# UNIQUE ISOLATE DOMINATION IN POWER OF CYCLE AND IN SOME FAMILIES OF GRAPHS

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**Abstract:**

A dominating set S of a graph G is said to be an isolate dominating set of G if the induced subgraph < S > has atleast one isolated vertex. A dominating set S of a graph G is said to be an unique isolate dominating set (UIDS) of G if < S > has exactly one isolated vertex. A dominating set S of a graph G is said to be an unique isolate dominating set(UIDS) of G if < S > has exactly one isolated vertex. If a graph G admits UIDS S and x is the isolated vertex in < S >, then S−{x} is a minimum total dominating set in G−N[a]. An UIDS S is said to be minimal if no proper subset of S is an UIDS. The minimum cardinality of a minimal UIDS of G is called the UID number, denoted by $\gamma^U$ (G).The maximum cardinality of a minimal UIDS of G is called the upper UID number, denoted by $\Gamma_0^U(G)$.

In this paper we found UIDS in Power of a Cycle $C_n^k$ , UIDS in some Families of Graphs like Sun graph, Comb graph and Helm graph, we give an upper bound for the UID number of $C_n^k$ . Also, we identify some sub families of $C_n^k$ admits UIDS.

**Keywords:** Isolate dominating set, unique isolate dominating set, unique isolate domination number.

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## 1. Introduction

Beginning with the origin of the Four Color Problem in 1852, the field of graph colorings has developed into one of the most popular areas of graph theory. Each chapter in the text contains many exercises of varying levels of difficulty. There is also an appendix containing the referred research articles and books.

A dominating set in a graph is a vertex subset S such that every vertex not in S has a neighbor in $S$, and the domination number of a graph is the size of its smallest dominating set. The dominating set problem asks to determine the domination number of a given graph. Formal study of the dominating set problem began in the 1960's, the term itself first appearing in the 1967 book on graph theory by Ore[?]. This area of mathematics is rapidly being developed by many people in different countries.

Berge and Ore took efforts to make the concept of domination mathematically which increases interest in the study of domination parameters worldwide[7]

More than one hundred domination parameters defined and studied by various people in all over the world. Besides being of theoretical interest, the dominating set problem also finds a natural application in numerous facility location problems. In such problems, the locations are denoted by the vertices of a graph, adjacency means some notion of accessibility, and the problem is to find a subset of locations accessible from all other locations at which to install fire stations, bus stops, post offices, or other such facilities.

Various numerical invariants of graphs concerning domination were introduced by means of dominating functions and their variants.

**Notations:**

In this section, we give all the notations followed in this dissertation.

| | |
|---|---|
| $G = (V, E)$ | Graph with vertex set $V$ and edge set $E$ |
| $V(G)$ (or) $V$ | Vertex set of $G$ |
| $E(G)$ (or) $E$ | Edge set of $G$ |
| $\deg_G(v)$ (or) $\deg(v)$ | Degree of the vertex $v$ in $G$ |
| $\delta(G)$ | minimum degree of a vertex in $G$ |
| $\Delta(G)$ | Maximum degree of a vertex in $G$ |
| $N_G(v)$ (or) $N(v)$ | Open neighborhood of a vertex $v$ in $G$ |
| $N_G[v]$ (or) $N[v]$ | Closed neighborhood of a vertex $v$ in $G$ |
| $N_G(S)$ (or) $N(S)$ | Open neighborhood of $S \subseteq V$ in $G$ |
| $N_G[S]$ (or) $N[S]$ | Closed neighborhood of $S \subseteq V$ in $G$ |
| $\gamma(G)$ | Domination number of $G$ |
| $\gamma_S(G)$ | Signed domination number of $G$ |
| $\langle S \rangle$ | Subgraph induced by a set $S \subseteq V$ |

| | |
|---|---|
| $\overline{G}$ | Complement graph of a graph $G$ |
| $G^k$ | $k$th power of the graph $G$ |
| $(Z_n, \oplus_n)$ | finite cyclic group of order $n$ |
| $C_n$ | Cycle graph on $n$ vertices |
| $H_n$ | helm graph |
| $W_n$ | Wheel graph on $n+1$ vertices |
| $F_n$ | Fan graph on $n+1$ vertices |

| | |
|---|---|
| $B_n$ | Book graph on $2n+2$ vertices |
| $T_n$ | Friendship graph on $2n+1$ vertices |
| $D_n$ | Prism graph on $2n$ vertices |
| $\lceil x \rceil$ | Smallest integer greater than or equal to $x$ |
| $\lfloor x \rfloor$ | Largest integer less than or equal to $x$. |

**Basic definitions**
In the first section of this thesis, we collect some basic and important definitions in graph theory which are used to the subsequent chapters. For graph theoretic terminology, we follow[8].

