Fixed points results for α_* −admissible mapping with new approach of (α_*, ψ_{S_b}) −contractions in complete S_b −metric spaces with application to Fredholm integral equation

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Abstract

The aim of this manuscript is to prove some fixed point results for α_* −admissible mapping with a new approach of (α_*, ψ_{S_b}) −contraction in complete S_b −metric space. As a work that extends and enhances the existing investigation of α_* − admissible mappings with regard to ψ. finally, in order to demonstrate the relevance of our ideas and to bolster our findings, we also provide an application to the Fredholm integral equation and the existing solution of functional equations.

Keywords: Fixed points; contraction mappings; S_b −metric space;

1. Introduction

Due to the advancement and generalization of several conclusions in fixed point theory, scientists are now paying close attention to the generalization of metric spaces in diverse forms. Recent research has established fixed point theorems in full S_b −metric space and presented the idea of a S_b −metric space as a generalization of b −metric spaces and S −metric spaces [12]. However, Tas and Ozur [10] just recently investigated several relationships between S_b −metric spaces and other metric spaces. References to groundbreaking work on fixed point theorems for mappings on S_b −metric spaces that meet specific contractive requirements may be found in [6,8,11,12,13,15,16].

Three new concepts triangular α −orbital admissible, α −orbital admissible, and α −orbital attractive mappings were developed by Haji [3] in 2022. The idea of triangular

α −orbital admissible mappings with respect to $\eta -$ was first suggested in 2016 by Chuadchawna $et\ al.$ [2]

In the context of S_b −metric spaces and $\alpha_* -$ admissible mappings with regard to ψ. this paper aims to demonstrate unique fixed point theorems for generalized (α_*, ψ_{S_b}) −contractions self-mapping. Additionally, we may provide relevant applications for Fredholm integral equation.

2. Preliminaries and definitions

In 2012, sedghi, shobe and Aliouche [10] introduced S −metric on nonempty set X as follows.

Definition 2.1 ([**10**]) Let X be a nonempty set. A S −metric on X is a function $S: X^3 \to [0, \infty)$ that satisfies the following conditions:

(1.1) $0 < S(x, y, z), \forall\, x, y, z \in X, with\ x \neq y \neq z.$

(1.2) $S(x, y, z) = 0\ if\ x = y = z$

(1.3) $S(x, y, z) \leq [S(x, x, a) + S(y, y, a) + S(z, z, a)\ \forall\, x, y, z, a \in X,$

The pair (X, S) is called S −metric space.

Definition 2.2. ([**8**]) Let X be a non empty set and $s \geq 1$ be a real number. A function $d : X \times X \times X \to [0, \infty)$ is said to be b −metric on X if and only if satisfies the following properties.

(1) $d(x, y) = 0\ if\ and\ only\ if\ x = y.$

(2) $d(x, y) = d(y, x), \forall\, x, y \in X,$

(3) $(x, y) \leq s[d(x, z) + d(z, y)]\ \forall\, x, y, z \in X]$

Then the order pair (X, d) is called a b −metric space with $s \geq 1.$

Here we note that the class of b −metric spaces is larger class than the class of metric spaces, since (X, d) is a metric when s=1.

Inspired by the work of Bakhtin [2] and Sedghi et al., [10] Souayah and Mlaiki [11] introduced the concept of S_b −metric space. Tas and Ozur [12] modified the definition of S_b −metric spaces.

Definition 2.3. ([**8**]) Let X be a non empty set and $s \geq 1$ be a given real number. A function $S_b : X \times X \times X \to [0, \infty)$ is said to be S_b −metric if and only if $\forall\, x, y, z, t \in X$: the following properties hold:

(S_b1) $S_b(x, y, z) = 0 \text{ if and only if } x = y = z,$

(S_b2) $S_b(x, x, y) = S_b(y, y, x), \forall\, x, y \in X,$

(S_b3) $S_b(x, y, z) \leq s[S_b(x, x, t) + S_b(y, y, t) + S_b(z, z, t), \forall\, x, y, z, t \in X]$

The pair (X, S_b) is called a S_b −metric space.

Remark 2.4. Note that the class of S_b −metric spaces is larger than the class of $S-$ metric spaces. Indeed, every S −metric space is and S_b −metric space with $s = 1$. However, the converse is not always true.

