The Steiner Global Domination Number of Degree Splitting Graph of a Graph

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Abstract

Let $G = (V, E)$ be graph with $V = S_1 \cup S_2 \cup S_3 \cup \dots S_t \cup T$, where each $S_i (1 \leq i \leq t)$ is a set of vertices having at least two vertices of the same degree and $T = V \backslash \cup S_i$. The degree splitting graph of G denoted by $DS(G)$ is obtained from G by adding vertices $w_1, w_2, \dots, w_t$ and joining to each vertex of S_i $(1 \leq i \leq t)$. In this article, we studied the concept of the Steiner global domination number of degree splitting graph of a graph.

Keywords: splitting graph of a graph, Steiner global dominating number.
global dominating number, dominating number, Steiner number distance, Steiner distance.

AMS Subject Classification: 05C12,05C69

1.Introduction

By a graph $G = (V, E)$, we mean a finite,undirected connected graph without loops or multiple edges. The *order* and *size* of G are denoted by n and m respectively. For basic graph theoretic terminology, we refer to [1]. Two vertices u and v are said to be *adjacent* if uv is an edge of G. Two edges of G are said to be adjacent if they have a common vertex. The *neighbors* of a vertex v are the vertices that are adjacent to v, it is denoted by $N(v)$. The degree of a vertex $v \in G$ is $deg_G(v) = \lfloor N(v) \rfloor$. A vertex $v \in G$ is said to be a universal vertex if $\deg(v) = n - 1$. For any set S of vertices of G, the *induced subgraph* $G[S]$ is the maximal subgraph of G with vertex set S. A vertex $v \in G$ is said to be *extreme* if the $G[N(v)]$ is complete.

Let $G = (V, E)$ be graph with $V = S_1 \cup S_2 \cup S_3 \cup \dots S_t \cup T$, where each $S_i (1 \leq i \leq t)$ is a set of vertices having at least two vertices of the same degree and $T = V \backslash \cup S_i$. The degree splitting graph of G denoted by $DS(G)$ is obtained from G by adding vertices $w_1, w_2, \dots, w_t$ and joining to each vertex of S_i $(1 \leq i \leq t)$. This concepts were studied in [25].

The *distance* $d(u, v)$ between two vertices u and v in a connected graph G is the length of a shortest $u - v$ path in G. An $u - v$ path of length $d(u, v)$ is called an $u - v$ *geodesic*. For a nonempty set W of vertices in a connected graph G, the *Steiner distance* $d(W)$ of W is the minimum size of a connected subgraph of G containing W. Necessarily, each such subgraph is a tree and is called a *Steiner tree* with respect to W or a *Steiner* W*-tree*. For a given set $W \subseteq V(G)$, there may be more than one Steiner W-tree in G. Let $S(W)$ denotes the set of vertices that lies in Steiner W-trees. Let G be a connected graph with at least 2 vertices. A set $W \subseteq V(G)$ is called a Steiner set of G if $S(W) = V(G)$. The *Steiner number* $s(G)$ is the minimum cardinality of its Steiner sets and any Steiner set of cardinality $s(G)$ is a *minimum Steiner* set of G. The Steiner number of a graph was studied in [5,6,7,11-15,18,19, 21-24,26-29.30].

A set $D \subseteq V(G)$ is a dominating set of G if every vertex in $V \setminus D$ is adjacent to some vertex in D. The *domination number* $\gamma(G)$ is the minimum order of its dominating sets and any *dominating set* of order $\gamma(G)$ is called γ-set of G. The domination number of a graph was studied in [3,4,5,8, 9, 16,17,19,20,31,33]. A subset $D \subseteq V$ is called a global dominating set in G if D is a dominating set of both G and $\bar{G}$. The global domination number $\overline{\gamma}_m(G)$ is the minimum cardinality of a minimal global dominating set in G. The global domination number of a graph was studied in [34]. A set $S \subseteq V$ is said to be a Steiner global dominating set of G if S is both a Steiner set and a global dominating set of G. The minimum cardinality of a Steiner global dominating set of G is the Steiner global domination number of G and is denoted by $\bar{\gamma}_s(G)$. A Steiner global dominating set of cardinality $\bar{\gamma}_s(G)$ is called a $\bar{\gamma}_s$-set of G. The Steiner global domination number of a graph was studied in [32].The following theorems are used in sequel.

Theorem 1.1.[32] Each extreme (universal) vertex of a connected graph G belongs to every Steiner global dominating set of G.