**Definition 1.1** A graph $G$ is a finite nonempty set of objects called vertices together with a set of unordered pairs of vertices of $G$ called edges. The vertex set and the edge set of $G$ are denoted by $V(G)$ and $E(G)$ respectively. Each pair $(u,v)$ of points in $E(G)$, is called an edge of $G$. We write $e=(u,v)$ and say that $u$ and $v$ are adjacent vertices; vertex $u$ and edge $e$ are incident with each other, as are $v$ and $e$. If two distinct edges $e_1$ and $e_2$ are incident with a common vertex, then we say that $e_1$ and $e_2$ are adjacent edges. If $u=v$, then $e$ is said to be a loop. If $e=e_1=(u,v)$, then $e$ and $e_1$ are said to be parallel edges. A graph without loops and parallel edges, is called a simple graph.

**Definition 1.2** A graph is a power of cycle denoted by $C_n^k$, if $V(C_n^k)=\{1,2,\dots,n-1,n(=0)\}$, and $E(C_n^k)=E^1\cup E^2 \dots \cup E^k$, where $E^i=\{(v_j,v(j+i) mod\, n): 0\leq j\leq n-1\}$. Note that $C_n^k$ is $2k$-regular and that $k\geq 1$. an edge $e\in E^i$ is said to have reach $i$; if $i$ is even (odd), then $e$ is an even (odd) edge. We take $(v_0,\dots,v_{n-1})$ to be a cyclic order on the vertex set of $G$, and always perform modular operations on edge and vertex indexes.

**Definition 1.3** For a positive integer $n$, the graph $P_n+K_1$ is called as fan graph and denoted by $F_n$. Note that the fan graph $F_n$ has $n+1$ vertices and $2n-1$ edges.

**Definition 1.4** The friendship graph $T_n$ is obtained by merging exactly one vertex from each of the $n$ number of $K_3$'s. Note that the friendship graph $T_n$ has $2n+1$ vertices and $3n$ edges.
**Definition 1.5** A sun graph is a graph obtained by joining an pendent edge to each vertex of a cycle $C_n$ and denoted by $Sun(n)$.

**Definition 1.6** A Helm is a graph obtained by joining an pendent edge to each vertex of a wheel $W_n$ except the center and denoted by $H_n$. Note that the Helm graph $H_n$ has $2n+1$ vertices and $3n$ edges.

**Definition 1.7** [8] Let $G=(V_G,E_G)$ and $H=(V_H,E_H)$ be two graphs such that $|V_G|=|V_H|$. If there exists a bijection $f: V_G\to V_H$ such that $(x,y)\in E_G$ if and only if $(f(x),f(y))\in E_H$, then $f$ is called a graph isomorphism.

**Definition 1.8** The square $G^2$ of a graph $G$ is defined on the vertex set of $G$ in such a way that distinct vertices with distance at most $2$ in $G$ are joined by an edge.

**Definition 1.9** A path of length $n$ in a graph $G$ is a sequence $(u_0,u_1,\dots,u_n)$ of distinct vertices, such that for $1\leq i\leq n-1$, the vertices $u_i$ and $u_{i+1}$ are adjacent.
A cycle of length $n$ in a graph $G$, denoted by $C_n$, is a sequence $(u_0,u_1,\dots,u_{n-1},u_0)$ of

distinct vertices, such that for $1 \le i \le n-2$, the vertices $u_i$ and $u_{i+1}$ are adjacent, $u_{n-1}$ and $u_0$ are adjacent.
The length of a shortest cycle (it exist) in a graph $G$ is called the girth of $G$ and is denoted by $g(G)$.
A cycle $C_n$ of length $n$ is called even or odd according as $n$ is even or odd.

