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# REGULAR NUMBER OF COMPLEMENTARY PRISM OF FERRERS GRAPH

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**Abstract**

The regular number r(G) of a graph G is the minimum number of subsets into which the edge set of G is partitioned so that the subgraph induced by each subset is regular. In this paper, we examine the regular number r(G) of complementary prism of a Ferrers graph.

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## 1. Introduction

Graph theory notation and terminology are not given here we refer it from [1]. The complement of a graph G is a graph $\overline{G}$ on the same set of vertices as of G such that there will be an edge between two vertices in $\overline{G}$, if and only if there is no edge in between in G. The complementary prism of a graph G, denoted by $G\overline{G}$, as the graph formed from the disjoint union of G and its complement $\overline{G}$ by adding the edges of the perfect matching between the corresponding vertices of G and $\overline{G}$, Where $V\left(G\overline{G}\right)=V(G)\cup V(\overline{G})$. The regular number of G is defined to be the minimum number of subsets into which the edge set of G can be partitioned so that the subgraph induced by each subset is regular.

In this paper we find the regular number of complementary prism of a Ferrers graph. Also we find the diameter and radius of the complementary prism of a Ferrers graph.

**Theorem 1.1 [2]** If G = (V, E) is a Ferrers graph iff for all distrinct *x*, y, z, w ∈ E then
d(*x*, w) + d(y, z) ≤ 4.

**Theorem 1.2 [5]** For any path $P_n$, $r(P_n) = 2$.

**Theorem 1.3 [5]** For any graph G, r(G) = 1 if and only if G is regular.

**Theorem 1.4 [1]** For any complete bipartite graph $K_{3,n}$, where n ≥ 1, $r(K_{3,n}) = \frac{n}{3}$ if n ≡ 0 (mod 3), and $r(K_{3,n}) = \left\lfloor \frac{n}{3} \right\rfloor + 3$ if n ≡ 1, 2 (mod 3)

**Theorem 1.5 [3]** If G is a Ferrers graph, then d(u,v) ≤ 3 for all u, v ∈ V(G).

**Theorem 1.6 [4]** For a graph G, G is a Ferrers tree if and only if G has two internal vertices.

## 2. Main results

**Theorem 2.1.** For any path G = $P_n$ , $G\bar{G}$ is a Ferrers graph for n = 2 and non-Ferrers graph otherwise.

**Proof. Case (i) When n = 2**

In this case $G\bar{G}$ = $P_4$, and by Theorem 1.6, $P_4$ is a Ferrers graph.

**Case (ii) When n ≥ 3**

Consider a path $P_n$ with n (≥ 3) vertices. Let $x_1, x_2, ...., x_n$ be the vertices of $P_n$ and $\overline{x_1}, \overline{x_2}, ..., \overline{x_n}$ be the vertices of $\overline{P_n}$. Consider the four vertices $x_1, \overline{x_1}, x_{n-1}, x_n$ in $G\overline{G}$. By the definition of complementary prism, $\overline{x_1}$ is adjacent to $x_1, \bar{x}_2, \bar{x}_3, , \dots \dots , \bar{x}_n$ Hence $d(x_1, x_n) = 3$ and $d\left(\overline{x_1}, x_{n-1}\right) = 2$ (or) $d(x_1, x_{n-1}) = 2$ and $d\left(\overline{x_1}, x_n\right) = 3$. In both the cases, $d(x_1, x_n) + d\left(\overline{x_1}, x_{n-1}\right) > 4$ and $d(x_1, x_{n-1}) + d\left(\overline{x_1}, x_n\right) > 4$. Therefore, by Theorem 1.1, $G\overline{G}$ is a non-Ferrers graph.

**Theorem 2.2.** For any path $P_2$, the regular number $r\left(P_2\overline{P_2}\right) = 2$

**Proof.** Consider $P_2$. Clearly $P_2\overline{P}_2 = P_4$, which is a path. Therefore by Theorem 1.2, $r\left(P_2\overline{P_2}\right) = r(P_4) = 2$.

**Theorem 2.3.** For a cycle G = $C_n$ , $G\overline{G}$ is a Ferrers graph for n = 3 and non-Ferrers for n ≥ 4.

**Proof. When n = 3**

Clearly $C_3$ is an infringe Ferrers graph and $\overline{C_3}$ is a null graph. Let *x*, *y*, *z* be the vertices of $C_3$ and $\bar{x}, \bar{y}, \bar{z}$ be the vertices of $\overline{C_3}$. Consider any two disjoint edges $x\bar{x}$ and $y\bar{y}$ in $C_3\overline{C_3}$. Then we find $d(x,\bar{y}) + d(\bar{x}, y) = 4$. Hence by Theorem 1.1, $C_3\overline{C_3}$ is Ferrers graph.