Theorem 1.2.[15] Let G be either the complete graph$K_n (n \geq 2)$ or the wheel graph $W_n (n \geq 4)$. Then $\bar{\gamma}_s(G) = n$.

Theorem 1.3.[15] For the complete bipartite graph $G = K_{r,s}$ $(1 \leq r \leq s)$, $\bar{\gamma}_s(G) = r + s$.

2.The Steiner Global Domination Number of Degree Splitting Graph of a Graph

Theorem 2.1. Let G be the path $P_n (n \geq 4)$. Then $\bar{\gamma}_s(DS(G)) = \begin{cases} 2 & \text{for } n = 4 \\ 3 & \text{for } n = 5 \\ 4 & \text{for } n = 6 \\ n-2 & \text{for } n \geq 7 \end{cases}$.

Proof. Let G be the path P_n $(n \geq 4)$, and $V(G) = \{v_1, v_2, \dots, v_n\}$ and $V(DS(G) \setminus V(G)) = \{u_1, u_2\}$. Let $E(DS(G)) = \{u_1v_1, u_1v_n\} \cup \{u_2v_i : 2 \leq i \leq n-1\}$. Hence it follows that $|E(DS(G)| = 2n - 1$.

First assume that $n = 4$. Then u_2 is an extreme vertex of G. Then by Theorem 1.1, u_2 is a subset of every Steiner global dominating set of G. Let $W = \{u_1, u_2\}$. Then $S(W) = V(G)$ and so W is a Steiner set of $DS(G)$. Since W is a dominating set in $DS(G)$ and $\overline{DS(G)}$. W is a global dominating set of $DS(G)$. Hence it follows that W is a Steiner global dominating set of $DS(G)$ so that $\bar{\gamma}_s(DS(G)) = 2$.

Next assume that $n = 5$. Since there is no pairs of antipodal vertices x, y such that every vertex of antipodal vertices x, y such that every vertex lies on x-y diametral path $\bar{\gamma}_s(DS(G)) \geq 3$. Let $W_1 = \{u_1, u_2, v_3\}$. Then $S(W_1) = V(DS(G))$ and so W_1 is a Steiner set in $DS(G)$ and $\overline{DS(G)}$. W_1 is a global dominating set of $DS(G)$. Hence it follows that W_1 is a Steiner global dominating set of $DS(G)$ so that $\bar{\gamma}_s(DS(G)) = 3$.

Next assume that $n = 6$. Let $Z = \{u_1, u_2, v_3, v_4\}$. We prove that Z is a subset of every $\bar{\gamma}_s$-set of $DS(G)$. On the contrary suppose that Z is not a subset of any $\bar{\gamma}_s$-set $DS(G)$. Then there exists is $\bar{\gamma}_s$-set W' of $DS(G)$ such that $Z \not\subset W'$. Let $x \in W$ such that $x \notin Z$. Then either W' not a Steiner set of $DS(G)$ or W' is not a dominating set of $DS(G)$, which is a contradiction. Hence it follows that Z is a subset of every $\bar{\gamma}_s$-set of $DS(G)$. Since $S(Z) = V\big(DS(G)\big)$, Z is a Steiner set of $DS(G)$. Since Z is a dominating in $DS(G)$ and $\overline{DS(G)}$. Z is a dominating set in $DS(G)$. Hence it follows that Z is a Steiner global dominating set of $DS(G)$ so that $\bar{\gamma}_s(DS(G)) = 4$.

Next assume that $n \geq 7$. Let $Z_1 = \{u_1, u_2, v_3, v_4, \dots, v_{n-2}\}$. We prove that Z_1 is a subset of every $\bar{\gamma}_s$-set of $DS(G)$. On the contrary suppose that Z_1 is not a subset of any $\bar{\gamma}_s$-set $DS(G)$. Then there exists is $\bar{\gamma}_s$-set W_1' of $DS(G)$ such that $Z_1 \not\subset W_1'$. Let $x \in W_1'$ such that $x \notin Z_1$. Then either W_1' not a Steiner set of $DS(G)$ or W_1' is not a dominating set of $DS(G)$, which is a contradiction. Hence it follows that Z_1 is a subset of every $\bar{\gamma}_s$-set of $DS(G)$ and so $\bar{\gamma}_s(DS(G)) \geq n - 2$. Since $S(Z_1) = V\big(DS(G)\big)$, Z_1 is a Steiner set of $DS(G)$. Since Z_1 is a dominating in $DS(G)$ and $\overline{DS(G)}$. Z_1 is a global dominating set in $DS(G)$. Hence it follows that Z_1 is a Steiner global dominating set of $DS(G)$ so that $\bar{\gamma}_s(DS(G)) = n - 2$.