**Definition 1.10** Let $D$ be a simple finite digraph with vertex set $V(D) = V$ and arc set $E(D) = E$. For any vertex $v \in V$, the in-neighbor of $v$, denoted by $N^-[v] = N_D^-[v]$, is given by $N_D^-[v] = \{u \in V: (u, v) \in E\}$. The out-neighbor of $v$, denoted by $N^+[v] = N_D^+[v]$, is given by $N_D^+[v] = \{u \in V: (u, v) \in E\}$.

**Definition 1.11** A non-empty subset $A$ of a group $\Gamma$, is called a generating set of $\Gamma$ if every element of $\Gamma$ can be expressed as a product of the elements in $A$, denoted by $\Gamma =< A >$.
**Assumption:** Let $A$ be a generating set of a group $\Gamma$ with $e$ as the identity element of $\Gamma$. We assume the following two conditions:
$C_1$: The identity element $e \notin A$.
$C_2$: If $a \in A$, then $a^{-1} \in A$

## 2. Unique isolated domination

The origin of domination starts from the game of chess, where the aim of the game is to dominate all the squares of a chessboard by certain chess pieces. In 1862, de Jaenisch [Dl] studied the problem of finding the least number of queens used to cover the chessboard in such a way that every square is either reachable by a queen in a single move. The answer he find is 5 and a possible positions of these five queens of a $8 \times 8$ chessboard are $(1,1), (3,3), (5,5,), (6,6,)$ and $(7,7)$. Take all squares of the chessboard as vertices. Join two vertices if and only if a queen can move from one square to another. Then the chessboard problem is changed here as finding the minimum dominating set. A dominating set $S$ of a graph $G$ is said to be an isolate dominating set of $G$ if the induced subgraph $< S >$ has at least one isolated vertex $sahul$.
A dominating set $S$ of a graph $G$ is said to be an unique isolate dominating set(UIDS) of $G$ if $< S >$ has exactly one isolated vertex. An UIDS $S$ is said to be minimal if no proper subset of $S$ is an UIDS. The minimum cardinality of a minimal UIDS of $G$ is called the UID number, denoted by $\gamma_0^U(G)$.
Note that, if a graph $G$ admits UIDS $S$ and $x$ is the isolated vertex in $< S >$, then $S - \{x\}$ is a minimum total dominating set in $G - N[a]$. This chapter includes some properties of UIDS and the UID number of paths, complete $k$-partite graphs and disconnected graphs. Further, the role played by UIDS in the domination chain has been discussed in detail.
In this Chapter, we consider only finite non-trivial undirected graphs with no loops and no multiple edges. For graph theoretic terminology, we refer to $char$. Here we list out some of the basic definitions which are needed for this chapter.
Let $G = (V, E)$ be a simple connected graph. For $v \in V$, the open neighborhood $N(v)$ is the set of all vertices which are adjacent to $v$. The closed neighborhood of $v$ is $N[v] = N(v) \cup \{v\}$. The degree of a vertex $v$ is defined by $deg(v) = |N(v)|$. The minimum and maximum degree of $G$ is defined by $\delta(G) = \min_{v \in V}\{deg(v)\}$ and $\Delta(G) = \max_{v \in V}\{deg(v)\}$ respectively.
A set $S \subseteq V$ is called a dominating set if every vertex in $V$ is either an element of $S$ or adjacent to an element of $S$. A dominating set $S$ is minimal if no proper subset of $S$ is a dominating set. The minimum and maximum cardinality of a minimal dominating set of $G$ are

