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Acyclic Coloring of Mycielskian of Graphs

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Abstract

An acyclic coloring of a graph is a proper coloring without bichromatic cycles such that the union of any two color classes results in a forest. The concept of acyclic coloring is crucial in the computation of Hessians, classification of kekule structures, statistical mechanics and coding theory. In this paper, the acyclic coloring parameters for the Mycielskian of graphs have been computed. Moreover, the relation between the acyclic coloring parameters have been determined for the graphs under consideration. Also, algorithms have been developed for the acyclic coloring of the above-mentioned graphs.

Keywords: Acyclic chromatic number, Acyclic chromatic index, Star graph, Banana tree graph, Firecracker graph.

AMS Subject Classification: 05C15, 05C85

1 Introduction

Computer networks can be visualized as graphs, with hubs or servers acting as vertices and the edges used as a connecting medium. The concept of acyclic vertex coloring for a graph was first proposed by Grünbaum [9]. Fiamčik [5] introduced the idea of acyclic edge coloring. It is an NP-complete problem to calculate the number of cycles and also to determine the acyclic chromatic index for any graph [1,2]. Therefore, finding the acyclic chromatic index is challenging. For a given G and an integer k, Kostochka [11] proved that it is an NP-complete problem to determine whether the acyclic chromatic number of G is at most k. Applications of acyclic colorings include multivariable calculus [6], chemistry [3,7], statistical mechanics [13], and so on.

A simple undirected graph is represented by $G = (V, E)$, where V and E denotes the vertex set and the edge set respectively. It is a mathematical representation of a network describing the relationship between V and E. A *k-vertex coloring* of G can be considered as a labeling $f: V(G) \longrightarrow [k]$ in which the labels denote the colors. Further, the set of vertices having a single color is called a color class. A k-vertex coloring is said to be a *proper* coloring, if adjacent vertices are assigned different colors. A coloring of a graph G is *acyclic*, if it is a proper coloring and does not have any bichromatic cycles in G [9]. The *acyclic chromatic number* of G, denoted by $\chi_a(G)$, is the least number of colors used in an acyclic coloring of G. A labeling $f: E(G) \longrightarrow [k]$ is a *k-edge coloring* of a graph G. Here, the labels are colors, and a color class is a set of edges with one color. The k-

edge coloring of G is said to be *proper*, if edges incident to a vertex have distinct colors. The chromatic index of a graph G is denoted by $\chi'(G)$, and is defined as the minimum number of colors required for a proper edge coloring of G. The proper coloring of the edges of a graph G is a*cyclic*, if there are no bichromatic cycles in G [5]. The least number of colors required to acyclically edge-color G, denoted by $\chi'_a(G)$, is the *acyclic chromatic index* of G.

Theorem 1.1. [10] *If $\Delta(G)$ represents the maximum degree of the graph G, then*

$\chi'_a(G) \geq \chi'(G) \geq \Delta(G)$.

There have been a lot of research into the acyclic vertex and edge coloring of planar graphs, leading to the discovery of several bounds and their improvements. Some of the current works include acyclic coloring of certain graphs [8], local conditions for planar graphs of acyclic edge coloring [18], acyclic coloring of graphs with maximum degree 7 [17], acyclic edge coloring of chordal graphs with bounded degree [12], a new bound on the acyclic edge chromatic number [4], acyclic edge coloring of IC-planar graphs [15], and acyclic coloring of graphs with maximum degree at most six [16]. An open and challenging problem is to determine the acyclic coloring parameters for graphs which are non-planar.

Mycielski introduced a Mycielskian graph to construct a graph with small clique number and high chromatic number. The Mycielski construction [14] of triangle-free graphs of arbitrary chromatic number is one of the earliest and simplest constructions. In this study, we have estimated the acyclic coloring parameters, namely the acyclic chromatic number and its edge version called the acyclic chromatic index, for the Mycielskian of star graph, banana tree graph and firecracker graph which are non-planar graphs. Also, the relation between the acyclic coloring parameters have been found. Further, algorithms are also developed for the graphs under consideration.

Definition 1.2. [14] *Consider G to be a graph with the vertex set and edge set represented by V and E respectively. The Mycielskian of G is denoted by $\mu(G)$. It is a graph in which the vertex set is represented by $V \cup V' \cup \{u\}$ and the edge set is given by $E' = E \cup \{xy', x'y: xy \in E\} \cup \{x'u: x' \in V'\}$. Here, $V' = \{x': x \in V\}$ is disjoint from V. The root vertex of $\mu(G)$ is the vertex u.*

The rest of the paper is structured as follows. The key findings are laid up in section 2. Conclusion is given in section 3.

2 Main Results

2.1 Acyclic coloring parameters of Mycielskian of star graph $\mu(S_k)$

The star graph S_k, also known as a k-star, is a tree with k vertices. It has a single vertex of degree $k-1$ called the central vertex (say v_1). The remaining $k-1$ vertices, denoted by $v_2, v_3, \dots, v_k$ are of degree 1.