Definition 2.5. Let (X, S_b) be an S_b −metric space and $\{x_n\}_{n=0}^{\infty}$ be a sequence in X. Then

i. A sequence $\{x_n\}_{n=0}^{\infty}$ is called convergent if and only if there exists $z \in X$ such that $S_b(x_n, x, z) \to 0$, as $n \to \infty$. in this case we write $\lim_{n\to\infty} x_n = z$.
ii. A sequence $\{x_n\}_{n=0}^{\infty}$ in (X, S_b) is called as S_b −cauchy sequence if for all $\in > 0$, there exists $n \in \mathbb{N}$ such that $S_b(x_n, x_n, x_m) \to 0$ for all $n, m \to \infty$.
iii. A space X is said to be S_b −converges if for every S_b −cauchy sequence in X is S_b −convergent.
iv. (X, S_b) is said to be a complete S_b −metric space if every Cauchy sequence $\{x_n\}_{n=0}^{\infty}$ converges to a point $x \in X$ such that

$$\lim_{n,m\to\infty} S_b(x_n, x_n, x_m) = \lim_{n,m\to\infty} S_b(x_n, x_n, x) = S_b(x, x, x).$$

3. Main results

Definition 3.1. ([1]) Denote by Ψ the family of nondecreasing functions $\psi: [0, \infty) \to \psi: [0, \infty)$ such that $\sum_{n=1}^{+\infty} \psi^n(t) < +\infty$ for each $t > 0$, where ψ^n is the n −th iterate of ψ.

Now we define the (α_*, ψ_{S_b}) −contractions self mapping in S_b −metric space.

Definition 3.2. Let T be a self mapping on a complete S-metric spaces (X, S). we say that T is α_*, ψ_{S_b} −contraction self mapping if there exists a function $\boldsymbol{\alpha}_*: X^3 \to [0, \infty)$ and $\psi \in \Psi$ such that for all $x, y \in X$ we have

$$\boldsymbol{\alpha}_*(x, y, z) S_b(Tx, Ty, Tz) \leq \psi_{S_b}\big(S_b(x, y, z)\big).$$

Definition 3.3. Let (X, S) be a S_b −metric space and T be a self mapping on X. we say that T is α_* −admissible if $x, y \in X, \alpha_*(x, y, z) \geq 1$ implies that $\alpha_*(Tx, Ty, Tz) \geq 1$.

Also, this next lemma is very useful for our purpose.

Lemma 3.4. Let $T : X \to (X)$ be α_* −admissible mapping with respect to ψ_{S_b}. assume that there exists $x_0 \in X$ such that $\alpha_*(x_0, Ta_1) \geq \psi_{S_b}(x_0, Ta_1)$. Define a sequence $\{x_n\}$ by $x_{n+1} = Ta_1$. then we have $\alpha_*(x_n, x_m) \geq \psi_{S_b}(x_n, x_m)\ \forall\ m, n \in N$ with $n < m$.

Proof. Since $\alpha_*(x_0, x_0, Ta_1) \geq \psi_{S_b}(x_0, x_0, Ta_1)$ and T is α_* −admissible mapping with respect to ψ_{S_b}. we obtain that

$$\alpha_*(x_1, x_1, a_2) = \alpha_*(Tx_1, Tx_1, T(Ta_1)) \geq \psi_{S_b}(Tx_1, Tx_1, T(a_1)) = \psi_{S_b}(x_2, x_2, a_3)$$

By continuing the process as above, we have $\alpha_*(x_n, x_n, a_{n+1}) \geq \psi_{S_b}(x_n, x_n, a_{n+1})\ \forall\ n \in N$. suppose that

$$\alpha_*(x_n, x_n, a_m) \geq \psi_{S_b}(x_n, x_n, a_m) \qquad (3.4.1)$$

And we will prove that

$$\alpha_*(x_n, x_n, a_{m+1}) \geq \psi_{S_b}(x_n, x_n, a_{m+1})\ .where\ m > n.$$

Since $\alpha_*(x_m, x_m, a_{m+1}) \geq \psi_{S_b}(x_m, x_m, a_{m+1})$. we obtain that

$$\alpha_*(x_m, x_m, Ta_{m+1}) = \alpha_*(x_m, x_m, a_{m+1}) \geq \psi_{S_b}(x_m, x_m, a_{m+1}) \qquad (3.4.2)$$

By (3.4.1), (3.4.2) and α_* −admissible of T, we have

$\alpha_*(x_n, x_n, Ta_m) \geq \psi_{S_b}(x_n, x_n, Ta_m)$. This implies that $\alpha_*(x_n, x_n, a_{m+1}) \geq \psi_{S_b}(x_n, x_n, a_{m+1})$.