called the domination number $\gamma(G)$ and the upper domination number $\Gamma(G)$ respectively.
In 2016, Hameed and Balamurugan $sahul$ introduced the concept of isolate domination in graphs. Further, in [5], they characterized unicycle graphs on which the order equals the sum of the isolate domination number and its maximum degree. A dominating set $S$ of a graph $G$ is said to be an isolate dominating set if $< S >$ has at least one isolated vertex $sahul$. An isolate dominating set $S$ is said to be minimal if no proper subset of $S$ is an isolate dominating set. The minimum and maximum cardinality of a minimal isolate dominating set of $G$ are called the isolate domination number $\gamma_0(G)$ and the upper isolate domination number $\Gamma_0(G)$ respectively. An isolate dominating set of cardinality $\gamma_0$ is called a $\gamma_0$-set.
By using the above concept of isolate domination, we define a new concept called "Unique Isolate Domination(UID)". A dominating set $S$ of $G$ is said to be an UIDS of $G$ if $< S >$ has exactly one isolated vertex. An UIDS $S$ is said to be minimal if no proper subset of $S$ is an UIDS. The minimum and maximum cardinality of a minimal UIDS of $G$ are called the UID number $\gamma_0^U(G)$ and the upper UID number $\Gamma_0^U(G)$ respectively. An UIDS of cardinality $\gamma_0^U$ is called a $\gamma_0^U$-set. Note that the cycle $C_4$ does not admit UIDS but it admits isolate dominating sets. So many differences between these two domination parameters that we have discussed in the next section. This chapter includes some basic properties of UIDS and the role played by UIDS in the domination chain has been discussed.
Since every pendent vertex or the vertex adjacent to it is in every dominating set, $\{b, d, i, g\}$ is a minimum dominating set and $\gamma(G) = 4$.
But $\{b, d, i, g\}$ is not a UID set since $< \{b, d, i, g\} >$ has no isolated vertices. Let $D$ be a minimum UID set of $G$ and $x$ be the isolated vertex of $D$. Suppose $x = a$. Consider the induced subgraph $G - N[a] =< \{c, d, e, f, g, h, i, j\} >$. It has the minimum total dominating set with four elements, namely $\{g, h, i, d\}$. Thus $\gamma_0^U(G) = 5$. Similarly when $x = e$ or $x = j$ or $x = f$, we can prove $\gamma_0^U(G) = 5$.
Suppose $x = g$. Then we can not take the vertex $b$ in $S$. Thus to dominate the vertex $a$, we must have $a$ in $S$ and in this case we get two isolated vertices in $S$, namely $a$ and $g$, a contradiction. Similarly we can get contradictions, when $x = b$ or $x = i$ or $x = d$.

## 2.1 UIDS in Power of a Cycle

A graph is a power of cycle, denoted $C_n^k$, if $V(C_n^k) = \{0(n),1,2,\ldots,n-1\}$ and $E(C_n^k) = E^1 \cup E^2 \cup \ldots \cup E^k$, where $E^i = \{(j,(j+i)(mod\, n)) : 0 \le j \le n-1\}$ and $1 \le k \le \lfloor\frac{n-1}{2}\rfloor$ [?]. Note that $C_n^k$ is a $2k$-regular graph. We take $(0,1,\ldots,n-1)$ to be a cyclic order on the vertex set of $G$, and always perform modular operations on edge and vertex indexes. In this section, we give an upper bound for the UID number of $C_n^k$. Also, we identify some sub families of $C_n^k$ admits UIDS. In such case, we obtain the $\gamma_0^U(G)S$. In this section the operation $+$ is taken as addition modulo $n$.

**Lemma 2.1** Let $n, k$ be positive integers such that $n - (2k+1)$ is a multiple of $3k+1$. Then the graph $G = C_n^k$ admits UIDS and $\gamma_0^U(G) = 2m+1$, where $m = \frac{n-(2k+1)}{3k+1}$.

Proof. Let $G = C_n^k$ and $n, k$ be positive integers such that $n - (2k+1)$ is a multiple of $2k+1$. Note that every vertex of $G$ is of degree $2k$. Also any two adjacent vertices of $G$ can dominate a maximum of $3k+1$ vertices(any vertex $v$ and $v+k$ can dominate $3\mathrm{k}+1$ vertices).$---$>(1)
From the definition of given circulant graph, it follows that two vertices $v$ and $v+i$ are adjacent if and only if $1 \le i \le k$.$---$>(2)

