DETOUR SELF-DECOMPOSITION OF GRAPHS

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Abstract

Every vertex x of a graph that is connected has two corresponding vertices u and v in the vertex subset S such that x is on some $u-v$ detour paths in G. This set S is named as a detour set and the detour number of G, indicated by the symbol $dn(G)$, is the least cardinality of a detour set. If every subgraph $G_i, 1 \leq i \leq n$ of G has the same detour number as the graph G, then G has a detour self-decomposition. The detour self-decomposition number of a graph G, denoted as $\pi_{sdn}(G)$, is the highest cardinality of the detour self-decomposition $\Pi = (G_1, G_2, \ldots, G_n)$. This decomposition's few bounds and a few general features are investigated here.

Keywords: Detour number, Detour self-decomposition, Detour self-decomposition number.

AMS Subject Classification: 05C12, 05C99.

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INTRODUCTION

The graphs utilized in this work are all finite, simple, connected and undirected $G = (V, E)$ graphs. We refer [4] for introductory terms in graph theory. G. Chartrand, G.L. Johns, and P. Zhang [1] introduced the idea of detour number of graphs. In G between any two vertices u and v the longest $u-v$ path's length is the detour distance $D(u, v)$. All vertices located on a $u-v$ detour of G are included in the closed interval $I_D[u, v]$, whereas for $S \subset V, I_D[S] = \cup_{\{u,v \in S\}} I_D[u, v]$. A vertex subset S is a detour set if $I_D[S] = V$ and the cardinality of smallest such detour set is the detour number $dn(G)$. In other words, if x lies on P which is an $u-v$

detour in G, then x is said to be on a $u - v$ detour P. If every vertex v in G lies on a detour path connecting a pair of vertices of S, then the set $S \subset V$ is referred to as a detour set. This number has application in Channel Assingment problems in radio technologies[10].

Graph decomposition is an interesting area of research due to its contributions in Structural graph theory and Combinatorics [8]. A graph decomposition $(G_1, G_2, \ldots, G_n)$ of G is a set of subgraphs of whose edge set is disjoint and each edge of G is a part of exactly one $G_i, 1 \leq i \leq n$ [6]. In literature various graph decompositions have been studied by giving some restrictions to the above definition. The idea behind H-decomposition was introduced by P. Erdos, A. W. Goodman and L. Posa [3] and various problems related to H-decomposition has been studied in recent years. E.E.R. Merly and M. Mahiba's "Steiner decomposition number of graphs" [7] and J.John and D.Stalin's "Edge geodetic self-decomposition in graphs" [5] concepts motivated to introduce the concept detour self-decomposition of graphs. Throughout this paper S_n denote the star graph $K_{1,n}$ with n −edges.

Theorem: 1.1 [1]

Every end vertex of a non-trivial connected graph G belongs to every detour set of G.

Theorem:1.2 [1]

If T is a tree with k end-vertex, then $dn(T) = k$.

Theorem:1.3 [9]

For any graph G of order $p \geq 2, 2 \leq dn(G) \leq p$.

II. METHODOLOGY

Definition: 2.1

Detour self-decomposition in the graph G is a decomposition $\Pi = (G_1, G_2, \ldots, G_n)$ in G with $dn(G) = dn(G_i), 1 \leq i \leq n$. The greatest cardinality of such Π is represented by the symbol $\pi_{sdn}(G)$ is the detour self decomposition number of G .