Hence $\alpha_*(x_n, x_m) \geq \psi_{S_b}(x_n, x_m)\ \forall\ m, n \in N$ with $n < m$.

Theorem 3.5. Let (X, S_b) be a complete S_b −metric space. Suppose that (α_*, ψ_{S_b}) contraction and α_* − admissible mapping $T : X \to (X)$ satisfies the following conditions:

(i) T is α_* −admissible;

(ii) There exist $x_0 \in X$ such that $\alpha_*(x_0, x_0, a_1) \geq 1$ for each $a_1 \in T(x_0)$,

(iii) If $\{x_n\} \in X$ is a sequence such that $\alpha_*(x_n, x_{n+1}, Tx_{n+1}) \geq 1\ \forall\, n\ and\ \{x_n\} \to x \in X$ as $n \to \infty$, then $\alpha_*(x_n, x, x) \geq 1\ \forall\, n \in \mathbb{N}$. then T has a fixed point.

Proof. Let $x_1 \in T(x_0)$ then by the hypothesis (ii) $\alpha_*(x_0, x_0, a_1) \geq 1$. from Lemma 3.4 there exist $x_2 \in T(x_1)$ such that

$$S_b(x_1, x_1, x_2) \leq S_b\big(T(x_0), T(x_0), T(x_1)\big) + \psi_{S_b}\big(S_b(x_0, x_0, x_1)\big)$$

Similarly, there exist $x_3 \in T(x_2)$ such that

$$S_b(x_2, x_2, x_3) \leq S_b\big(T(x_1), T(x_1), T(x_2)\big) + \psi^2{}_{S_b}\big(S_b(x_0, x_0, x_1)\big)$$

Following the similar arguments, we obtain a sequence $\{x_n\} \in X$ such that there exist $x_{n+1} \in T(x_n)$

Satisfying the following inequality

$$S_b(x_n, x_n, x_{n+1}) \leq S_b\big(T(x_{n-1}), T(x_{n-1}), T(x_n)\big) + \psi^n{}_{S_b}\big(S_b(x_0, x_0, x_1)\big) \qquad (3.5.1)$$

Since T is α_* −admissible, therefore $\alpha_*(x_0, x_0, x_1) \geq 1 \Longrightarrow \alpha_*(x_1, x_1, x_2) \geq 1$. using mathematical induction, we get

$$\alpha_*(x_n, x_n, x_{n+1}) \geq 1 \qquad (3.5.2)$$

By (3.5.1) and(3.5.2), we have

$$S_b(x_n, x_n, x_{n+1}) \leq S_b\big(T(x_{n-1}), T(x_{n-1}), T(x_n)\big) + \psi^n{}_{S_b}\big(S_b(x_0, x_0, x_1)\big)$$

$$\leq \alpha_*(x_n, x_n, x_{n+1}) S_b\big(T(x_{n-1}), T(x_{n-1}), T(x_n)\big) + \psi^n{}_{S_b}\big(S_b(x_0, x_0, x_1)\big)$$

$$\leq \psi_{S_b}\big(S_b(x_{n-1}, x_{n-1}, x_n)\big) + \psi^n{}_{S_b}\big(S_b(x_0, x_0, x_1)\big)$$

$$= \psi_{S_b}\Big[\big(S_b(x_{n-1}, x_{n-1}, x_n)\big) + \psi^{n-1}{}_{S_b}\big(S_b(x_0, x_0, x_1)\big)\Big]$$

$$\leq \psi_{S_b}\begin{bmatrix}\big(S_b(T(x_{n-2}), T(x_{n-2}), T(x_{n-1})\big) \\ +\psi^{n-1}{}_{S_b}\big(S_b(x_0, x_0, x_1)\big)\end{bmatrix}$$

$$\leq \psi_{S_b}\begin{bmatrix}\alpha_*(x_n, x_n, x_{n+1}) S_b(T(x_{n-1}), T(x_{n-1}), T(x_n) \\ +\psi^{n-1}{}_{S_b}\big(S_b(x_0, x_0, x_1)\big)\end{bmatrix}$$

$$\leq \psi_{S_b}\begin{bmatrix}\psi_{S_b}\big(S_b(T(x_{n-2}, x_{n-2}, x_{n-1})\big) + \psi^{n-1}{}_{S_b}\big(S_b(x_0, x_0, x_1)\big) \\ +\psi^{n-1}{}_{S_b}\big(S_b(x_0, x_0, x_1)\big)\end{bmatrix}$$

$$\leq \psi^2{}_{S_b}\big(S_b(x_{n-2}, x_{n-2}, x_{n-1})\big) + 2\psi^n{}_{S_b}\big(S_b(x_0, x_0, x_1)\big)$$