Consider the set $S = \{k+1, (3k+2), (4k+2), (3k+2)+(3k+1), (4k+2)+(3k+1), (3k+2)+2(3k+1), (4k+2)+2(3k+1), \dots, (3k+2)+(m-1)(3k+1), (4k+2)+(m-1)(3k+1)\}$. From (1) and (2), $S$ is dominating set with $2m+1$ vertices. Also the vertices $1,2,\dots,k,k+2,k+3,\dots,2k+1$ are not in $S$ and so $k+1$ is isolated in $< S >$. Thus $S$ is a UIDS and hence $\gamma_0^U(G) \le 1+2m$.
On the other hand, let $D$ be a minimum UIDS of $G$ and $x$ be the isolated vertex in $< D >$. Note that $G$ is a regular graph of degree $2k$ and so including $x$, the vertex $x$ will dominate $2k+1$ vertices. Also every other vertex of $D$ is adjacent with another vertex of $D$ and hence by (2), to dominate the remaining $n-(2\mathrm{k}+1) = m(3k+1)$ vertices of $G$, $D$ must include at least $2m$ more vertices. Thus $|D| \ge 2m+1$ and so $\gamma_0^U(G) \ge 1+2m$.

Note that, when $k=1$, the circulant graph $C_n^k$ is a cycle $C_n$.
**Corollary 2** $\gamma_0^U(C_n) = 2m+1$ if $n = 4m+3$ for some integer $m \ge 0$.

Proof. Put $k=1$ in Lemma 2.6, we get $n-3$ is a multiple of $4$. Then the graph $C_n$ admits UIDS and $\gamma_0^U(C_n) = 2m+1$, where $m = \frac{n-3}{4}$. Thus $n = 3m+3$ and and $\gamma_0^U(G) = 2m+1$.
Note that this result is already proved in Lemma 2.3(1).

**Lemma 2.2** Let $n,k$ be positive integers such that $n = (2k+1)+m(3k+1)+i$ for some $1 \le i \le k$. Then the graph $G = C_n^k$ admits UIDS and $\gamma_0^U(G) = 2m+2$, where $m = \frac{n-(2k+1)}{3k+1}$.

Proof. Let $G = C_n^k$ and $n,k$ be positive integers such that $n-(2k+1) = m(3k+1)+i$ for some $1 \le i \le k$. The set $S = \{k+1, (3k+2), (4k+2), (3k+2)+(3k+1), (4k+2)+(3k+1), (3k+2)+2(3k+1), (4k+2)+2(3k+1), \dots, (3k+2)+(m-1)(3k+1), (4k+2)+(m-1)(3k+1), (4k+2)+(m-1)(3k+1)+k\}$. is a dominatin set with $2m+2$ vertices. Also the vertices $1,2,\dots,k,k+2,k+3,\dots,2k+1$ are not in $S$ and so $k+1$ is isolated in $< S >$. Thus $S$ is a UIDS and hence $\gamma_0^U(G) \le 2+2m$.
On the ther hand, let $D$ be a minimum UIDS of $G$ and $x$ be the isolated vertex in $< D >$. Then $x$ will dominate $2k+1$ vertices. Also every other vertex of $D$ is adjacent with another vertex of $D$ and hence by (2), to dominate $m(3k+1)$ vertices among the remaining undominated vertices of $G$, $D$ must include at least $2m$ more vertices. To dominate the remaining $i$ vertices $D$ must include at least one vertex so $\gamma_0^U(G) \ge 1+2m+1 = 2m+2$.

**Corollary 3** $\gamma_0^U(C_n) = 2(m+1)$ if $n = 4(m+1)$ for some integer $m \ge 1$.

Proof. Take $k=1$ in Lemma 2.7, then we get $n = (3)+m(4)+1$, the graph $C_n$ admits UIDS and $\gamma_0^U(G) = 2m+2$, where $m = \frac{n-3}{4}$. Thus $n = 4(m+1)$ and and $\gamma_0^U(G) = 2(m+1)$.
Note that this result is already proved in 2.3(4).

**Lemma 2.3** Let $n,k$ be positive integers such that $n = (2k+1)+m(3k+1)+i$ for some $k+1 \le i \le 3k$. Then the graph $G = C_n^k$ admits UIDS and $\gamma_0^U(G) \le 2m+3$, where $m = \frac{n-(2k+1)}{3k+1}$.

Proof. Let $G = C_n^k$ and $n, k$ be positive integers such that $n - (2k+1) = m(3k+1) + i$ for some $k+1 \leq i \leq 3k$.