**When n ≥ 4**

Consider a cycle $C_n$ with n ( ≥ 4) vertices. Let $x_1, x_2, ..., x_n$ be the vertices of $C_n$ and $\overline{x_1}, \overline{x_2}, ..., \overline{x_n}$ be the vertices of $\overline{C_n}$. Consider the four vertices $\overline{x_1}, \overline{x_3}, x_4, \overline{x_4}$ in $G\overline{G}$. Clearly $d(\overline{x_1}, \overline{x_4}) = 3$ and $d(\overline{x_3}, x_4) = 2$ (or) $d(\overline{x_1}, x_4) = 2$ and $d(\overline{x_3}, \overline{x_4}) = 3$. In both the cases $d(\overline{x_1}, \overline{x_4}) + d(\overline{x_3}, x_4) > 4$ and $d(\overline{x_1}, x_4) + d(\overline{x_3}, \overline{x_4}) > 4$. By theorem 1.1, $G\overline{G}$ is non-Ferrers.

**Theorem 2.4.** For any complementary prism of $C_3$, the regular number $r(C_3\overline{C_3}) = 2$

**Proof.** Consider the complementary prism of $C_3$. To prove $r(C_3\overline{C_3}) = 2$. Suppose $r(C_3\overline{C_3}) \neq 2$. Let *x*, *y*, *z*, $\bar{x}, \bar{y}, \bar{z}$ be the vertices in $C_3\overline{C_3}$ and *xy*, *yz*, *xz*, $x\bar{x}, y\bar{y}, z\bar{z}$ be the edges in $C_3\overline{C_3}$ and is shown in Figure 1.

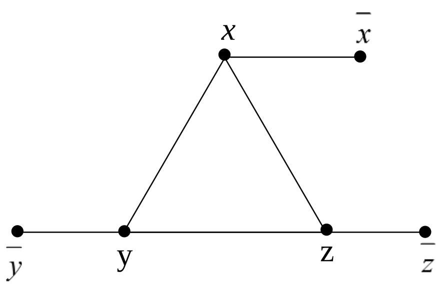


**Figure 1**

Clearly $C\overline{C_3}$ is not a regular graph. Hence $r(C_3\overline{C_3}) \neq 1$. Also $C_3\overline{C_3}$ contains one $C_3$ cycle and the remaining non adjacent edges are in one set. Hence $r(C_3\overline{C_3}) = 2$

**Theorem 2.5,** For a wheel graph $G = W_n$, $G\overline{G}$ is ferrers for n = 4 and non-Ferrers for n ≥ 5.

Proof**.** Let $G = W_n$ be a wheel graph on n vertices.

**When n = 4**

Clearly $W_4$ is a regular graph with degree 3, and $\overline{W_4}$ is a null graph. Let *x*, y, z, w be the vertices in $W_4$ and $\bar{x}, \bar{y}, \bar{z}, \bar{w}$ be the vertices in $\overline{W_4}$. By the definition of complementary prism $xy, yz, xz, xw, yw, zw, x\bar{x}, y\bar{y}, z\bar{z}, w\bar{w}$ are the edges in $W_4\overline{W_4}$. Consider any two non adjacent edges $x\bar{x}, z\bar{z} \in W_4\overline{W_4}$. Then $d(x,\bar{z}) + d(\bar{x}, z) = 4$. for every $x, \bar{x}, z$ and $\bar{z} \in V(W_4\overline{W_4})$. Therefore by Theorem 1.1, $W_4\overline{W_4}$ is a Ferrers graph.

**When n ≥ 5**

Consider a wheel graph $W_n$ with n (≥ 5) vertices. Let $x_1$, $x_2$, …, $x_n$ be the vertices of $W_n$ and $\overline{x_1}, \overline{x_2}, ..., \overline{x_n}$ be the vertices in $\overline{W_n}$. Let *z* be the center vertex of $W_n$. Clearly deg (z) = n-1 and the remaining vertices have degree 3.Also $\bar{z}$ is an isolated vertex in $\overline{W_n}$ and the remaining vertices have degree n – 4 in $\overline{W_n}$, $\overline{W_n}$ is a disconnected graph. But by the definition of complementary prism $W_n\overline{W_n}$ is a connected graph. Consider the four vertices $x_1$, $x_2$, $x_4$, $\overline{x_4}$ in $G\overline{G}$. Clearly $d(x_1, \overline{x_4}) = 3$ and $d(x_2, x_4) = 2$. Then we find $d(x_1, \overline{x_4}) + d(x_2, x_4) > 4$. By theorem 1.1, $G\overline{G}$ is non-Ferrers.