Example: 2.2

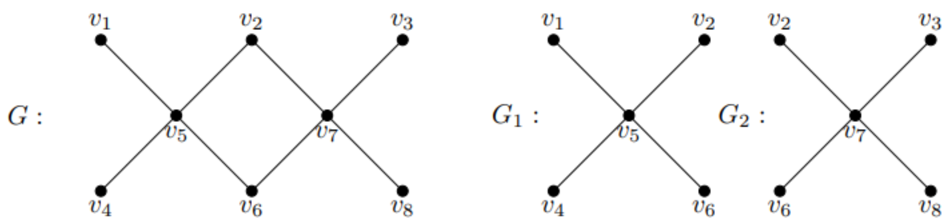

Figure:2.1 A graph G and its detour self-decomposition

Consider the graph G. Here $S = \{v_1, v_3, v_8, v_4\}$ is the smallest detour set, so $dn(G) = 4$. Also the smallest detour set of G_1 is $\{v_1, v_2, v_4, v_6\}$ and the smallest detour set of G_2 is$\{v_2, v_3, v_6, v_8\}$. Therefore $dn(G_1) = dn(G_2) = dn(G) = 4$. Hence $\Pi = (G_1, G_2)$ is a detour self-decomposition of G.

III. RESULTS AND DISCUSSION

Theorem:3.1

For every graph G with $p > 2$ and $dn(G) = 2$ has detour self-decomposition $(G_1, G_2, \dots, G_n)$ if and only if $\pi_{sdn}(G) = q$.

Proof:

Consider G as a graph with $p > 2$.

Assume that $dn(G) = 2$. Since G is connected, any pair of vertices u, v in G is connected by a $u - v$ path. Also $dn(P_n) = 2, n \geq 2$. Since G contains more than two vertices it has more than one edge and each edge is isomorphic to P_2. Clearly each edge is a subgraph of G , so we get a detour self-decomposition $(G_1, G_2, \dots, G_q)$ of G. Hence G admits detour self-decomposition and $\pi_{sdn}(G) = q$.

Conversely, assume that G admits detour self-decomposition and $\pi_{sdn}(G) = q$. Since $\pi_{sdn}(G) = q$, each $G_i \cong P_2$ and $dn(G_i) = 2$, then $dn(G_i) = 2, 1 \leq i \leq q$. Since $dn(G) = dn(G_i)$, we get $dn(G) = 2$.

Corollary: 3.2

If G, a graph with $p > 2$, has a Hamiltonian path then G has detour self-decomposition and $\pi_{sdn}(G) = q$.

Proof:

Consider G as a graph with $p > 2$.

Suppose a Hamiltonian $u - v$ path exists on G. This Hamiltonian path is the detour path of G. The set $S = \{ u, v\}$ satisfies the condition for the detour set and hence $dn(G) = 2$.

From the theorem: 3.1, G admits detour self-decomposition and $\pi_{sdn}(G) = q$.

Corollary: 3.3

A Hamilton graph of order $p > 2$ has detour self-decomposition and its detour self-decomposition number is q.

Proof:

Consider G as a Hamilton graph of order $p > 2$. Then a Hamiltonian cycle is present in G, which implies a Hamiltonian path is present in G. From corollary: 3.2, G admits detour self-decomposition and $\pi_{sdn}(G) = q$.

Theorem:3.4

The inequality $1 \leq \pi_{sdn}(G) \leq q$ holds for all graph G with $p \geq 2$.

Proof:

Consider G as a graph with $p > 2$. Suppose G cannot be decomposed into subgraphs with same detour number, then $\pi_{sdn}(G) = 1$. Otherwise $\pi_{sdn}(G) > 1$. Hence $\pi_{sdn}(G) \geq 1$. From theorem: 3.1, when $dn(G) = 2,\ \pi_{sdn}(G) = q$ and this is the maximum possibility for any graph G.

Hence $1 \leq \pi_{sdn}(G) \leq q$.

Remark 1: For a given detour number $n > 2$, the only possible connected graph with the least number of cardinality of the edge set is S_n.

Theorem:3.5

For every graph G with $dn(G) > 2, \pi_{sdn}(G) = \frac{q}{dn(G)}$ if and only if $G_i = S_{dn(G)}$, for all i.

Proof:

Consider G where $dn(G) > 2$.