Theorem 2.1. *If $\mu(S_k)$ represents the Mycielskian of a star graph S_k, then*

$\chi_a(\mu(S_k)) = 3, k \geq 2$.

Proof. For an acyclic coloring, every cycle should be given at least three colors. Thus, we have, $\chi_a(\mu(S_k)) \geq 3$. To prove that $\chi_a(\mu(S_k)) \leq 3$. Let $V = \{v_i: 1 \leq i \leq k\}$ denote

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the vertex set of S_k. Consider the central vertex v_1 to be in level 1 and the vertices $v_2, v_3, \dots, v_k$ in level 2. Let the vertices in $V' = \{v_i' : 1 \leq i \leq k\}$ belong to level 3 and the root vertex u in level 4. Denote by $c[V(L_p)]$, the color assigned for the vertices in each level L_p, where $p = 1, 2, 3, 4$. Then, $c[V(L_1)] = 1$, $c[V(L_2)] = 2$, $c[V(L_3)] = 3$ and $c[V(L_4)] = 2$. The procedure for assigning such colors is as follows: $c[V(L_1)] = 1$, since $V(L_1) = \{v_1\}$. $V(L_2)$ are adjacent to $V(L_1)$ and should be assigned a color different from 1. Also, the vertices in L_2, being disjoint, are assigned color 2. Next, assume $c[V(L_3)] = 1$. There are 4-cycles between the levels 1, 2 and 3. Hence, each cycle would have been colored using only 2 colors, contradicting the definition of acyclic coloring. Hence, $V(L_3)$ should be assigned a third distinct color (say 3). Further, $c[V(L_4)] = j$, where $j \in \{1, 2, 3\}$. There are 4-cycles when we consider the levels 1, 3 and 4. Therefore, assign a color distinct from $c[V(L_1)]$ and $c[V(L_3)]$. Moreover, $c[V(L_4)] \neq 3$, since $V(L_4)$ and $V(L_3)$ are adjacent. Hence, $c[V(L_4)] = 2$. Thus, we get $\chi_a(\mu(S_k)) \leq 3$. Therefore, $\chi_a(\mu(S_k)) = 3$.

Claim: Coloring is proper

Vertices in levels 1, 2 and 3 have been assigned three distinct colors. Further, there are no edges connecting the vertices in levels 2 and 4, which have been assigned the same color.

Claim: Coloring is acyclic

There are no cycles of length 3 in the graph. Hence, we consider 4-cycles and cycles of length greater than four. 4-cycles are present between the levels 1, 2 and 3. Similarly, 4-cycles are present between the levels 1, 3 and 4. They have been colored using 3 distinct colors by the coloring scheme mentioned in the proof, thus ensuring the absence of bichromatic cycles in the graph. Any other cycle must pass through the 4-cycles, if its length is greater than 4. Hence, there are no two-colored cycles in the graph.

Further, if we take the union of any two color classes, we get a forest. Thus, the coloring coloring is proper and acyclic. □

Algorithm 1: Acyclic vertex coloring of Mycielskian of star graph $\mu(S_k)$

Input: u, v_i, v_i', where u represents the root vertex of $\mu(S_k)$, v_i represents the vertices of S_k, v_i' represents the remaining vertices of $\mu(S_k)$

Output: Acyclic chromatic number of $\mu(S_k)$

Initialize $k = 2$
Label v_1 as 1
Label u as 2
for $2 \leq i \leq k$ **do**
  Label v_i as 2
for $1 \leq i \leq k$ **do**
  Label v_i' as 3

Theorem 2.2. *If $\mu(S_k)$ represents the Mycielskian of a star graph S_k, then*

$\chi_a'(\mu(S_k)) = 2k - 2, k \geq 3$.

Proof. By Theorem 1.1, $\chi_a'(\mu(S_k)) \geq \Delta$. Let v_1 represent the central vertex of S_k. Then, by

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Definition 1.2, $deg(v_1) = 2k-2, deg(v_1') = deg(u) = k$ and $deg(v_i) = deg(v_i') = 2$, for $2 \leq i \leq k$. Thus, the maximum degree is $2k-2$. Hence, to prove that $\chi_a'(\mu(S_k)) \leq \Delta$. Let $c(v_x, v_y)$ denote the color assigned to the edge (v_x, v_y). We divide the edge set E of $\mu(S_k)$ into three parts for the coloring and consider the following cases:

Case 1: Edges incident on the central vertex v_1

Each edge of S_k is assigned a unique color, since they are incident on a common central vertex v_1. Hence, for $2 \leq i \leq k$,

$$c(v_1, v_i) = i - 1 \quad (2.1)$$

Thus, $k-1$ colors have been assigned to the edges of S_k (say $1, 2, \dots, k-1$). Since, $deg(v_1) = 2k-2$, the remaining edges incident to v_1 are assigned $k-1$ distinct colors (say $k, k+1, \dots, 2k-2$). Hence, for $2 \leq i \leq k$,

$$c(v_1, v_i') = k + i - 2 \quad (2.2)$$

Thus, in this case, $2k-2$ colors have been used.