...

$$S_b(x_n, x_n, x_{n+1}) \leq \psi^n{}_{S_b}\big(S_b(x_0, x_0, x_1)\big) + n\psi^n{}_{S_b}\big(S_b(x_0, x_0, x_1)\big)$$

$$S_b(x_n, x_n, x_{n+1}) \leq (n+1) + \psi^n{}_{S_b}\big(S_b(x_0, x_0, x_1)\big)$$

Let us assume that $\epsilon > 0$, then there exist $n_0 \in N$ such that

$$\sum_{n \geq n_0} (n+1)\psi^n{}_{S_b}\big(S_b(x_0, x_0, x_1)\big) < \epsilon$$

By the definition (3.3), we get

$$\lim_{n\to\infty} S_b(x_n, x_n, x_{n+1}) = 0 \qquad (3.5.3)$$

Using the above inequality and we deduce that

$$\lim_{n\to\infty} S_b(x_n, x_n, x_n) = \lim_{n\to\infty} \min\{m(x_n, x_n, x_n), m(x_{n+1}, x_{n+1}, x_{n+1})\}$$

$$= \lim_{n\to\infty} S_b(x_n, x_n, x_{n+1})$$

$$\leq \lim_{n\to\infty} S_b(x_n, x_n, x_{n+1}) = 0.$$

Owing to limit, we have $\lim_{n\to\infty} S_b(x_n, x_n, x_{n+1}) = 0$,

$$\lim_{n,m\to\infty} S_b(x_n, x_n, x_m) = 0$$

Now, we prove that $\{x_n\}$ is S_b −cauchy in X. For $m, n\ in\ N$ with $m > n$ and using triangle inequality of an S_b −metric we get

$$S_b(x_n, x_n, x_m) - \psi_{S_b}(x_n, x_n, x_m)$$

$$\leq S_b(x_n, x_n, x_{n+1}) - \psi_{S_b}(x_n, x_n, x_{n+1}) + S_b(x_{n+1}, x_{n+1}, x_m) - \psi_{S_b}(x_{n+1}, x_{n+1}, x_m)$$

$$\leq S_b(x_n, x_n, x_{n+1}) - \psi_{S_b}(x_n, x_n, x_{n+1}) + S_b(x_{n+1}, x_{n+1}, x_{n+2}) - \psi_{S_b}(x_{n+1}, x_{n+1}, x_{n+1}) + S_b(x_{n+2}, x_{n+2}, x_m) - \psi_{S_b}(x_{n+2}, x_{n+2}, 2x_m)$$

$$\leq S_b(x_n, x_n, x_{n+1}) - \psi_{S_b}(x_n, x_n, x_{n+1}) + S_b(x_{n+1}, x_{n+1}, x_{n+2}) - \psi_{S_b}(x_{n+1}, x_{n+1}, x_{n+2}) + \cdots + S_b(x_{m-1}, x_{m-1}, x_m) - \psi_{S_b}(x_{m-1}, x_{m-1}, x_m)$$

$$\leq S_b(x_n, x_n, x_{n+1}) + S_b(x_{n+1}, x_{n+1}, x_{n+2}) + \cdots + S_b(x_{m-1}, x_{m-1}, x_m)$$

$$= \sum_{r=n}^{m-1} S_b(x_r, x_r, x_{r+1})$$

$$\leq \sum_{r=n}^{m-1} (r+1)\psi^r{}_{S_b}\big(S_b(x_0, x_0, x_1)\big)$$

$$\leq \sum_{r\geq n_0}^{m-1} (r+1)\psi^r{}_{S_b}\big(S_b(x_0, x_0, x_1)\big)$$

$$\leq \sum_{r\geq n_0}^{m-1} (r+1)\psi^r{}_{S_b}\big(S_b(x_0, x_0, x_1)\big) < \epsilon$$

$$S_b(x_n, x_n, x_m) - \psi_{S_b}(x_n, x_n, x_m) \to 0, as\ n \to \infty,$$

We obtain $\lim_{m,n\to\infty}\left(S_b(x_n, x_n, x_m) - \psi_{S_b}(x_n, x_n, x_m)\right) = 0$. thus $\{x_n\}$ is a S_b −Cauchy sequence in X.

