ACCELERATION OF ALTERNATING SERIES

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1. ABSTRACT;-

In this paper we shall describe about the series acceleration for Alternating series . We shall define a correction term and error function for an Alternating series. Using that correction term and error functions, we can deduce a rapidly convergent series. Here we compare the series deduced by Euler transformation method & Aitken's delta squared process with the series deduced by using correction term and error functions. We will show that our deduced series is better than the other two.

2 . KEY WORDS:-

Alternating series, correction term, error function, Euler transformation, Aitken's delta squared process, sequence of partial sums.

3 . INTRODUCTION;-

The theory of infinite series is an important branch of Mathematical analysis.The historical development of infinite series can be divided into 3 periods. The period of Newton and Leibnitz is considered as the first period. The period of Euler is the second period and third period is the modern period, the period beginning with Gauss

Indian Mathematicians studied infinite series around 1350.In 1715. Brook Taylor provided a general method for constructing Taylor series for all functions. Later, the theory of hypergeometric series is developed by Leonhard Euler. The criteria of convergence and the questions of remainders and the rate of convergence were initially stated by Gauss.

4. PRELIMINARIES

In this section we shall include the preliminary results required for the further work of this paper.

Definition 1

For a convergent series $\sum_{n=1}^{\infty} a_n$, the series $\sum_{k=n+1}^{\infty} a_k$ is called **remainder after n terms** of the series $\sum_{n=1}^{\infty} a_n$.

If $\sum_{n=1}^{\infty} a_n$ converges , then so does the remainder series,

Definition 2.

For a convergent series $\sum_{n=1}^{\infty} a_n$, the **remainder term** after n terms is denoted by **R_n**

and it is the term attached to the series after n terms for approximating the remainder series.

Thus $S_n + R_n = S$, where S_n is the sequence of partial sums and S denote the sum of the series $\sum_{n=1}^{\infty} a_n$.

It is the **error** obtained when the series is approximated by its sequence of partial sums.

Definition 3

A **telescoping series** is a series of the form $\sum_{k=1}^{\infty} \frac{1}{k(k+p)}$ where k is any positive integer.

Definition 4

An **Alternating series** is the sum of infinite number of positive terms which are arranged as alternatively positive and negative.

ie An Alternating series is a series of the form $\sum_{n=1}^{\infty} (-1)^{n-1} a_n$ where the terms $a_n > 0$.

Theorem 5 (Leibnitz Test for convergence of alternating series)(**Alternating Series Test**)

If (a_n) is a monotone decreasing sequence converging to 0 , then the Alternating series$\sum_{n=1}^{\infty} (-1)^{n-1} a_n$ converges

Definition 6

A **continued fraction** is an expression of the form $a_1 + \cfrac{b_1}{a_2 + \cfrac{b_2}{a_3 + \cdots + \cfrac{b_{n-1}}{a_n} + \cdots}}$

where a_i 's and b_i 's may be real or complex numbers.

Usually a continued fraction is denoted as $[a_1; a_2, a_3, \ldots, a_n]$.

Definition 7

A **simple continued fraction** is a continued fraction in which each b_i is 1 and each a_i is a positive integer for $i > 1$. Here a_1 can be positive integer or negative integer or zero.

Here a_i's are called **terms** or **partial quotients** of the continued fraction.

That is $[a_1] = \frac{a_1}{1}$,

$[a_1, a_2] = \frac{a_1 a_2 + 1}{a_2} = a_1 + \frac{1}{a_2}$

$[a_1, a_2, \ldots, a_n] = [a_1, a_2, \ldots, a_{n-1} + \frac{1}{a_n}$

$[a_1, a_2, \ldots, a_n] = a_1 + \frac{1}{[,a2,\ldots,an]}$

Definition 8

If the number of terms of a continued fraction is finite, it is called a **finite simple continued fraction.**
If the number of terms is infinite, the continued fraction is called an **infinite simple continued fraction**

Definition 9

For a simple continued fraction (finite or infinite) $[a_1; a_2,a_3,\ldots\ldots\ldots,a_n,\ldots]$, the **successive convergents** of the continued fraction are defined as follows.