Case 2: Edges incident on the vertex u

Each edge incident on u should be assigned a distinct color. Further, the edges (v_1, v_i') and (u, v_i') are adjacent for all i. By case 1, the edges (v_1, v_i') have received the colors $k, k+1, \dots, 2k-2$. Hence, the edges (u, v_i') could be assigned the colors $1, 2, \dots, k-1, k$ such that adjacent edges get distinct colors. The number of colors assigned to the edges incident on u is k, since $deg(u) = k$. Therefore, for $1 \leq i \leq k$,

$$c(u, v_i') = i \quad (2.3)$$

Here, the colors used in case 1 have been reused.

Case 3: Edges incident on the vertex v_1' except (u, v_1')

For $2 \leq i \leq k$, $c(v_1', v_i) \neq 1$, since $c(u, v_1') = 1$. Also, $deg(v_1') = k$. Hence, $k-1$ colors must be assigned to the edges (v_1', v_i), where $i \neq 1$. In order to assign distinct colors to adjacent edges, we consider, for $2 \leq i \leq k$,

$$c(v_1', v_i) = c(v_1, v_i) + 1 = i \quad (2.4)$$

by equation (2.1). Here, the colors used in cases 1 and 2 have been reused.

Thus, in total, $2k-2$ colors have been assigned to the edges of $\mu(S_k)$, which represents the maximum degree of the graph. Hence, $\chi_a'(\mu(S_k)) \leq \Delta$. Therefore, $\chi_a'(\mu(S_k)) = \Delta = 2k-2$.

Claim: Coloring is proper

By the cases 1, 2 and 3, the edges (v_1, v_i), (v_1, v_i'), (u, v_i') and (v_1', v_i) have been assigned distinct colors. Further, the adjacent edges (v_1', v_i) and (v_1, v_i) have distinct colors by equation (2.4). Moreover, the adjacent edges (v_1, v_i') and (u, v_i') have distinct colors, by the argument stated in case 2. Thus, we can see that there is a proper coloring as per the coloring scheme mentioned in the above three cases.

Claim: Coloring is acyclic

There are no cycles of length 3 in the graph. Hence, we consider 4-cycles and cycles of length greater than 4. The 4-cycles present between the vertices v_1 and v_1' is of the form $(v_1, v_i, v_1', v_j, v_1)$, where $i \neq j$. Here, (v_1, v_i) and (v_j, v_1) have two distinct colors. Further, $c(v_1', v_i) = c(v_1, v_i) + 1$ giving a third distinct color. Similarly, the 4-cycles present between the vertices v_1 and u is of the form $(v_1, v_i', u, v_j', v_1)$, where $i \neq j$. Here, (v_1, v_i') and (v_j', v_1) have two distinct colors. Further, by equations (2.2) and (2.3), (v_1, v_i') and (v_j', u) have two different color assignments giving a third distinct color. Any other cycle

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must pass through the 4-cycles, if its length is greater than 4. Hence, there are no two-colored cycles in the graph.

Further, the union of any two color classes results in a forest. Thus, the coloring is proper and acyclic. □

Remark 2.3. *The Mycielskian of* S_2*, denoted by* $\mu(S_2)$*, is a 5-cycle. Hence,* $\chi'_a(\mu(S_2)) = 3$.

Remark 2.4. $\chi_a(\mu(S_k)) \le \chi'_a(\mu(S_k))$.

Algorithm 2: Acyclic edge coloring of Mycielskian of star graph $\mu(S_k)$

Input: u, v_i, v'_i, where u represents the root vertex of $\mu(S_k)$, v_i represents the vertices of S_k, v'_i represents the remaining vertices of $\mu(S_k)$

Output: Acyclic chromatic index of $\mu(S_k)$

Initialize $k = 3$
for $2 \le i \le k$ **do**
  Label (v_1, v_i) as $i - 1$
  Label (v_1, v'_i) as $k + i - 2$
  Label (v'_1, v_i) as i
for $1 \le i \le k$ **do**
  Label (u, v'_i) as i

2.2 Acyclic coloring parameters of Mycielskian of banana tree graph $\mu(B_{n,k})$

The banana tree graph, denoted by $B_{n,k}$, has $nk + 1$ vertices and nk edges. It is obtained by joining one leaf from each of the n copies of a k-star graph with a single root vertex that is different from all the stars.

Theorem 2.5. *If* $\mu(B_{n,k})$ *represents the Mycielskian of a banana tree graph* $B_{n,k}$*, then* $\chi_a\left(\mu(B_{n,k})\right) = 4$, $n \ge 2$, $k \ge 4$.

