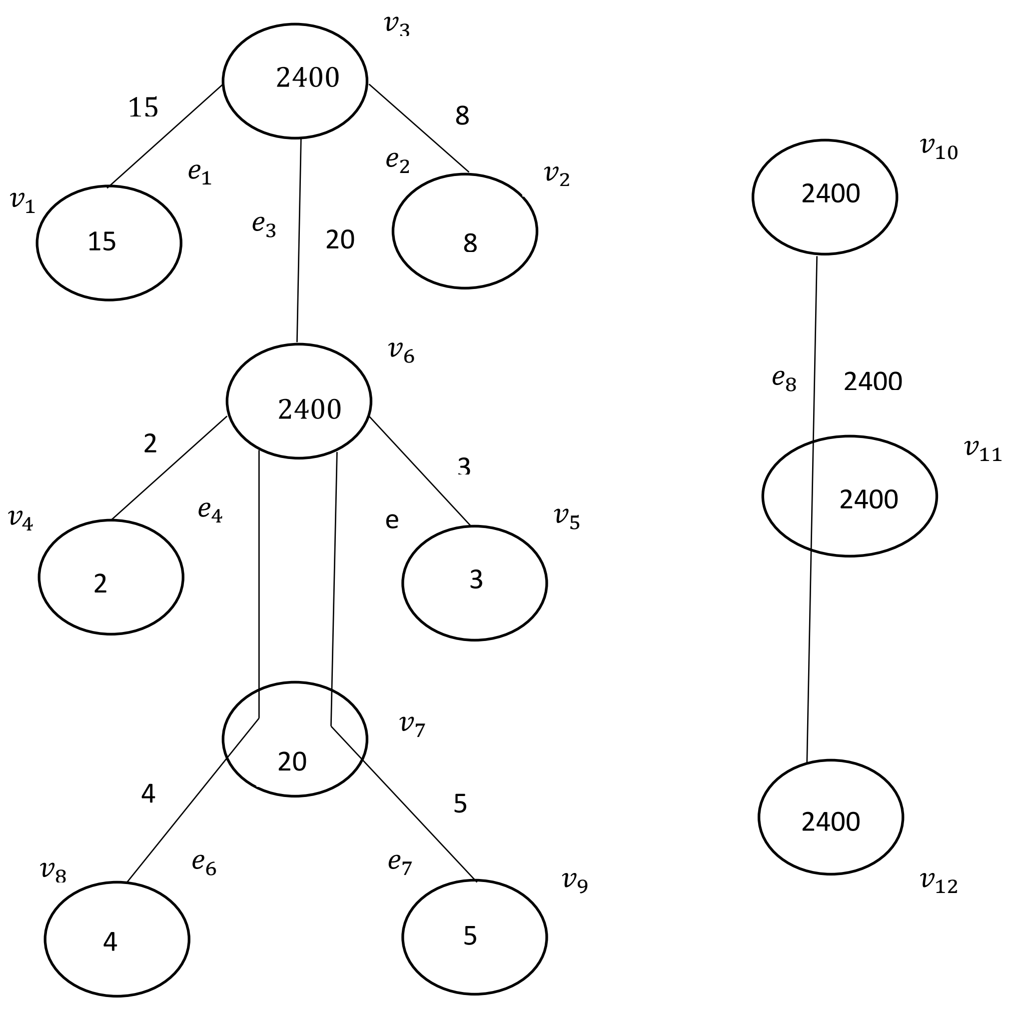


Figure 2

Figure 2, Virtual representation of figure 1, which is an edge product cluster hypergraph.

Definition 2.4. Let X be a nonempty set and let V_x be a subset of $P(X)$ such that $\emptyset \notin V_x$ and $X \subset V_x$. Now F be a multi-set whose element belong to $P(P(X))$ such that

(i) ∅∈E

(ii) for each element $e \in E$, there exist atleast one element $v \in V_x$ such that $v \in e$.