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Theorem 2.2. Let G be the cycle $C_n (n \geq 4)$. Then $\bar{\gamma}_s(DS(G)) = n$.

Proof. Since the degree splitting graph of $C_n (n \geq 4)$ is the wheel graph W_{n+1}. the follows from Theorem 1.2 ■

Theorem 2.4. Let G be the complete graph $K_n (n \geq 4)$. Then $\bar{\gamma}_s(DS(G)) = n. +1$

Proof. Since the degree splitting graph of $K_n (n \geq 4)$ is the complete graph K_{n+1}. the follows from Theorem 1.2 ■

Theorem 2.6. Let G be the fan graph $F_n = K_1 + P_{n-1}$ $(n \geq 5)$. Then $\bar{\gamma}_s(DS(G)) = n - 2$.

Proof. Let G be the fan graph F_n $(n \geq 4)$ with central vertex x and $V(G) = \{x, v_1, v_2, \dots, v_n\}$. Let $V(DS(G) \backslash V(G)) = \{u_1, u_2\}$. Let $E(DS(G)) = \{u_1v_1, u_1v_{n-1}\} \cup \{u_2v_i : 2 \leq i \leq n-2\}$. Hence it follows that $|E(DS(G)| = 3n - 4$.

Let $Z_1 = \{u_1, u_2, x, v_3, v_4, \dots, v_{n-3}\}$. We prove that Z_1 is a subset of every $\bar{\gamma}_s$-set of $DS(G)$. On the contrary suppose that Z_1 is not a subset of any $\bar{\gamma}_s$-set $DS(G)$. Then there exists is $\bar{\gamma}_s$-set W_1' of $DS(G)$ such that $Z_1 \not\subset W_1'$. Let $y \in W_1'$ such that $y \notin Z_1$. Then either W_1' not a Steiner set of $DS(G)$ or W_1' is not a dominating set of $DS(G)$, which is a contradiction. Hence it follows that Z_1 is a subset of every $\bar{\gamma}_s$-set of $DS(G)$ and so $\bar{\gamma}_s(DS(G)) \geq n - 2$. Since $S(Z_1) = V\big(DS(G)\big)$, Z_1 is a Steiner set of $DS(G)$. Since Z_1 is a dominating in $DS(G)$ and $\overline{DS(G)}$. Z_1 is a global dominating set in $DS(G)$. Hence it follows that Z_1 is a Steiner global dominating set of $DS(G)$ so that $\bar{\gamma}_s(DS(G)) = n - 2$. ■

Theorem 2.6. Let G be the star graph $K_{1,n-1}$ $(n \geq 4)$. Then $\bar{\gamma}_s(DS(G)) = n + 1$.

Proof. Since $DS(G)$ is the complete bipartite graph $K_{2,n-1}$ the result follows from Theorem 1.3. ■

Theorem 2.7. Let G be the complete bipartite graph $K_{r,s}$ $(2 \leq r \leq s)$. Then

$$\bar{\gamma}_s(DS(G)) = \begin{cases} r + s + 1, & if\ r = s \\ 2, & if\ r \neq s \end{cases}.$$

Proof: Let $X = \{x_1, x_2, \dots, x_r\}$ and $Y = \{y_1, y_2, \dots, y_s\}$ be the bipartite sets of G.

Case (1) $r = s$

Let u_1 be the additional vertex of $DS(G)$. Then either X or Y is a Steiner set of G but not a global dominating set of G and so $\bar{\gamma}_s(DS(G)) \geq r + 1$. Since either $G[X \cup \{y\}]$ or $G[Y \cup \{x\}]$, where $x \notin X$ and $y \notin Y$ is connected, $W = V(DS(G))$ is the unique Steiner global dominating set of $DS(G)$ so that $\bar{\gamma}_s(DS(G)) = r + s + 1$.

Case (2) $r \neq s$

Let u_1 and u_2 be the additional vertices of $DS(G)$. Then $W = \{u_1, u_2\}$ be the set of antipodal vertices of G. Then each vertex of $DS(G)$ lies on the $u_1 - u_2$ diametral path of G. Hence it follows that W is a Steiner set of $DS(G)$. Since W is a dominating set in $DS(G)$ and $\overline{DS(G)}$, W is a Steiner global dominating set of G so that $\bar{\gamma}_s(DS(G)) = 2$.

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