Suppose $G_i = S_{dn(G)}$, for all i. Since $dn(S_{dn(G)}) = dn(G)$, G is detour self-decomposable and $\pi_{sdn}(G) = \frac{q}{dn(G)}$.

The converse is obvious.

Theorem:3.6

For every graph G with $dn(G) > 2, \pi_{sdn}(G) \leq \left\lfloor \frac{q}{dn(G)} \right\rfloor$

Proof:

From remark 1, the least number of edges required in each subgraph for the detour self-decomposition is either equal or more than $dn(G)$.

Hence $\pi_{sdn}(G) \leq \left\lfloor \frac{q}{dn(G)} \right\rfloor$.

Remark 2: If G with $diam(G) = 2$ has cut vertex v then $deg(v) = p - 1$ *i.e.*, each vertex of G is a neighbour of v.

Lemma:3.7

Every graph G with $diam(G) = 1$ and $p > 1$ has a detour number 2.

Proof:

Since G contains atleast two vertices and $diam(G) = 1$, each vertex of G is a neighbour to the remaining vertices of G. Then $G \cong K_p$. Every complete graph with more than one vertex is a Hamiltonian graph. Then $dn(G) = 2$.

Theorem:3.8

Consider G as a graph with $diam(G) = 2$ and a cutvertex v. Let each components of $G - \{v\}$ have atleast two vertices and detour number 2. If $\Delta(G)$ is a multiple of $dn(G)$, then G is detour self-decomposable and $\pi_{sdn}(G) \geq \frac{\Delta(G)}{dn(G)}$.

Proof:

Suppose G is a graph with $diam(G) = 2$ and a cut vertex v. Let the components of $G - \{v\}$ be $H_1, H_2, \ldots, H_m$. Given each H_i has more than one vertex and $dn(H_i) = 2$, for all i. Let $V(H_i) = \{u_{i,1}, u_{i,2}, \ldots, u_{i,r_i}\}$, where $r_i = |V(H_i)|, 1 \leq i \leq m$. Since $diam(G) = 2$ and v is a cut vertex of G, each vertex of G is a neighbour to v.

Claim 1: $dn(G) = m$

For this we have to first prove that the detour set of the graph G has atleast one vertex from every components of $G-\{v\}$. Suppose there exists a detour set S that contains no vertex of H_1 (say). Then $V(H_1) \subseteq I_D[S]$. Therefore $u_{i,1} \in I_D[S]$. Let P be the detour path containing $u_{i,1}$. In this path P, the cut vertex v lies twice which means P cannot be a path so this is a contradiction. So the set S contains atleast one vertex from every H_i. Since $dn(H_i) = 2, 1 \leq i \leq m$, there exist a set $\{u_{i,a}, u_{i,b}\}$ where $a, b \in \{1,2,\dots,r_i\}$ such that $I_D[u_{i,a}, u_{i,b}] = V(H_i)$, for all i. Rename the vertices of $H_i, 1 \leq i \leq m$ so that the detour set of H_i is named as $u_{i,1}$ and $u_{i,2}$. Also every vertex of G has v as a neighbour. Thus in G between two components H_i, H_j there exists some $u_{i,1} - u_{i,2} - v - u_{j,2} - u_{j,1}$ detour path that contains all the vertices of H_i, H_j and $\{v\}$ in G. In this way, taking the set S=$\{u_{1,1}, u_{2,1}, \dots, u_{m,1}\}$ as the detour set and this set is minimum. Hence $dn(G) = m$.

Since every vertex of G is a neighbour of the the cut vertex v, v has the maximum degree, hence $deg(v) = n \times dn(G)$, where $n \in \mathbb{N}$.

Now, decompose G as follows:

First construct G_1 by taking $H_1, H_2, \dots, H_m$ and joining them by adding the edges $vu_{i,1}, 1 \leq i \leq m$.

Claim 2: $dn(G_1) = m$.