Since (X, S_b) is S_b −complete, there exist $x^* \in X$ such that

$$\lim_{n\to\infty}\left(S_b(x_n, x_n, x^*) - \psi_{S_b}(x_n, x_n, x^*)\right) = 0$$

And

$$\lim_{n\to\infty}\left(S_b(x_n, x_n, x^*) - \psi_{S_b}(x_n, x_n, x^*)\right) = 0$$

Also, $\lim_{n\to\infty} S_b(x_n, x_n, x_n) = 0$ gives that

$$\lim_{n\to\infty} S_b(x_n, x_n, x^*) = 0\ \ and\ \ \lim_{n\to\infty} \psi_{S_b}(x_n, x_n, x^*) = 0$$

$$\lim_{n\to\infty}\{max(S_b(x_n, x_n, x^*) = 0\,, \psi_{S_b}(x^*, x^*, x^*))\} = 0$$

Which implies that $S_b(x^*, x^*, x^*) = 0$ and hence we obtain $\psi_{S_b}(x^*, x^*, T(x^*)) = 0$. By using (3.5.1) and (3.5.3) with

$$\lim_{n\to\infty} \alpha_*(x_n, x_n, x^*) \geq 1$$

Thus

$$\lim_{n\to\infty} S_b\left(T(x_n), T(x_n), T(x^*)\right) \leq \lim_{n\to\infty} \psi_{S_b}\big(S_b(x_n, x_n, x^*)\big) \leq \lim_{n\to\infty} S_b(x_n, x_n, x^*).$$

$$\lim_{n\to\infty} S_b\left(T(x_n), T(x_n), T(x^*)\right) = 0. \qquad (3.5.4)$$

Now from (3.5.3)and (3.5.4), and $x_{n+1} \in T(x_n)$, we have

$$S_b\big(x_{n+1}, x_{n+1}, T(x^*)\big) \le S_b\big(T(x_n), T(x_n), T(x^*)\big) = 0. \quad (3.5.5)$$

Taking limit as $n \to \infty$ and using(3.5.4), we obtain that

$$\lim_{n\to\infty} S_b\left((x_{n+1}), (x_{n+1}), T(x^*)\right) = 0 \qquad (3.5.6)$$

Using the condition (iii), we obtain

$$S_b\big(x^*, x^*, T(x^*)\big) - \sup_{y\in T(x^*)} S_b(x^*, x^*, y) \le S_b\big(x^*, x^*, T(x^*)\big) - \psi_{S_b}\big(x^*, x^*, T(x^*)\big)$$

$$\le S_b(x^*, x^*, x_{n+1}) - \psi_{S_b}(x^*, x^*, x_{n+1}) + S_b(x_{n+1}, x_{n+1}, T(x^*)) - \psi_{S_b}(x_{n+1}, x_{n+1}, T(x^*))$$

Applying limit as $n \to \infty$ and using (3.5.3) and (3.5.6), we have

$$S_b\big(x^*, x^*, T(x^*)\big) \le \sup_{y\in T(x^*)} S_b(x^*, x^*, y) \qquad (3.5.7)$$

From (ii), $S_b(x^*, x^*, y) \le \psi_{S_b}(x^*, x^*, y)$ for each $y \in T(x^*)$ which implies that

$$S_b(x^*, x^*, y) - \psi_{S_b}(x^*, x^*, y) \le 0$$

Hence

$$\sup\big\{S_b(x^*, x^*, y) - \psi_{S_b}(x^*, x^*, y) :\ y \in T(x^*)\big\} \le 0$$

Then

$$\sup_{y\in T(x^*)} S_b(x^*, x^*, y) - \inf_{y\in T(x^*)} \psi_{S_b}(x^*, x^*, y) \le 0$$

Thus

$$\sup_{y\in T(x^*)} S_b(x^*, x^*, y) \le \psi_{S_b}(x^*, x^*, T(x^*)) \qquad (3.5.8)$$