$C_1 =[a_1] = a_1$

$C_2 = [a_1,a_2] = a_1+$

$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

$$C_k = [a_1; a_2,a_3,\ldots\ldots\ldots,a_k]= a_1+\frac{1}{a_{2}+\frac{1}{a_{3}+\cdots\ldots+\frac{1}{a_k}}}$$

$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

5. MAIN RESULTS AND DISCUSSIONS

In this section we shall discuss about the main results of this paper work

Definition.10

For an alternating series $\sum_{n=1}^{\infty} (-1)^{n-1} a_n$ the remainder term $R_n = (-1)^n G_n$ where G_n is called **correction term** after n terms of the series.

Theorem 11

The correction term after n terms for the Alternating series $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{an^2+ bn+c}$ where a, b ,c R with a 0 is $G_n = \frac{1}{2an^2+(2b+2a)n+(2c+b+2a)}$

Proof

If G_n denotes the correction term after n terms of the series $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{an^2+ bn+c}$, then

$$G_n + G_{n+1} = \frac{1}{an^2+ bn+c}$$

Choose $G_n(r_1 , r_2) = \frac{1}{2an^2+ (4a+2b-r_1)n+(2a+2b+2c-r_2)}$

We may choose and in such a way that the error function

$E_n(r_1, r_2) = G_n(r_1, r_2) + G_{n+1}(r_1, r_2) - \frac{1}{an^2+bn+c}$ is a minimum.

It can easily be prove,d that $E_n(r_1, r_2)$ is minimum when $r_1 = 2a$ and r_2 $r_2 = 2b$.
When $r_1 = 2a$ and $r_2 = 2b$, we have both $G_n(r_1, r_2)$ and $E_n(r_1,)$ are functions of a single variable n and so we denote them as G_n and E_n respectively.

Thus the correction term G_n for the series $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{an^2+bn+c}$ is

$G_n = \frac{1}{2an^2+(2b+2a)n+(2c+b+2a)}$ and the corresponding error function is

$|E_n| = \left|\frac{a^2}{(2an+2b+a)+(2an+2b+3a)(an++b+a)}\right|$

<u>Theorem</u> 12

The correction term for generalized alternating harmonic series $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{an+b}$ where a, b $\in$ R with $a \neq 0$ are successive convergents of the infinite continued fraction

$$\cfrac{\mathbf{1}}{\cfrac{}{\mathbf{2an+b+2a+}\cfrac{\mathbf{a^2}}{\mathbf{2an+b+2a+}\cfrac{\mathbf{4a^2}}{\mathbf{2an+b+2a+\cdots\ldots}}}}}$$

<u>Proof</u>

The correction term for $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{an+b}$ is $G_n = \frac{\mathbf{1}}{\mathbf{2an+b+2a}}$ by Theorem 11
By applying the minimisation procedure for finding the correction term in the previous theorem repeatedly, it can be proved that the correction term follows an infinite continued fraction.

The k^{th} successive convergent of the infinite continued fraction is the k^{th} **order correction** term and it is denoted as **$G_n[k]$ f**or k=1,2,3,.....

$G_n[k]$=

$$\cfrac{\mathbf{1}}{\cfrac{}{\mathbf{2an+b+2a+}\cfrac{\mathbf{a^2}}{\mathbf{2an+b+2a+}\cfrac{\mathbf{4a^2}}{\mathbf{2an+b+2a+\cdots\ldots+}\cfrac{\mathbf{(k-1)^2a^2}}{\mathbf{2an+b+2a}}}}}}$$

Theorem 13

The correction terms for Alternating Telescoping series $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n(n+1)}$ are the successive convergents of the infinite continued fraction

$$\cfrac{1}{(2(n+1)^2+1^2)+\cfrac{1^2(1.3)}{(2(n+1)^2+3^2+\cfrac{2^2(3.5)}{(2(n+1)^2+5^2+\cfrac{3^2(5.7)}{(2(n+1)^2+7^2+}}}}$$

Proof:-

The proof is similar to the proof of Theorem 13.