Since G_1 is a subgraph of graph G, the detour set of G_1 has atleast one vertex from each $H_i, 1 \leq i \leq m$. Since $u_{i,1}, u_{i,2}$ is the detour set of H_i, for all i, in G_1 there exist some $u_{i,2} - u_{i,2} - v - u_{j,1} - u_{j,2}$ detour path that contains all the vertices of H_i, H_j and $\{v\}$ in the graph G_1. In this way, we take the set $\{u_{1,2}, u_{2,2}, \dots, u_{m,2}\}$ as the detour set of G_1 and this set is minimum.

Hence $dn(G_1) = m$.

Let $G' = G - G_1$.Then $G' \cong S_{(n-1)m}$. Therefore G' can be decomposed into $(n-1)$ copies of S_m and $dn(S_m) = m$. These $(n-1)$ copies of S_m is named from G_2 to G_n. Thus G is detour self-decomposable. Since H_i is any graph with detour number 2 and have more than one vertex, sometimes it can further be decomposed into subgraphs having detour number m. Thus $\pi_{sdn}(G) \geq \frac{\Delta(G)}{dn(G)}$.

Theorem:3.9

For any $a, b \in \mathbb{N}$ and $a > 1$ there exists a graph which is connected with $dn(G) = a$ and $\pi_{sdn}(G) = b$

Proof:

We prove this theorem in three cases.

Case 1: $a > 1$ *and* $b = 1$

The star graph S_a has detour number a and it has minimum number of edges so it cannot be further decomposed into a subgraph with the same detour number. Hence $\pi_{sdn}(G) = 1 = b$.

Case 2: $a = 2k$*(even) and* $b > 1$.

We try to construct G from b stars.

Let $H_1, H_2, \ldots, H_b$ be the stars with a pendant vertices and let $V(H_i) = \{ u_i, v_{(i-1)k+1}, v_{(i-1)k+2}, \ldots, v_{(i-1)k+k}, v_{ik+1}, v_{ik+2}, \ldots, v_{ik+k}\}, 1 \leq i \leq b$ and $G = \bigcup_{i=1}^{b} H_i$. Then G is shown in figure: 3.1.

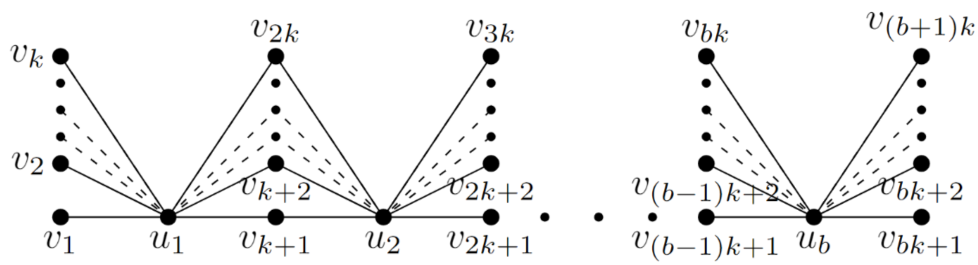

Figure:3.1 A graph G with $dn(G) = a = 2k$ (even) and $\pi_{sdn}(G) = b$

From figure:3.1, the set $v_1, v_2, \ldots, v_k, v_{bk+1}, v_{bk+2}, \ldots, v_{(b+1)k}$ is the set of pendant vertices of G. By theorem:1.1, every detour set of G contains this set but this set itself satisfies the condition for detour set. Thus the above set is minimum detour set and hence $dn(G) = 2k = a$. Also

$dn(H_i) = dn(G), 1 \leq i \leq b$. Since there are ab edges in G and each subgraph is a star by theorem:3.5, $\pi_{sdn}(G) = b$.