Now, from (3.5.7) and (3.5.8), we obtain

$$S_b(T(x^*), T(x^*), x^*) = \sup_{y\in T(x^*)} S_b(x^*, x^*, y)$$

Consequently, owing to lemma (3.4) we have $x^* \in T(x^*) = T(x^*)$

Corollary 3.6. Let (X, S_b) be a complete S_b −metric space. Suppose that (α_*, ψ_{S_b}) contraction and α_*- admissible mapping $T : X \to (X)$ satisfies the following conditions:

(i) T is α_* −admissible;

(ii) There exist $x_0 \in X$ such that $\alpha_*(x_0, x_0, a_1) \ge 1$ for each $a_1 \in T(x_0)$,

If $\{x_n\} \in X$ is a sequence such that $\alpha_*(x_n, x_{n+1}, Tx_{n+1}) \ge 1\ \forall\ n\ and\ \{x_n\} \to x \in X$ as $n \to \infty$, then $\alpha_*(x_n, x, x) \ge 1\ \forall\ n \in \mathbb{N}$. then T has a fixed point

Example 3.7. Let $X = [0,1]$, and define the S_b −metric space by $S_b: X \times X \times X \to (-\infty, +\infty)$ then by $S_b(x, y, z) = \max\{x, y, z\}$ and $S_b(x, y, z) = |x - z| + |y - z|\ if\ \{x, y, z\} \subset [0,1]$. Then (X, S_b) is a complete S_b −metric space with s=$\frac{1}{2}$. Now we define $T: X \to X$ and $\alpha_*: X^3 \to X\ by\ Tx = \frac{1}{2}(x + 1) if\ 0 \le x \le 1. Also, define\ \alpha_*$ as follows

$$\alpha_*(x, x, y) = \max\{S_b(x, x, y) - S_b(x, y, y)\}$$

We have

$$S_b(Tx, Tx, Ty) \le \alpha_*(x, x, y) S_b(Tx, Tx, Ty) \le \frac{1}{2} S_b(x, y, y)$$

Moreover, there exists $x_0 \in X$ such that $\alpha_*(x_0, Tx_0, y) \le 1$. we have

$$\alpha_*(x, x, y) S_b(Tx, Tx, Ty) \le \frac{1}{2} \psi_{S_b}(x, x, y)$$

Obviously T is continuous and so it remains to show that T is α_* −admissible. Let $x, y \in X$, such that $\alpha_*(x, x, y) \le 1$. this implies that $x, y \in [0,1]$ and by the definition of T and α_*, we have

$$Tx = \frac{1}{2} x \in [0,1], \qquad Ty = \frac{1}{2} y \in [0,1]\ and\ \alpha_*(Tx, Tx, Ty) = 1$$

Then T is α_* −admissible.

Finally, let $\{x_n\}$ be a sequence in X such that $\alpha_*(x_n, x_{n+1}, y) \le 1\ \forall\ n \in N\ and\ x_n \to x\ as\ n \to \infty$, then the sequence $\{x_n\}$ is convergent i.e. $\{x_n\}$ is a Cauchy sequence and $\lim_{n\to\infty} S_b\ \alpha_*(x_n, x_{n+1}, y) = 1$.

Now all the hypotheses of theorem 3.5 are satisfied.

4. An application to Fredholm integral equation

In the present section, we discuss the existence of solution for the Fredholm integral equation

$$S_b(t) = \int_0^1 S_b\left(t(x), r(x), S_b(r)\right) \psi_{S_b} r, \qquad (4.1)$$

Define $\psi_{S_b}: [0,1] \times [0,1] \times \mathbb{R} \to \mathbb{R}^+$ is a continuous function.

Throughout this section, Let $S_b = C[0,1]$ be the set of real continuous function defined on $[0,1]$ and

$$S_b\big(v(x), h(x), g(x)\big) = \max \psi_{S_b}\left(x, t(x), t(x) - S_b(x, x, x)\right)$$

Such that there is an $x_n \in C$ and for all $v, h, g \in S_b$, we have

$$\alpha_*(x_n, x_n, Tx_n), \int_0^1 S_b\left(v(x), h(x), g(x)\right) \geq \psi_{S_b} \int_0^1 S_b\left(v(x_n), h(x_n), g(x_n)\right)$$

It is clear that (X, S_b) is a complete S_b −metric space. Now, we prove the following result.