Theorem 14

The correction terms for Dirichlet's series $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n^2}$ are successive convergents of the infinite continued fraction $\cfrac{\mathbf{1}}{(2n^2+2n+2)+\cfrac{4}{(2n^2+2n+10)+\cfrac{10}{(2n^2+2n+26).+\cdots\ldots}}}$

Proof:-

Proof is similar to the proof of Theorem 13.

Remark:-

For the series $\sum_{n=1}^{\infty}\ (-1)^{n-1}\, a_n$, if $G_n[k]$ denotes the k th order correction term ., then the corresponding error function is $E_n[k] = G_n[k] + G_{n+1}[k] - a_n$, for k= 1,2,3,....

5.1 RAPIDLY CONVERGENT SERIES USING CORRECTION TERM AND ERROR FUNCTIONS

Definition 15

Given a convergent sequence (s_n) having limit , an **accelerated series** is a faster convergent sequence (s_n') with same limit in the sense that $\lim_{n\to\infty} \frac{s_n'-l}{s_n-l} = 0.$

Two classical techniques used for series acceleration are **Euler's transformation of series** and **Aitken's delta squared process .**

Definition 16

The speed at which a convergent sequence approaches its limit is called **rate of convergence** of that sequence.

Theorem 17

For a convergent alternating series $\sum_{n=1}^{\infty} (-1)^{n-1} a_n$ with sum S, if G_n denotes the correction function and if E_n denotes the error function after n terms of the series , then the deduced series

a_1 - G_1) + $E_{3\_}$ …… is also convergent with same sum S .

Proof

Consider the alternating series $\sum_{n=1}^{\infty} (-1)^{n-1} a_n$ with sum S.

Let $\partial_n = a_1 - a_2 + a_3 - \quad + (-1)^{n-1}a_n + (-1)^n G_n$ where G_n denotes the correction function applied to the series after n terms.

Let $\varepsilon_n = \partial_{n+1} - \partial_n$ so that $\partial_{n+1} = \partial_n + \varepsilon_n$

Put n = 1,2,3, …..n-1 in succession in the place of n and add to get

$\partial_n = \partial_1 + \varepsilon_1 + \varepsilon_2 + 3 + \ldots + \varepsilon_{n-1}$

$= a_1 - G_1 + \varepsilon_1 + \varepsilon_2 + 3 + \ldots + \varepsilon_{n-1}$ since $\partial_1 = a_1 - G_1$

$\lim_{n\to\infty} \partial_n = a_1 - G_1 + \varepsilon_1 + \varepsilon_2 + 3 + \ldots + \varepsilon_{n-1} + \ldots$

We have $\lim_{n\to\infty} \partial_n = S$ and so $S = a_1 - G_1 + \varepsilon_1 + \varepsilon_2 + 3 + \ldots + \varepsilon_{n-1} + \ldots$

But $\varepsilon_n = (-1)^n E_n$ where E_n is the error function, $E_n = G_n + G_{n+1} - a_{n+1}$

Thus $S = = (a_1 - G_1) + \varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \ldots$ can be written as

$S = (a_1 - G_1) + E_1 - E_2 + E_3 - E_4 + \ldots$

Hence the proof.

Corollary 18

The series (a_1 - G_1) + E- E_2 +E_3- …… converges more rapidly than the original series $\sum_{n=1}^{\infty} (-1)^{n-1} a_n$

Corollary 19

If $G_n[k]$ is the k^{th} order correction function and if $E_n[k]$ is the k^{th} order error function after n terms of the series $\sum_{n=1}^{\infty} (-1)^{n-1} a_n$, then the deduced series a_1- $G_1[k]$+ $E_1[k]$ - E_2 [k] + $E_3[k]$ - …… converges …… for k ∈ N is an accelerated series.

5.2 TRANSFORMATION OF SERIES

Two classical techniques used for series acceleration are Euler transformation of series and Aitken's delta squared process.

5.2.1..EULER TRANSFORMATION OF SERIES AND DEDUCED SERIES USING CORRECTION TERM & ERROR FUNCTIONS.