Proof. For an acyclic coloring, every cycle should be given at least three colors. Thus, we have, $\chi_a\left(\mu(B_{n,k})\right) \ge 3$. Let $V = \{v_i : 1 \le i \le nk + 1\}$ denote the vertex set of $B_{n,k}$ where the number of vertices $|V| = nk + 1$. Denote by $c(v_t)$, the colors assigned for the vertices v_t. Let v_p, $2 \le p \le n + 1$ denote the central vertices of the k-star present in $B_{n,k}$. Let v_1 denote the root vertex of $B_{n,k}$. By definition of $B_{n,k}$, v_p and v_1 are non-adjacent. Hence, $c(v_p) = c(v_1) = 1$. The remaining vertices of $B_{n,k}$ could be assigned a second color (say 2). Next, consider the Mycielskian of $B_{n,k}$. Then, the vertices in $V' = \{v'_i : 1 \le i \le nk + 1\}$ could be assigned a third distinct color (say 3). Finally, the root vertex u of $\mu(B_{n,k})$ is assigned a fourth distinct color (say 4). The procedure for assigning such colors is as follows: Since $B_{n,k}$ is a tree, two distinct colors 1 and 2 could be assigned for its vertices. A third color should be introduced for the vertices in V', since 3-cycles and 4-cycles are formed by the edges $E \cup \{xy', x'y : xy \in E\}$, where E represents the edge set of $B_{n,k}$. Also, V' is disjoint from V. Therefore, $c(v'_i) = 3$. Next, consider the following three cases for the assignment of colors to u.

Case 1: $c(u) = 1$

There exists 4-cycles of the form $(v_i, v_i', u, v_j', v_i)$, where $i \neq j$. The color assignment will be either of the form $1-3-1-3-1$ or $2-3-1-3-2$. Hence, if $c(v_i) = 1$, for some i, then bichromatic cycles exists. Thus, $c(u) \neq 1$.

Case 2: $c(u) = 2$

There exists 4-cycles of the form $(v_i, v_i', u, v_j', v_i)$, where $i \neq j$. The color assignment will be either of the form $1-3-2-3-1$ or $2-3-2-3-2$. Hence, if $c(v_i) = 2$, for some i, then bichromatic cycles exists. Thus, $c(u) \neq 2$.

Case 3: $c(u) = 3$

Since $c(v_i') = 3$, for all i, $c(u) \neq 3$, by the definition of proper coloring.

Thus, three colors are not sufficient. Hence, $c(u) = 4$. Therefore, $\chi_a\left(\mu(B_{n,k})\right) = 4$.

Claim: Coloring is proper

Adjacent vertices in $B_{n,k}$ have been assigned distinct colors 1 and 2. Vertices in V' have been assigned color 3 and they are adjacent to the vertices in $B_{n,k}$. Further, the vertex u and the vertices in V' are adjacent but have been assigned distinct colors 3 and 4. Hence, by the coloring scheme mentioned in the proof, the coloring is proper.

Claim: Coloring is acyclic

The girth of the graph is 3, where the 3-cycles are of the form (v_i, v_j', v_k, v_i). There are no bichromatic cycles, since $c(v_j') = 3$, $c(v_i) = 1$ and $c(v_k) = 2$. Next, consider cycles of length 4. The 4-cycles present between $B_{n,k}$ and v_i', of the form $(v_i, v_l, v_i', v_m, v_i)$, have been assigned three distinct colors by the coloring scheme mentioned in the proof. Further, the 4-cycles involving the vertex u, present between $B_{n,k}$, v_i' and u, have been colored using 3 distinct colors as mentioned in cases 1 and 2, ensuring the absence of bichromatic cycles in the graph. Moreover, any other cycle must pass through the 3-cycles and 4-cycles, if its length is greater than 4. Hence, there are no two-colored cycles in the graph.

Further, if we take the union of any two color classes, we arrive at a forest. Thus, the coloring is proper and acyclic. □

Algorithm 3: Acyclic vertex coloring of Mycielskian of banana tree graph $\mu(B_{n,k})$

Input: v_1, v_p, v_i, u, v_i', where v_1 represents the root vertex of $B_{n,k}$, v_p represents the central vertices of S_k, v_i represents the vertices of $B_{n,k}$ other than v_1 and v_p, u represents the root vertex of $\mu(B_{n,k})$ and v_i' represents the remaining vertices of $\mu(B_{n,k})$

Output: Acyclic chromatic number of $\mu(B_{n,k})$

1 Initialize $n = 2, k = 4$

2 Label v_1 as 1

3 **for** $2 \leq p \leq n+1$ **do**

4 | Label v_p as 1

5 **for** $1 \leq i \leq nk+1$ **do**

6 | Label v_i as 2

7 Label v_i' as 3
8 Label u as 4

Theorem 2.6. *If $\mu(B_{n,k})$ represents the Mycielskian of a banana tree graph $B_{n,k}$, then $\chi_a'\left(\mu(B_{n,k})\right) = nk+1$, $n \geq 2$, $k \geq 4$.*

Proof. Let $V = \{v_i : 1 \leq i \leq nk+1\}$ denote the vertex set of $B_{n,k}$, where the number of vertices $|V| = nk+1$. We label the root vertex as v_1 and consider it to be in level 1. In level 2, consider the vertices adjacent to v_1 and denote them by v_j, $j \in \{n+2, n+3, \ldots, 2n+1\}$. We consider the central vertices of S_k, labeled v_p, $2 \leq p \leq n+1$, to be in level 3. In level 4, consider the remaining vertices of $B_{n,k}$, denoted by v_q, $q \in \{2n+2, 2n+3, \ldots, nk+1\}$. The vertices in the level 4 are labeled in order by first considering the adjacent vertices of v_2, followed by v_3 till v_{n+1}. Next, consider the Mycielskian of $B_{n,k}$. The root vertex of $\mu(B_{n,k})$ is denoted by u. The remaining vertices belong to the vertex set $V' = \{v_i' : 1 \leq i \leq nk+1\}$.