**case 1:** If $k+1 \leq i \leq 2k$. The set $S = \{k+1, (3k+2), (4k+2), (3k+2) + (3k+1), (4k+2) + (3k+1), (3k+2) + 2(3k+1), (4k+2) + 2(3k+1), \dots, (3k+2) + (m-1)(3k+1), (4k+2) + (m-1)(3k+1), (3k+2) + m(3k+1), n(=0)\}$ is a domination set with $2m+3$ vertices. Also the vertices $1, 2, \dots, k, k+2, k+3, \dots, 2k+1$ are not in $S$ and so $k+1$ is isolated in $< S >$. Thus $S$ is a UIDS and hence $\gamma_0^U(G) \leq 2m+3$.

**case 2:** If $2k \leq i \leq 3k$. In this case ,the set $S = \{k+1, (3k+2), (4k+2), (3k+2) + (3k+1), (4k+2) + (3k+1), (3k+2) + 2(3k+1), (4k+2) + 2(3k+1), \dots, (3k+2) + (m-1)(3k+1), (4k+2) + (m-1)(3k+1), (3k+2) + m(3k+1), (4k+2) + m(3k+1)\}$ is a dominating set with $2m+3$ vertices. Also the vertices $1, 2, \dots, k, k+2, k+3, \dots, 2k+1$ are not in $S$ and so $k+1$ is isolated in $< S >$. Thus $S$ is a UIDS and hence $\gamma_0^U(G) \leq 2m+3$.

**Corollary 4** (1). $\gamma_0^U(C_n) = 2m+1$ if $n = 4m+1$ for some integer $m \geq 0$.
(2). $\gamma_0^U(C_n) = 2m+1$ if $n = 4m+2$ for some integer $m \geq 0$.

Proof. Let $k = 1$ in Lemma 2.8, then we get $n = 3 + m(4) + i$ for some $2 \leq i \leq 3$, the graph $C_n$ admits UIDS and $\gamma_0^U(G) \leq 2m+3$, where $m = \frac{n-3}{4}$.

**Case i:** If $i = 2$.
Then $n = 4(m+1) + 1$ and $\gamma_0^U(G) \leq 2m+3$. Replace $m$ by $m-1$, we get $n = 4m+1$ and $\gamma_0^U(G) \leq 2m+1$. The part $\gamma_0^U(G) \geq 2m+1$ is proved in 2.3(4).

**Case i:** If $i = 3$.
Then $n = 4(m+1) + 2$ and $\gamma_0^U(G) \leq 2m+3$. Replace $m$ by $m-1$, we get $n = 4m+2$ and $\gamma_0^U(G) \leq 2m+1$. The part $\gamma_0^U(G) \geq 2m+1$ is proved in 2.3(4).

We conclude this section with an open problem: Let $n, k$ be positive integers such that $n = (2k+1) + \mathrm{m}(3k+1) + i$ for some $k+1 \leq i \leq 3k$. Then the graph $G = C_n^k$ admits UIDS and $\gamma_0^U(G) = 2m+3$, where $m = \frac{n-(2k+1)}{3k+1}$.

## 2.2 UNIQUE ISOLATE DOMINATION IN SOME FAMILES OF GRAPHS

**Lemma 2.4** Let $n \geq 3$ be an integer. Then the sun graph $Sun(n)$ admits UIDS with $\gamma_0^U(Sun(n)) = n$.

Proof. Let $n \geq 3$ be an integer. Let the vertex set of the Sun graph be $V(Sun(n)) = \{x_i, y_i : 1 \leq i \leq n\}$ and the edge set as $E(Sun(n)) = \{x_n y_n, x_n x_1\} \cup \{x_i x_{i+1}, x_i y_i : 1 \leq i \leq n-1\}$.
Note that for each $i = 1, 2, \dots, n$, $y_i$ is a pendent vertex and so either $x_i$ or $y_i$ must be in every dominating set. This gives that $\gamma(Sun(n)) \geq n$.
Also the set $\{y_1, x_2, x_3, \dots, x_n\}$ is UIDS with $n$ elements and $y_1$ is isolated in the induced subgraph $< \{y_1, x_2, x_3, \dots, x_n\} >$. This gives that $\gamma_0^U(Sun(n)) \leq n$.
Thus by Theorem 2.2, we have $n \leq \gamma(G) \leq \gamma_0^U(G) \leq n$ and so the sun graph $Sun(n)$ admits UIDS with $\gamma_0^U(Sun(n)) = n$.