Euler transformation on an Alternating series is given by

$$\sum_{n=0}^{\infty} (-1)^n a_n = \sum_{n=0}^{\infty} \frac{\Delta^n a_0}{2^{n+1}} \quad \text{where} \quad \Delta^k a_0 = \sum_{m=0}^{k} (-1)^m kC_m a_{k-m}$$

The series on the right side is also convergent and has the same sum as . $\sum_{n=0}^{\infty} (-1)^n a_n$

If the original series in LHS is a slowly convergent series, the forward differences will tend to become small quite rapidly,the additional power of 2 further improves the rate at which the RHS converges. The terms on RHS become much smaller, much more rapidly, thus allowing rapid numerical summation.

Here we compare the series obtained by Euler transformation with the series deduced using correction function and error function.

(1) ALTERNATING HARMONIC SERIES

Alternating harmonic series that converges to log2 is given by

log2 = $1 - \frac{1}{2} + \frac{1}{3} -$......

By Euler transformation , the series becomes

$$S = \frac{a_0}{2} - \frac{\Delta a_0}{2^2} + \frac{\Delta^2 a_0}{2^3} - \ldots\ldots\ldots\ldots\ldots$$

$$= \frac{1}{2} - \frac{(-\frac{1}{2}}{4} + \frac{(\frac{1}{3})}{8} - \frac{(-\frac{1}{4})}{16} + \ldots\ldots\ldots$$

Let S_n be the sequence of partial sums of original series , S_n^* denote the sequence of partial sums of the transformed series by Euler and let $S_n^{[k]}$ be the sequence of partial sums of the series deduced using correction function and error function

Number of terms (n)	S_n	S_n^*
10	0.**6**456349206	**0.693**0648561

For a pre-designed accuracy, the number of terms required from the two series are shown below.

Accuracy	Number of terms from original series	Number of terms from the transformed seriesby Euler
0.**693**0971831	10000	10

We have S_{10}^* = **0.693**0648561

We shall show that the series deduced using correction function and error function is rapidly convergent than the series transformed by Euler.

From the series deduced , we have

$S_{10}^{[1]}$ = 0.6932530683

$S_5^{[2]}$ = **0.6931**693989

$S_2^{[3]}$ = **0.693**333333

The result is tabulated as follows.

S_{10}^*	$S_5^{[2]}$	$S_2^{[3]}$
0.6930648561	**0.6931**693989	**0.693**333333

The table shows that the series deduced using correction functions and error function is rapidly convergent than the series transformed by Euler. The rapidity of convergence is increased with the increase in order of correction function.

(2). **<u>LEIBNITZ-GREGORY SERIES</u>**

The Leibniz –Gregory series that converges to is given by

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \ldots\ldots$$

By Euler transformation , the series becomes

$$S = \frac{a_0}{2} - \frac{\Delta a_0}{2^2} + \frac{\Delta^2 a_0}{2^3} - \ldots\ldots\ldots\ldots\ldots$$

$$= \frac{1}{2} - \frac{(-\frac{2)}{3}}{4} + \frac{\frac{8}{15}}{8} - (-\frac{\frac{16}{35}}{16}) + \ldots\ldots\ldots\ldots\ldots$$

Let S_n be the sequence of partial sums of original series, S_n^* denote the sequence of partial sums of the transformed series by Euler and let $S_n^{[k]}$ be the sequence of partial sums of the series deduced using correction function and error function.

Number of terms (n)	S_{10}	S_6^*
10	**3.**0418396189	**3.1**51568252

For a predesigned accuracy, the number of terms required from the two series are shown below.

Accuracy	Number of terms from original series	Number terms from the transformed series by Euler
3.1315929035	100	6

We have $S_6^* =$ **3.1**51568252

We shall show that the series deduced in section 2 using correction function and eroor function is rapidly convergent than the series transformed by Euler

From the series deduced in chapter 5 , we have

$\mathbf{S_{10}^{[1]}}$ **= 3.14**2657343

$\mathbf{S_6^{[2]}}$ = **3.1415**63418

$\mathbf{S_2^{[3]}}$ = **3.14**2857143

The result is tabulated as follows.

S_6^*	$S_6^{[2]}$	$S_2^{[3]}$
3.151568252	**3.1415**63418	**3.14**2857143

The table shows that the series deduced correction functions and error function is rapidly convergent than the series transformed by Euler. The rapidity of convergence is increased with the increase in order of correction function.