By the definition of banana tree graph, we have

$$\Delta(B_{n,k}) = \begin{cases} k-1, & if\ n < k \\ n, & otherwise \end{cases} \qquad (2.5)$$

Since, $\Delta(B_{n,k}) < |V(B_{n,k})|$, we have $\Delta\left(\mu(B_{n,k})\right) = nk+1$. By Theorem 1.1, $\chi_a'\left(\mu(B_{n,k})\right) \geq \Delta$. Therefore, to prove that $\chi_a'\left(\mu(B_{n,k})\right) \leq \Delta$.

Let $c(v_x, v_y)$ denote the color assigned to the edge (v_x, v_y). We divide the edge set E of $\mu(B_{n,k})$ into three parts for the coloring and consider the following cases:

Case 1: Edges incident on u.

By Definition 1.2, $deg(u) = nk+1$. Each edge incident on u should be assigned $nk+1$ distinct colors. Hence, for $1 \leq i \leq nk+1$,

$$c(u, v_i') = i \qquad (2.6)$$

Case 2: Edges of $B_{n,k}$

By equation (2.5), the number of colors assigned to the edges of $B_{n,k}$ could be $k-1$ or n, depending on whether $n < k$ or $n \geq k$. Thus, $B_{n,k}$ can be properly and acyclically colored using just $\Delta(B_{n,k})$ colors. For the edge (v_1, v_j), $j \neq 1$,

$$c(v_1, v_j) = x \qquad (2.7)$$

where $x \in \{1, 2, \ldots, n\}$. Further, the edges (v_1, v_j) and (v_j, v_p) are adjacent. Therefore,

$$c(v_j, v_p) = x \bmod(n) + 1 \qquad (2.8)$$

Moreover, the edges (v_p, v_q) must be assigned a color distinct from $c(v_j, v_p)$, since they are adjacent. Hence,

$$c(v_p, v_q) = z \qquad (2.9)$$

where $z \in \{1, 2, \ldots, \Delta(B_{n,k})\}$ and $z \neq c(v_j, v_p)$.

Case 3: Edges incident on the vertices v_i'

Let v_l denote the vertices of $B_{n,k}$. Then, consider the coloring for the edges (v_i', v_l) as follows:

$$c(v_i', v_l) = ((i + l - 1)\ mod(nk+1)) + 1 \qquad (2.10)$$

However, if $v_l \in V(S_k)$, then replace the colors $1, 2, \ldots, \Delta(B_{n,k})$ by the colors $\Delta(B_{n,k}) + 1, \Delta(B_{n,k}) + 2, \ldots, 2\,\Delta(B_{n,k})$. This ensures that edges incident on the vertices v_i' and the vertices of $B_{n,k}$ are distinct.

Hence, $\chi_a'\left(\mu(B_{n,k})\right) \leq \Delta$. Therefore, $\chi_a'\left(\mu(B_{n,k})\right) = \Delta = nk + 1$.

Claim: Coloring is proper

By the cases 1, 2 and 3, it is clear that the edges incident on u, v_i' and the edges of $B_{n,k}$ have been assigned distinct colors. Further, by equation (2.10), the adjacent edges of $B_{n,k}$ and v_i' have distinct colors. Moreover, the adjacent edges (u, v_i') and the edges connecting v_i' and $B_{n,k}$ have distinct colors by the equations (2.6) and (2.10).

Claim: Coloring is acyclic

There are no cycles of length 3 in the graph. Hence, we consider 4-cycles and cycles of length greater than 4. The 4-cycles present between the vertices of $B_{n,k}$ and v_i' is of the form $(v_i', v_r, v_s, v_t, v_i')$, where $v_r, v_s, v_t \in V$. Here, (v_r, v_s) and (v_s, v_t) have two distinct colors. Further, by equation (2.10), the edges (v_i', v_r) and (v_t, v_i') have the third and fourth color. Similarly, the 4-cycles present between the vertices of $B_{n,k}$, v_i' and u is of the form (u, v_e', v_l, v_f', u), where $v_e', v_f' \in V'$ and $v_l \in V$. Here, (u, v_e') and (v_f', u), have two distinct colors. Further, by equation (2.10), (v_e', v_l) and (v_l, v_f') have the third and fourth color. Thus, every 4-cycle have been assigned four distinct colors. Any other cycle must pass through the 4-cycles, if its length is greater than 4. Hence, there are no two-colored cycles in the graph.