Then $H = (V_x, E)$ is said to be Cluster hypergraph where V_x is said to be a vertex set and E is said to be multi-hyper edge set [11].

3 Main Results

Remark 3.1. Every cluster hypergraph can be form an edge product cluster hypergraph by adding an edge of cardinality less than or equal to $V_x(H)$ or adding a subset of $V_x(H)$ of cardinality two.

Definition 3.2. The cluster hypergraph $H = (V_x, E)$ is said to be connected if for every pair of vertices are connected by a path. If H is not connected, then it consits of two or more components, each of which is a connected cluster hypergraph [11].

Theorem 3.1. Let I be the set of positive integers with $1 \notin I$. Then every edge product cluster hypergraph is not connected.

Proof. Let $H = (V_x, E)$ be an edge product cluster hypergraph. Let $f : E \to I$ be an edge product function and F be an edge product function of f. Since $1 \notin I$ shows that $f(e_i) > 1$ for every edge $e_i \in E$ and $i \in E$. Without loss of generality, we assume that *j* be the largest element in I. Since $f : E \to I$ be a bijection, there exists an edge $e \in E$ such that $(e) = j$. To prove H is not connected. Suppose on the contrary, assume H is connected then e must be adjacent to some edge m in . This shows that there is a vertex $v \in e \cap m$ such that $F(v) = f(e).f(m) > f(e) = j$. This implies that $F(v) \notin I$ and so that H is not an edge product cluster hypergraph which is a contradiction. Hence H is not connected.

Definition 3.3. Let $H = (V_x, E)$ be an edge product cluster hypergraph. Then H is said to be a **Unit edge Product cluster hypergraph** if there exists an edge function $f : E \to I$ such that $1 \in I$.

Example 3.2. Let $H = (Vx, E)$ be a cluster hypergraph with vertex set $V_x(H) = \{v_1, v_2, v_3 = \{v_1, v_2\}, v_4, v_5, v_6 = \{v_4, v_5\}, v_7, \cdots, v_{16}\}$ and edge set $E(H) = \{e_1, e_2, \dots, e_8\}$. Here the edges of H are define as follows $e_1 = \{v_3, v_6, v_7, v_8, v_9, v_{10}, v_{11}, v_{12}, v_{13}, v_{14}, v_{15}, v_{16}\}$, $e_2 = \{v_1, v_3\}$, $e_3 = \{v_2, v_3\}$, $e_4 = \{v_4, v_6\}$, $e_5 = \{v_5, v_6\}$, $e_6 = \{v_7, v_8, v_9\}$, $e_7 = \{v_{10}, v_{11}, v_{12}\}$, $e_8 = \{v_{13}, v_{14}, v_{15}\}$. Now we define the edge function $f : E \to I$ by $f(e_1) = 1, f(e_2) = 3, f(e_3) = 11, f(e_4) = 2, f(e_5) = 5, f(e_6) = 7, f(e_7) = 10, f(e_8) = 33$.

The edge product function F of f is defined by $F(v_1) = 3, F(v_2) = 11, F(v_3) = 33, F(v_4) = 2, F(v_5) = 5, F(v_6) = 10, F(v_7) = F(v_8) = F(v_9) = 7, F(v_{10}) = F(v_{11}) = F(v_{12}) = 10, F(v_{13}) = F(v_{14}) = F(v_{15}) = 33, F(v_{16}) = 1$.

Theorem 3.2. Every unit edge product cluster hypergraph is always connected.

Proof. Let $H = (V_x, E)$ be a unit edge product cluster hypergraph. Let $f : E \to I$ be an edge product function and F be an edge product function of f . Let e be an edge of H with $(e) = 1$. To prove that H is connected. Suppose H is not connected. Then H has more than one component. Let H_1 and H_2 be two components of H, where H_1 and H_2 are connected cluster hypergraphs. Let m be any edge in H_1 such that $(m) \in I$. Without loss of generality, assume that the edge e in H . Since H_1 and H_2 are not connected, there is no vertex common to both e and m . But now, $(m) = f(m).1$. This implies that, $(m) = f(m).f(e) \in I$. This shows that e and m are incident with a vertex $v \in V_x$, and so H_1 and H_2 are connected by a path, which is a contradiction. Hence H is connected.