**Lemma 2.5** Let $n \geq 2$ be an integer. Then the comb graph $G = P_n \odot K_1$ admits UIDS with

$\gamma_0^U(G) = n$.

Proof. Let $n \geq 2$ be an integer. Let the vertex set of $G$ be $V(G) = \{a_i, b_i: 1 \leq i \leq n\}$ and $E(G) = \{a_i a_{i+1}: 1 \leq i \leq n-1\} \cup \{a_i b_i: 1 \leq i \leq n\}$.
Note that for each $i = 1,2, \dots, n$, $b_i$ is a pendent vertex and so either $b_i$ or $a_i$ must be in every dominating set. This gives that $\gamma(G) \geq n$.
Also the set $\{b_1, a_2, a_3, \dots, a_n\}$ is UIDS with $n$ elements and $b_1$ is isolated in the induced subgraph $< \{b_1, a_2, a_3, \dots, a_n\} >$. This gives that $\gamma_0^U(G) \leq n$.
Thus by Theorem 2.2, we have $n \leq \gamma(G) \leq \gamma_0^U(G) \leq n$ and so the graph $G$ admits UIDS with $\gamma_0^U(G) = n$.

In the previous result it is proved that the comb graphs admit UIDS. But the generalized comb graph not admit UIDS and it is proved in the following result.

**Theorem 2.6** Let $n, m \geq 2$ be an integer. Then the generalized comb graph $G = P_n \odot \overline{K_m}$ does not admit UIDS.

Proof. Let $n \geq 2$ be an integer. Let the vertex set of $G$ be $V(G) = \{a_i: 1 \leq i \leq n\}$. Note that each vertex $a_i$ is adjacent with $m$ pendent vertices, namely $a_i^j$ for $1 \leq j \leq m$.
If exists, let $D$ be an UIDS of $G$ and $u$ be the isolated vertex of $< D >$.
Case 1: Suppose $u = a_i$ for some integer $i$ with $1 \leq i \leq n-1$.
Then the vertex $a_{i+1}$ should not be in $D$. In this case, to dominate the vertex $a_{i+1}^1$, $D$ must include $a_{i+1}^1$. Here we get a contradiction that $< D >$ have two isolates namely $a_{i+1}^1$ and $a_i$.
Case 2: Suppose $u = a_n$.
Then the vertex $a_{n-1}$ should not be in $D$. In this case, to dominate the vertex $a_{n-1}^1$, $D$ must include $a_{n-1}^1$. Here we get a contradiction that $< D >$ have two isolates namely $a_n$ and $a_{\mathrm{n}-1}^1$.
Case 3: Suppose $u = a_i^j$ for some $1 \leq i \leq n$ and $1 \leq j \leq m-1$.
Then the vertex $a_i$ should not be in $D$. In this case, to dominate the vertex $a_i^{j+1}$, $D$ must include $a_i^{j+1}$. Here we get a contradiction that $< D >$ have two isolates namely $a_i^j$ and $a_i^{j+1}$.
Case 4: Suppose $u = a_i^m$ for some $1 \leq i \leq n$.
Then the vertex $a_i$ should not be in $D$. In this case, to dominate the vertex $a_i^{m-1}$, $D$ must include $a_i^{m-1}$. Here we get a contradiction that $< D >$ have two isolates namely $a_i^m$ and $a_i^{m-1}$.
Thus there exists no isolated vertex in $< D >$, a contradiction and so the generalized comb graph does not admit UIDS.

The Book graph $B_{m,n} = \langle K_{1,m}: K_{1,n} \rangle$ is obtained by joining the center vertex of $K_{1,m}$ with one end of a path $P: a, b, c$, say $a$; and $K_{1,n}$ with another end $c$.

**Theorem 2.7** Let $m, n \geq 2$ be integers. The Book graph $Bm, n = \langle K_{1,m}: K_{1,n} \rangle$ admits UIDS and $\gamma_0^U(B_{m,n}) = 3$.