Further, if we take the union of any two color classes, we get a forest. Thus, the coloring is proper and acyclic. □

Algorithm 4: Acyclic edge coloring of Mycielskian of banana tree graph $\mu(B_{n,k})$

Input: $u, v_1, v_j, v_p, v_q, v_i'$, where u represents the root vertex of $\mu(B_{n,k})$, v_1 represents the root vertex of $B_{n,k}$, v_j represents the vertices of $B_{n,k}$ adjacent to v_1, v_p represents the central vertices of S_k, v_q represents the remaining vertices of $B_{n,k}$, v_i' represents the remaining vertices of $\mu(B_{n,k})$

Output: Acyclic chromatic index of $\mu(B_{n,k})$

Initialize $n = 2$, $k = 4$
for $j \in \{n+2, n+3, \ldots, 2n+1\}$, $j \neq 1$ **do**
  Label (v_1, v_j) as x, $x \in \{1, 2, \ldots, n\}$
  for $2 \leq p \leq n+1$, $j \neq p$ **do**
    Label (v_j, v_p) as $x \bmod (n) + 1$, $x \in \{1, 2, \ldots, n\}$
    for $q \in \{2n+2, 2n+3, \ldots, nk+1\}$ **do**
      Label (v_p, v_q) as z, where $z \in \{1, 2, \ldots, \Delta(B_{n,k})\}$ and $z \neq$ label of (v_j, v_p)
for $1 \leq i \leq nk+1$, $v_l \in V(B_{n,k})$ **do**
  Label (u, v_i') as i
  Label (v_i', v_l) as $((i + l - 1)\ mod(nk+1)) + 1$
if $v_l \in V(S_k)$ **then**
  Replace the labels $1, 2, \ldots, \Delta(B_{n,k})$ of (v_i', v_l) by

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$\Delta(B_{n,k})+1, \Delta(B_{n,k})+2, \ldots, 2\,\Delta(B_{n,k})$

Remark 2.7. $\chi_a\left(\mu(B_{n,k})\right) < \chi'_a\left(\mu(B_{n,k})\right)$.

2.3 Acyclic coloring parameters of Mycielskian of firecracker graph $\mu(F_{n,k})$

The firecracker graph, denoted by $F_{n,k}$, has nk vertices and $nk-1$ edges. It is obtained by the concatenation of n k-stars by linking one leaf from each.

Theorem 2.8. *If $\mu(F_{n,k})$ represents the Mycielskian of a firecracker graph $F_{n,k}$, then $\chi_a\left(\mu(F_{n,k})\right) = 4$, $n \geq 3$, $k \geq 3$.*

Proof. For an acyclic coloring, every cycle should be given at least three colors. Thus, we have, $\chi_a\left(\mu(F_{n,k})\right) \geq 3$. Let $V = \{v_i : 1 \leq i \leq nk\}$ denote the vertex set of $F_{n,k}$, where the number of vertices $|V| = nk$. Denote by $c(v_t)$, the colors assigned for the vertices v_t. Let v_p, $1 \leq p \leq n$ denote the central vertices of the k-star present in $F_{n,k}$. Then, for $1 \leq p \leq n$, $c(v_{2p-1}) = 1$, and $c(v_{2p}) = 2$.Then, the remaining vertices adjacent to v_{2p-1} and v_{2p} are assigned the colors 2 and 1 respectively. Next, consider the Mycielskian of $F_{n,k}$. Then, the vertices in $V' = \{v'_i : 1 \leq i \leq nk\}$ could be assigned a third distinct color (say 3). Finally, the root vertex u of $\mu(F_{n,k})$ is assigned a fourth distinct color (say 4). The procedure for assigning such colors is as follows: $F_{n,k}$ being a tree could be colored with two distinct colors (say 1 and 2). The vertices of $F_{n,k}$ are adjacent to the vertices v'_i. Also, V' is disjoint from V. Therefore, $c(v'_i) = 3$. Next, consider the following three cases for the assignment of colors to u.

Case 1: $c(u) = 1$

There exists 4-cycles of the form $(v_i, v'_e, u, v'_f, v_i)$, where $v'_e, v'_f \in V'$. The color assignment will be either of the form $1-3-1-3-1$ or $2-3-1-3-2$. Hence, if $c(v_i) = 1$, for some i, then bichromatic cycles exist. Thus, $c(u) \neq 1$.

Case 2: $c(u) = 2$

There exists 4-cycles of the form $(v_i, v'_e, u, v'_f, v_i)$, where $v'_e, v'_f \in V'$. The color assignment will be either of the form $1-3-2-3-1$ or $2-3-2-3-2$. Hence, if $c(v_i) = 2$, for some i, then bichromatic cycles exist. Thus, $c(u) \neq 2$.

Case 3: $c(u) = 3$

Since $c(v'_i) = 3$ for all i, $c(u) \neq 3$, by the definition of proper coloring.

Thus, we see that three colors are not sufficient. Hence, $c(u) = 4$. Therefore, $\chi_a\left(\mu(F_{n,k})\right) = 4$.

Claim: Coloring is proper

Adjacent vertices of $F_{n,k}$ have been assigned distinct colors 1 and 2. Vertices of $F_{n,k}$ and v'_i are adjacent and hence v'_i have been assigned a third color. Further, the adjacent vertices v'_i and u have the colors 3 and 4.