Theorem 3.3. Let $H = (V_x, E)$ be a unit edge product cluster hypergraph with an edge $\in E$. Then $f(e) = 1$ if and only if e must be adjacent to all the edges in H .

Proof. Let $H = (V_x, E)$ be a unit edge product cluster hypergraph. By theorem 3.2, H is connected. Let $e \in E$. Assume $f(e) = 1$. To prove e is adjacent to every edges in H. Suppose e is not adjacent to some edge e_j in H such that $f(e_j) \in I$. This shows that there is no vertex common to both e and e_j. Thus $e \cap e_j = \emptyset$. But $(e_j) = f(e_j s).1 = f(e_j).f(e) \in I$. If this is true then e and e_j are incident to a vertex $v \in V_x(H)$ and so $v \in e \cap ej$. Since $e \cap e_j = \emptyset$ shows that $v = \emptyset$. If $v = \emptyset$ then $\emptyset \notin V_x(H)$ which is not possible. Therefore $v \neq \emptyset$ and $e \cap e_j = \emptyset$ shows that $f(e) > 1$ which is a contradiction. Thus e is adjacent to every edges in H.

Conversely assume that e is adjacent to every edges in H. To show that $f(e) = 1$. Suppose $f(e) > 1$. Let j be the largest element in I. Since $f : E \to I$ be a bijection, there exists an edge $m \in E(H)$ such that $(m) = j$. By assumption there is a vertex $v \in V_x(H)$ such that e and m are incident with v. But $F(v) = f(e).f(m) > f(m) = j$. This implies that $F(v) > j$, which is not possible. Therefore $F(v) = j$ follows that e is not adjacent with m, which is a contradiction. Hence $(e) = 1$.

Definition 3.4. An edge e in a unit edge product cluster hypergraph whose edge function label as one is called a unit edge of that cluster hypergraph.

The edge which is not a unit edge is called a non-unit edge of that cluster hypergraph.

Example 3.4. Let $H = (V_x, E)$ be a unit edge product cluster hypergraph with vertex set $V_x(H) = \{v_1, v_2, v_3 = (v_1, v_2), v_4, v_5, v_6, v_7 = (v_4, v_5, v_6), v_8, v_9, v_{10}, v_{11}, v_{12}, v_{13}, v_{14}\}$ and edge set $E(H) = \{e_1, e_2, \ldots e_8\}$,where $e_1 = \{v_1, v_3\}$, $e_2 = \{v_2, v_3\}, e3 = \{v_3, v_7, v_8, v_{11}, v_{14}\}$, $e_4 = \{v_4, v_7\}, e_5 = \{v_6, v_7\}, e_6 = \{v_5, v_7\}, e_7 = \{v_8, v_9, v_{10}\}$, $e_8 = \{v_{11}, v_{12}, v_{13}\}$. Also the edge function $f : E \rightarrow I$ for H as $f(e_1) = 7, f(e_2) = 11, f(e_3) = 1, f(e_4) = 2$, $f(e_5) = 3$, $f(e_6) = 5$, $f(e_7) = 30$, $f(e_8) = 77$. Moreover the edge product function F of f is defined by $F(v_1) = 7, F(v_2) = 11, F(v_3) = 77, F(v_4) = 2$, $F(v_5) = 5, F(v_6) = 3, F(v_7) = F(v_8) = F(v_9) = F(v_{10}) = 30, F(v_{11}) = F(v_{12}) = F(v13) = 77$ and $F(v14) = 1$.