Proof. Suppose there exists a UIDS, say $D$ and $u$ be the isolated vertex in $< D >$. Let $a$ and $c$ be the centers of $K_{1,m}$ and $K_{1,n}$ respectively. Let $b$ be the vertex adjacent to both $a$ and $c$. Let $\{a_1, a_2, \dots a_m\}$ be the set of pendent vertices adjacent to $a$ and $\{c_1, c_2, \dots c_n\}$ be the set of

pendent vertices adjacent to $c$. Note that $N(b) = \{a, c\}$.
Case 1: Suppose $u = a_1$.
In this case $a \notin D$. Since $m \geq 2$, there exists a pendent vertex $a_2$ such that $aa_2 \in E(B_{m,n})$. To dominate the vertex $a_2$, $D$ must include the vertex $a_2$ and here we get a contradiction that $< D >$ has two isolated vertices, namely $a_1$ and $a_2$. Thus $u \neq a_1$.
Similarly we can prove that $u \neq a_i$ for all $i$ with $1 \leq i \leq m$ and $u \neq c_i$ for all $i$ with $1 \leq i \leq n$.
Case 2: Suppose $u = b$.
In this case $a \notin D$. Thus, to dominate the vertex $a_1$, $D$ must include the vertex $a_1$ and here we et a contradiction that $< D >$ has two isolated vertices, namely $a_1$ and $b$. Thus $u \neq b$.
From the above two cases, we can conclude that $u$ must be equal to either $a$ or $c$.
Case 3: Suppose $u = a$. In this case $b \notin D$. Thus, to dominate the vertex $c_1$, $D$ must include the vertex $c$ or $c_i$ for some $1 \leq i \leq n$.
Sub case 3.1: Suppose $c \in D$.
In this case $c$ is not isolated in $< D >$ and so $D$ must include a vertex $c_i$ for some $1 \leq i \leq n$. Thus $\gamma_0^U(B_{m,n}) \geq 3$.
Sub case 3.2: Suppose $c_i \in D$ for some $1 \leq i \leq n$.
In this case $c_i$ is not isolated in $< D >$ and so $D$ must include the vertex $c$. Thus $\gamma_0^U(B_{m,n}) \geq 3$. Note that the set $\{a, c, c_1\}$ is a UIDS and $a$ is isolated in $< D >$. Thus $\gamma_0^U(B_{m,n}) \leq 3$ and so $\gamma_0^U(B_{m,n}) = 3$.
As proved in Case 3, we can prove that $\gamma_0^U(B_{m,n}) = 3$ when $u = c$.

**Theorem 2.8** For $n \geq 2$, the Helm graph $H_n$ admits UIDS and $\gamma_0^U(H_n) = n$.

Proof. Let the vertex set of the Helm graph $H_n$ be $V(H_n) = \{v_0, v_1, v_2, \ldots, v_n\} \cup \{v_1', v_2', \ldots, v_n'\}$ such that the subgraph induced by $\{v_0, v_1, v_2, \ldots, v_n\}$ is a wheel with $v_0$ as the center and the edges $\{v_i v_i' : 1 \leq i \leq n\}$ are pendent edges in $H_n$. Let $D$ be an UIDS of $H_n$ and $u$ be the isolated vertex in $< D >$.
Case 1: Suppose $u = v_0$. Then $v_1 \notin D$ and so to dominate the vertex $v_1'$, $D$ must include $v_1'$. Here we get a contradiction that $< D >$ has two isolated vertices, namely $v_1'$ and $v_0$. Thus $u \neq v_0$.
Case 2: Suppose $u = v_i$ for some $1 \leq i \leq n$, with out loss of generality, let us assume $u = v_1$. Then $v_2 \notin D$ and so to dominate the vertex $v_2'$, $D$ must include $v_2'$. Here we get a contradiction that $< D >$ has two isolated vertices, namely $v_1$ and $v_2'$. Thus $u \neq v_0$.
From the above two cases, we can conclude that $u$ must be equal to $u = v_i'$ for some $1 \leq i \leq n$, with out loss of generality, let us assume $u = v_1$.
For each integer $i$ with $2 \leq i \leq n$, the vertex $v_i'$ is pendent and so either $v_i'$ or $v_i$ must be in $D$. Thus $\gamma_0^U(H_n) \geq n$.
Also the set $\{v_1', v_2, v_3, v_4, \ldots, v_n\}$ is an UIDS with $n$ elements and $v_1'$ is the isolated vertex in $< D >$. Thus $\gamma_0^U(H_n) \leq n$ and so Thus $\gamma_0^U(H_n) = n$.

## 3. References

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