Claim: Coloring is acyclic

There are no cycles of length 3 in the graph. Hence, we consider 4-cycles and cycles of length greater than 4. The 4-cycles present between $F_{n,k}$ and v'_i, of the form $(v_i, v_l, v'_i, v_m, v_i)$, have been assigned three distinct colors by the coloring scheme

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mentioned in the proof. Further, the 4-cycles involving the vertex u, present between $F_{n,k}$, v_i' and u have been colored using 3 distinct colors as mentioned in the cases 1 and 2, thereby ensuring the absence of bichromatic cycles in the graph. Any other cycle must pass through the 4-cycles, if its length is greater than 4. Hence, there are no two-colored cycles in the graph.

Further, the union of any two color classes results in a forest. Thus, the coloring is proper and acyclic. □

Algorithm 5: Acyclic vertex coloring of Mycielskian of firecracker graph $\mu(F_{n,k})$

Input: v_p, v_i, v_i', u, where v_p represents the central vertices of k-star, v_i represents the vertices of $F_{n,k}$ other than v_p, u represents the root vertex of $\mu(F_{n,k})$, v_i' represents the remaining vertices of $\mu(F_{n,k})$

Output: Acyclic chromatic number of $\mu(F_{n,k})$

```
Initialize n = 3, k = 3
for 1 ≤ p ≤ n do
    Label v_{2p−1} as 1
    Label v_{2p} as 2
    for n + 1 ≤ i ≤ nk + 1 do
        Label the v_i adjacent to v_{2p−1} as 2
        Label the v_i adjacent to v_{2p} as 1
        Label v_i' as 3
Label u as 4
```

Theorem 2.9. *If $\mu(F_{n,k})$ represents the Mycielskian of a firecracker graph $F_{n,k}$, then $\chi_a'\left(\mu(F_{n,k})\right) = nk$, $n \geq 3$, $k \geq 3$.*

Proof. Let $V = \{v_i : 1 \leq i \leq nk\}$ denote the vertex set of $F_{n,k}$, where the number of vertices $|V| = nk$. Label the vertices of $F_{n,k}$ as follows: We label the leaf of n copies of the k-star as $v_p = v_1, v_2, \ldots, v_n$ and it forms a path. The central vertices of the k-star are labeled as $v_q = v_{n+1}, v_{n+2}, \ldots, v_{2n}$. The remaining vertices are labeled as $v_r = v_{2n+1}, v_{2n+2}, \ldots, v_{nk}$ in order by first considering the adjacent vertices of v_{n+1} followed by v_{n+2} till v_{2n}. Next, consider the Mycielskian of $F_{n,k}$. Then, the root vertex of $\mu(F_{n,k})$ is denoted by u. The remaining vertices belong to the vertex set $V' = \{v_i' : 1 \leq i \leq nk\}$.

By the definition of firecracker graph, for $n \geq 3$,

$$\Delta(F_{n,k}) = \begin{cases} k, & if\ k = 3 \\ k-1, & if\ k \geq 4 \end{cases} \tag{2.11}$$

Since, $\Delta(F_{n,k}) < |V(F_{n,k})|$, we have $\Delta\left(\mu(F_{n,k})\right) = nk$. By Theorem 1.1, $\chi_a'\left(\mu(F_{n,k})\right) \geq \Delta$. Therefore, to prove that $\chi_a'\left(\mu(F_{n,k})\right) \leq \Delta$.

Let $c(v_x, v_y)$ denote the color assigned to the edge (v_x, v_y). We divide the edge set

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E of $\mu(F_{n,k})$ into three parts for the coloring and consider the following cases:

Case 1: Edges incident on u.

By Definition 1.2, $deg(u) = nk$. Each edge incident on u should be assigned nk distinct colors. Hence, for $1 \leq i \leq nk$,

$$c(u, v_i') = i \tag{2.12}$$

Case 2: Edges of $F_{n,k}$

By equation (2.11), the number of colors assigned to the edges of $F_{n,k}$ could be k or $k-1$, depending on whether $k = 3$ or $k \geq 4$. Thus, $F_{n,k}$ can be properly and acyclically colored using just Δ colors. For the edges incident on the vertices $v_p, 1 \leq p \leq n-2$,

$$c(v_p, v_{p+1}) = 1; c(v_{p+1}, v_{p+2}) = 2 \tag{2.13}$$

Since the vertices v_p and v_q are adjacent, the edges connecting them is assigned a third distinct color. Hence,

$$c(v_p, v_q) = 3 \tag{2.14}$$

Further, the vertices v_q and v_r are adjacent. So, the edges connecting them should be assigned a color distinct from 3. Therefore,

$$c(v_q, v_r) = z \tag{2.15}$$

where $z \in \{1, 2, \ldots, \Delta(F_{n,k})\}$ and $z \neq 3$.