For this unit edge product cluster hypergraph H the unit edge is e_3 and $E(H) - e_3$ are the set of non-unit edge of H.

Definition 3.5. An edge e in a unit edge product cluster hypergraph whose edge function label as the largest element in the set of positive integers I is called a maximal edge of that cluster hypergraph.

Example 3.5. Let $H = (V_x, E)$ be a unit edge product cluster hypergarph with vertex set $V_x(H) = \{v_1, v_2, v_3, v_4, v_5 = \{v_1, v_2, v_3, v_4\}, v_6, v_7, v_8, v_9\}$ and edge set $E(H) = \{e_1, e_2, \ldots e_6\}$ where $e_1 = \{v_1, v_5\}$, $e_2 = \{v_2, v_5\}$, $e_3 = \{v_3, v_5\}$, $e_4 = \{v_4, v_5\}$, $e_5 = \{v_5, v_6, v_9\}$, $e_6 = \{v_6, v_7, v_8\}$. Also the edge function $f : E \rightarrow I$ for H as $f(e_1) = 2$, $f(e_2) = 3$, $f(e_3) = 5$, $f(e_4) = 9, f(e_5) = 1, f(e_6) = 270$. Moreover the edge product function F of f is defined as $F(v_1) = 2, F(v_2) = 3, F(v_3) = 5, F(v_4) = 9, F(v_5) = 270, F(v_6) = F(v_7) = F(v_8) = 270$ and $(v_9) = 1$.

For this unit edge product cluster hypergraph H e_6 is the maximal edge of H.

Conclusion

The edge product of cluster hypergarph and some theorems related to this concept have been discussed here. In future studies, the edge product number can be determined for cluster hypergraphs.

References

[1] Berge.C, *Graphs and Hypergraph*, Elsevier, Amsterdam, The Netherlands, (1973).

[2] Berge.C, *Hypergraph*, North-Holland, Amsterdam, The Netherlands, (1989).

[3] Harary F, *Sum graphs over all the integers*, Discrete Math, 124: 99-105 (1994).

[4] Harary F, *Sum Graphs and difference graphs*, Congress Numer 72: 101-108 (1990).

[5] J.P. Thavamani and D.S.T Ramesh, *CYCLIC GRAPHS-EDGE PRODUCT NUMBER*, International Journal Sciences, Environment and Technology, Vol.1, No 5, 2012, 454-461.

[6] J.P. Thavamani and D.S.T Ramesh, *Edge product number of graphs in paths*, Journal of Mathematical Sciences and Computer Applications, 92011),78-84.

[7] Maity A, Samanta S, Mondal S, Dubey V, *A Study of Cluster Hypergarphs and Its Properties*, Social Network Analysis and Minning, (2021).

[8] Megha M. Jadhav and Kishor F. Pawar, *On Edge Product Number of Hypergraph*, Palestine Journal Of Mathematics, Vol. 11 (4) (2022), 183-194.

[9] Megha M. Jadhav and Kishor F. Pawar, *On Edge Product Hypergraphs,* Journal of Hyperstructures, 10(1) (2021), 1-12, ISSN: 2322-1666/2251-8436 online.

[10] Ore .O, *Theory of Graphs,* Amer. Maths. Soc. Collq. Publ., 38 (Amer. Math. Soc., Providence, RI), (1962).

[11] Samanta S, Lee JG, Naseem U, Khan SK, Das K, *Concept of Coloring of Cluster Hypergraphs*. Math Probl Eng (2020). https://doi.org/10.1155/2020/3705156.

[12] Samanta S, Muhiudin G, Alanazi AM, Das K, *A Mathematical approch on reprentations of Competition: Competition Cluster Hypergraphs*. Mat prob Eng, (2020).

[13] J.P. Thavamani and D.S.T Ramesh, *Edge product number of graphs and its properties*, The IUP Journal of Computatonal Mathematics, 4 (2011), 30-38.