Case3: $a = 2k + 1$*(odd),* $b > 1$

Subcase(i) : $b = 2m$*(even)*

We try to construct G by taking b stars with a pendant vertices. Let $H_1, H_2, \ldots, H_b$ be the stars with a pendant vertices. We have $b = 2m$, so take two vertices u_1, as the center vertex of the graphs $H_i, 1 \leq i \leq m$ and u_2, as the center vertex of the graphs $H_i, m+1 \leq i \leq b$. Thus we have two stars $\bigcup_{i=1}^{m} H_i$ and $\bigcup_{i=m+1}^{b} H_i$with am end vertices. Let the star with center vertex u_1 be named as S'_a and with center vertex u_2 be named as S_a''.

To construct G, let $V(S'_a) = \{u_1, v_{1,1}, v_{1,2}, \ldots, v_{1,k+1}, v_{k+2}, v_{k+3}, \ldots, v_{am}\}$ and

$$V(S''_a) = \{u_2, v_{2,1}, v_{2,2}, \ldots, v_{2,k+1}, v_{k+2}, v_{k+3}, \ldots, v_{am}\}$$

Then $S'_a \cup S''_a$ will be as shown in figure:3.2.

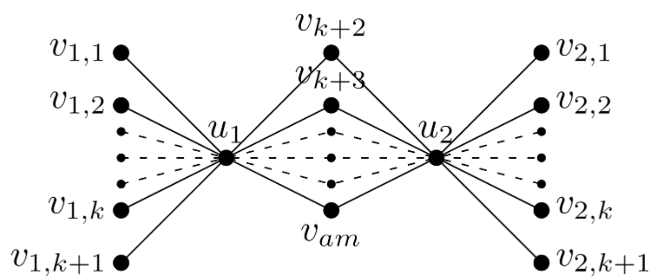

Figure: 3.2 $S'_a \cup S''_a$

The pendant vertices of $S'_a \cup S''_a$ is $\{v_{1,1}, v_{1,2}, \ldots, v_{1,k+1}, v_{2,1}, v_{2,2}, \ldots, v_{2,k+1}\}$ and clearly this set satisfies the condition for detour set, by theorem:1.1 this set is minimum and hence $dn(S'_a \cup S''_a) = 2k + 2$. But by adding an edge e between the vertex $v_{1,k+1}$ and the vertex u_2 to the above graph the number of pendant vertices reduces by one while the set of pendant vertices again satisfies the condition for detour set and so the resulting graph has detour number $2k + 1 = a$. In this way we can construct a graph G with $dn(G) = a$ as shown in figure:3.3.

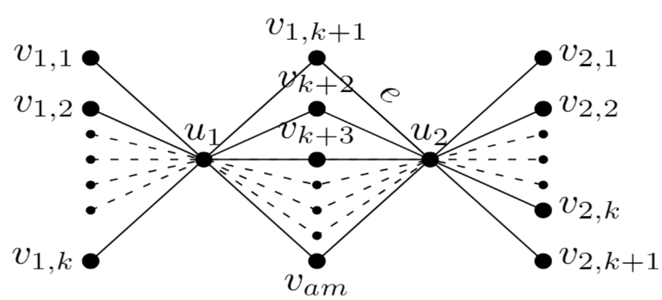

Figure:3.3 A graph G with $dn(G) = a = 2k + 1$(odd) and $\pi_{sdn}(G) = b = 2m$(even)

Now decompose G as follows:

$G_1 = H_1 + e$ and $G_i = H_i$, $2 \leq i \leq b$.

Clearly $dn(G_i) = a, 1 \leq i \leq b$.

Thus G is decomposed into b subgraphs$(G_1, G_2, \ldots, G_b)$. Also the number of edges of G is $ab + 1$ so the maximum bound in theorem:3.6 is attained.

Subcase(ii) : $b = 2m + 1$(odd)

Since S_a has minimum number of edges with detour number a, we try to construct G by taking b stars. Let $H_1, H_2, \ldots, H_b$ be the stars with a pendant vertices. We have $b = 2m + 1$, so take two vertices u_1 , as the center vertex of the subgraphs $G_i, 1 \leq i \leq m + 1$ and u_2, as the center vertex of the subgraphs $G_i, m + 2 \leq i \leq b$. Then we have two stars $\cup_{i=1}^{m+1} H_i$ and $\cup_{i=m+2}^{b} H_i$ with $a(m + 1)$ and am number of end vertices. Let the star with center vertex u_1 be named as S'_a and with center vertex u_2 be named as S''_a.