Case 3: Edges incident on the vertices v_i'

Consider the coloring for the edges (v_i', v_l), where $v_l \in V$ as follows:

$$c\left(v_{2n}', v_{\Delta\left(\mu(F_{n,k})\right)}\right) = 2n + 1; c(v_i', v_l) = ((i + l - 1) \, mod(nk)) + 1 \tag{2.16}$$

However, if $c(v_i', v_l) \in \{1, 2, \ldots, \Delta(F_{n,k}\}$, then consider $c(v_i', v_l)$ to belong to $\{\Delta(F_{n,k}) + 1, \Delta(F_{n,k}) + 2, \ldots, \Delta\left(\mu(F_{n,k})\right)\}$, such that $c(v_i', v_l) \neq c(v_i', u)$ and $c(v_i', v_r) \neq c(v_i', v_s)$, where $r \neq s$, $\{v_l, v_r, v_s\} \in V$. This ensures that edges incident on the vertices v_i' and the vertices of $F_{n,k}$ are distinct. Hence, $\chi_a'\left(\mu(F_{n,k})\right) \leq \Delta$.

Therefore, $\chi_a'\left(\mu(F_{n,k})\right) = \Delta = nk$.

Claim: Coloring is proper

By the cases 1, 2 and 3, it is clear that the edges incident on u, v_i' and the edges of $F_{n,k}$ have been assigned distinct colors. Further, by equation (2.16), the adjacent edges of $F_{n,k}$ and v_i' have distinct colors. Moreover, the adjacent edges (u, v_i') and the edges connecting v_i' and $F_{n,k}$ have distinct colors by the argument stated in cases 1 and 3.

Claim: Coloring is acyclic

There are no cycles of length 3 in the graph. Hence, we consider 4-cycles and cycles of length greater than 4. The 4-cycles present between the vertices of $F_{n,k}$ and v_i' is of the form $(v_i', v_r, v_s, v_t, v_i')$, where $v_r, v_s, v_t \in V$. Here, (v_r, v_s) and (v_s, v_t) have two distinct colors. Further, by equation (2.16), the edges (v_i', v_r) and (v_t, v_i') have the third and fourth color. Similarly, the 4-cycles present between the vertices of $F_{n,k}$, v_i' and u is of the form (u, v_e', v_l, v_f', u), where v_e', $v_f' \in V'$ and $v_l \in V$. Here, (u, v_e') and (v_f', u) have two distinct colors. Further, by equation (2.16), (v_e', v_l) and (v_l, v_f') have the third and fourth color. Thus, every 4-cycle have been assigned four distinct colors. Any other cycle must pass through the 4-cycles, if its length is greater than 4. Hence, there are no two-colored cycles in the graph.

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Further, if we take the union of any two color classes, we arrive at a forest. Thus, the coloring is proper and acyclic. □

Algorithm 6: Acyclic edge coloring of Mycielskian of firecracker graph $\mu(F_{n,k})$

Input: v_p, v_q, v_r, u, v_i', where v_p represents the leaves of n copies of the k-star, v_q represents the central vertices of the k-star, v_r represents the remaining vertices of $F_{n,k}$, u represents the root vertex of $\mu(F_{n,k})$, v_i' represents the remaining vertices of $\mu(F_{n,k})$

Output: Acyclic chromatic index of $\mu(F_{n,k})$

Initialize $n = 3,\ k = 3$
for $1 \le p \le n-2$ **do**
  Label (v_p, v_{p+1}) as 1
  Label (v_{p+1}, v_{p+2}) as 2
for $1 \le p \le n$ **do**
  for $n+1 \le q \le 2n$ **do**
    Label (v_p, v_q) as 3
    for $2n+1 \le r \le nk$ **do**
      Label (v_q, v_r) as z, where $z \in \{1, 2, \dots, \Delta(F_{n,k})\}$ and $z \neq 3$
Label $\left(v_{2n}', v_{\Delta(\mu(F_{n,k}))}\right)$ as $2n+1$
for $1 \le i \le nk$ **do**
  Label (v_i', v_l) as $((i + l - 1)\ mod(nk)) + 1$
  if label of $(v_i', v_l) \in \{1, 2, \dots, \Delta(F_{n,k}\}$, **then**
    replace the labels by $\{\Delta(F_{n,k}) + 1, \Delta(F_{n,k}) + 2, \dots, \Delta\left(\mu(F_{n,k})\right)\}$, such that $c(v_i', v_l) \neq c(v_i', u)$ and $c(v_i', v_r) \neq c(v_i', v_s)$, where $r \neq s$, $\{v_l, v_r, v_s\} \in V$

Remark 2.10. $\chi_a\left(\mu(F_{n,k})\right) < \chi_a'\left(\mu(F_{n,k})\right)$

3 Conclusion

We have determined the acyclic coloring parameters for the Mycielskian of three graphs namely, star graph, banana tree graph and firecracker graph. Algorithms are also provided for these graphs. Also, for the graphs under consideration, we have proved $\chi_a(G) \le \chi_a'(G)$. As a further research direction, the acyclic coloring of the Mycielskian of wheel graph, gear graph and helm graph will be investigated.

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