5.2.2 ..Aitken's delta squared process.

It is a series acceleration method used for accelerating the rate of convergence of a sequence. It was introduced by Alexander Aitken in 1926. It is a non- linear sequence transformation.

Let S_n ^ denote the sequence of partial sums of a series obtained by sequence transformation defined by Aitken's delta squared process.

$S_n^{\wedge} = s_{n+2} - \frac{(S_{n+2}-S_{n+1})^2}{S_{n+2}-2S_{n+1}+S_n}$ where S_n denote the sequence of partial sums of original series.

This transformation is commonly used to improve the rate of convergence of a slowly convergent sequence. It eliminates the largest part of the possible error.

(1) **ALTERNATING HARMONIC SERIES**

Let S_n be the sequence of partial sums of original series, $S_n^{\wedge}$ denote the sequence of partial sums of the transformed series by Aitken and let $S_n^{[k]}$ be the sequence of partial sums of the series deduced using correction function and error function

For n= 10, $S_{10}^{\wedge} = S_{12} - \frac{(S_{12}-S_{11})^2}{S_{n122}-2S_{11}+S_{10}}$

$S_{10}^{\wedge}$ = **0.693**065751

Number of terms (n)	S_{10}	$S_{10}^{\wedge}$
10	0.**6**456349206	**0.693**065751

For a pre-designed accuracy, the number of terms required from the two series are shown below.

Accuracy	Number of terms from original series	Number of terms from the transformed series by Aitken
0.693065751	10000	10

We shall show that the series deduced in section 2 using correction function and eroor function is rapidly convergent than the series transformed by Aitken's delta squared process.

From the series deduced in section 2 of this chapter, we have

$\mathbf{S_{10}^{[1]}}$ = **0.693**2530683

$\mathbf{S_5^{[2]}}$ = **0.6931**693989

$\mathbf{S_2^{[3]}}$ = **0.693**333333

The result is tabulated as follows.

$S_{10}^{\wedge}$	$S_5^{[2]}$	$S_2^{[3]}$
0.693065751	**0.6931**693989	**0.693**333333

The table shows that the series deduced correction functions and error function is rapidly convergent than the series transformed by Euler. The rapidity of convergence is increased with increase in order of convergence .

(2)**LEIBNITZ-GREGORY SERIES**

Let S_n be the sequence of partial sums of original series, $S_n^{\wedge}$ denote the sequence of partial sums of the transformed series by Euler and let $S_n^{[k]}$ be the sequence of partial sums of the series deduced using correction function and error function

For n= 10, $S_{10}^{\wedge} = S_{12}$

$S_{10}^{\wedge} =$ **3.141**2

Number of terms (n)	S_{10}	$S_{10}^{\wedge}$
10	**3**.0418396189	**3.141**2

We shall show that the series deduced using correction term and error function is rapidly convergent than the series transformed by Aitken's delta squared process.

From the series deduced using correction term and error functions, , we have

$\mathbf{S_{10}^{[1]}} =$ **3.14**2657343

$\mathbf{S_6^{[2]}} =$ **3.141**563418

$\mathbf{S_2^{[3]}} =$ **3.14**2857143

The result is tabulated as follows.

$S_{10}^{\wedge}$	$S_6^{[2]}$	$S_2^{[3]}$
3.1412	**3.141**563418	**3.14**2857143

The table shows that the series deduced correction functions and error function is rapidly convergent than the series transformed by Aitken. The rapidity of convergence is increased with the increase in order of correction function.

For a pre-designed accuracy, the number of terms required from the two series are shown below.

Accuracy	Number of terms from original series	Number terms from the transformed series by Aitken
3.1412	10000	10

The table shows that the series deduced by using correction functions and error function is rapidly convergent than the series transformed by Aitken. The rapidity of convergence is increased with increase in order of convergence .

CONCLUSION;-

In this paper we have discussed about the extraction of an accelerated series using correction functions and error functions. Also we have analysed the rate of convergence of this accelerated series with series obtained by Euler transformation method and Aitken's delta squared process. We see that the deduced series using correction term and error functions gives better approximation.

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