Theorem 4.1.1. Consider Eq (4.1) and suppose that

(i) $\psi_{S_b}: [0,1] \times [0,1] \times \mathbb{R} \to \mathbb{R}^+$ is a continuous function,

(ii) There exists a continuous function $S_b: [0,1]^2 \to \mathbb{R}^+$ such that $\int_0^1 S_b\left(t(x), t(x), s(x)\right) \leq 1$

(iii) there exists $x_0 \in X$ such that for all $t, r, s \in [0,1]$, we have

$$x_0(t) \leq \int_0^1 S_b\left(t(x), r(x), x_0(r)\right)\psi_{S_b} r$$

And

$$x_0(t) \leq \int_0^1 S_b\left(v(x), h(x), g(x) \int_0^1 S_b\left(t(x), r(x), x_0(r)\right)\psi_{S_b} r\right)$$

Then the Fredholm integral equation (4.1) has a solution in X.

Proof $\forall\, x \in X$ and $t, r, s \in [0,1]$, define the mapping $T : X^2$ by $Tx(t) = \int_0^1 \psi_{S_b}\left(t, r, x(r)\right)S_b r$

and

$\alpha_*(x, x, y) \geq 1, so\ x \leq y$ for all $t, r, s \in [0,1]$. thus by condition (i)

$$|Tx(t) + Ty(t)| = \int_0^1 \left|\psi_{S_b}(t, s, x_0(r)S_b r)\right| + \int_0^1 \left|\psi_{S_b}(t, s, y_0(r)S_b r)\right|$$

$$\leq \int_0^1 \left|\psi_{S_b}(t, s, x_0(r)\right|S_b r + \int_0^1 \left|\psi_{S_b}(t, s, y_0(r)\right|S_b r$$

$$\leq \int_0^1 \left|\psi_{S_b}(tx, sx, x_0(rx)\right|S_b r + \int_0^1 \left|\psi_{S_b}(tx, sx, y_0(rx)\right|S_b r$$

$$= \int_0^1 \left(\left|\psi_{S_b}(t, s, x_0(r)\right|S_b r + \left|\psi_{S_b}(t, s, x_0(r)\right|\right)S_b r$$

$$\leq \int_0^1 \psi_{S_b}(1 + \sup_{t\in[0,1]}\{x_0(r), y(r), t\}|x_0(r)| + |y_0(r)|)S_b r$$

$$\leq S_b \alpha_*(x, x, y)\psi_{S_b}(x, y, y)$$

Consequently, $\psi_{S_b}(Tx, Tx, Ty) \leq \alpha_*(x, x, y)S_b(x, x, y)$. On the other hand, let $n \in \mathbb{N}^*$ and $x \in X$

$$\psi_{S_b}(T^n x)(r) = T\left(T^{n-1}x_0(r)\right) = \int_0^1 \psi_{S_b}\left(t, s, T^{n-1}x_0(r)\right) S_b r$$

$$= \int_0^1 \psi_{S_b}\left(t, s, T(T^{n-2}x(r)\right) S_b r$$

$$= \int_0^1 \psi_{S_b}\left(t, s, \int_0^1 \psi_{S_b}\left(t, s, T^{n-2}x_0(r)\right)\right) S_b r$$

$$= \int_0^1 \psi_{S_b}\left(t, s, \int_0^1 \psi_{S_b}(t, s, T(T^{n-2}x_0(r)))\right) S_b r$$

$$\leq \int_0^1 \psi_{S_b}\left(t, s, T^{n-2}x_0(r)\right) S_b r = \left(T^{n-1}x_0(r)\right)$$

So for all $x, y \in X$ with $x \leq y$, we get $Tx(t) \leq Ty(t)$ for all $t, r, s \in [0,1]$, that is, if $\alpha_*(x, x, y) \geq 1$, we obtained $\alpha_*(Tx, Tx, Ty) \geq 1$. Moreover, the condition (iii) yields that there exists $x_0(r) \in X$ such that $\alpha_*(x_0, x_0, Tx_0) \geq 1, \alpha_*(x_n, x_{n+1}, Tx_{n+1}) \geq 1, \alpha_*(x_n, x_n, Tx_n) \geq 1$. Therefore all conditions of theorems 3.5 are verified and hence operator T has a fixed point, which is the solution to the Fredholm integral equation. (4.1) $in\ X$.

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