To construct G, let $V(S'_a) = \{u_1, v_{1,1}, v_{1,2}, \ldots, v_{1,a}, v_1, v_2, \ldots, v_{am}\}$,

$V(S''_a) = \{u_2, v_1, v_2, \ldots, v_{am}\}$. Then $S'_a \cup S''_a$ will be as shown in figure 3.4.

The pendant vertices of the above graph is $\{v_{1,1}, v_{1,2}, \ldots, v_{1,a}\}$ but this set does not satisfy the condition for detour set while addition of a vertex u_2 to the above set satisfies the condition for detour set and the obtained set will be minimum so $dn(S'_a \cup S''_a) = a + 1$. But by adding an edge e between the vertices $v_{1,a}$ and u_2 to the above graph the number of pendant vertices reduces by one while the set of pendant vertices together with vertex u_2 again satisfies the condition for detour set and so the graph produced has detour number $(a-1)+1 = a$. In this way a graph G with $dn(G) = a$ can be constructed as shown in figure:3.5.

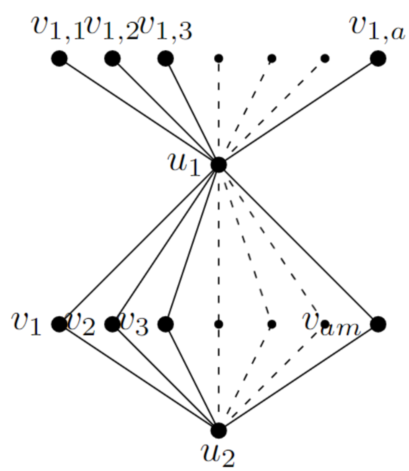

Figure: 3.4 $S'_a \cup S''_a$

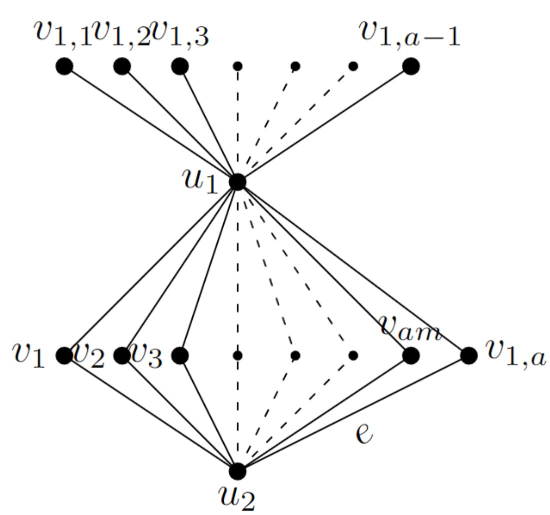

Figure: 3.5 A graph G with $dn(G) = a = 2k + 1$(odd) and $\pi_{sdn}(G) = b = 2m + 1$(odd)

Decompose G as follows:

$$G_1 = H_1 + e \text{ and } G_i = H_i, 2 \leq i \leq b$$

Clearly $dn(G_i) = a, 1 \leq i \leq b$. Thus G is decomposed into b subgraphs $(G_1, G_2, \dots, G_b)$. There are $ab + 1$ edges in G so the maximum bound in theorem 3.6 is attained. Hence the theorem.

IV. CONCLUSION

This work proposes a different decomposition parameter based on the detour number. Few bounds for the detour self-decomposition number and some properties of the graphs that satisfy such decomposition are discussed. Also a realization theorem for such decomposition is obtained. Further research can be obtained by analyzing the graphs satisfying this decomposition with other graph parameters.

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