**Theorem 2.6** For any complementary prism of a wheel graph $W_4$, the regular number $r(W_4\overline{W_4}) = 2$

**Proof** Consider the wheel G = $W_4$. By Theorem 2.5, $G\overline{G}$ is Ferrers. To prove that $r(W_4\overline{W_4}) = 2$. Let $v_1$, $v_2$, $v_3$, $v_4$, $\overline{v_1}, \overline{v_2}, \overline{v_3}, \overline{v_4}$ be the vertices in $W_4\overline{W_4}$ and $v_1v_2$, $v_2v_3$, $v_1v_3$, $v_1v_4$, $v_2v_4$, $v_3v_4$, $v_1\overline{v_1}, v_2\overline{v_2}, v_3\overline{v_3}$ be the edges in $W_4\overline{W_4}$ and is shown in Figure 2.

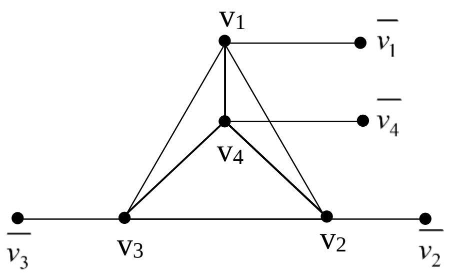


**Figure 2**

**Case(i)** Suppose $r(W_4\overline{W_4}) = 1$. By Theorem 1.3, $W_4\overline{W_4}$ is a regular graph. Which is a contradiction to $W_4\overline{W}_4$ is not regular. Hence $r(W_4\overline{W_4}) \neq 1$.

**Case(ii)** Suppose $r(W_4\overline{W_4}) > 2$. By Theorem 1.4, $W_4\overline{W_4}$ is a complete bipartite graph $K_{3,n}$ where n ≥ 1. From figure 2, $W_4\overline{W_4}$ is not a complete bipartite graph. Which is a contradiction. Hence $r(W_4\overline{W_4}) \ngtr 2$. Hence in both the cases $r(W_4\overline{W_4}) \neq 1$ and $r(W_4\overline{W_4}) \ngtr 2$. Thus $r(W_4\overline{W_4}) = 2$.

**Theorem 2.7** The complementary prism of a complete graph is a Ferrers graph.

**Proof** Consider a complete graph $G = K_n$ with $n$ vertices. And $\overline{K}_n$ is a complement of $K_n$ which is a null graph. But by the definition of complementary prism $G\overline{G}$ is a connected graph. Let $x_1, x_2, x_3 ...., x_n$ be the vertices of $K_n$ and $\overline{x_1}, \overline{x_2}, ..., \overline{x_n}$ be the vertices of $\overline{K}_n$. Consider any four vertices $x_1, \overline{x_1}, x_n, \overline{x_n}$ in $G\overline{G}$. Clearly $d(x_1, \overline{x_n}) = 2$ and $d(\overline{x_1}, x_n) = 2$ (or) $d(x_1, x_n) = 3$ and $d(\overline{x_1}, x_n) = 1$. In both the cases, $d(x_1, \overline{x_n}) + d(\overline{x_1}, x_n) \leq 4$. Therefore, by Theorem 1.1, $G\overline{G}$ is Ferrers.

**Theorem 2.8.** For any Complementary prism of $K_4$, the regular number $r(K_n\overline{K_n}) = 2$

**Proof.** Consider a Ferrers graph $K_n\overline{K_n}$ To prove that $r(K_n\overline{K_n}) = 2$. Let $v_1$, $v_2$, $v_3$, $v_4$, $\overline{v_1}, \overline{v_2}, \overline{v_3}, \overline{v_4}$ be the vertices in $K_n\overline{K_n}$ and $v_1v_2$, $v_2v_3$, $v_1v_3$, $v_1v_4$, $v_2v_4$, $v_3v_4$, $v_1\overline{v_1}, v_2\overline{v_2}, v_3\overline{v_3}$ be the edges in $K_n\overline{K_n}$ and is shown in Figure 3.

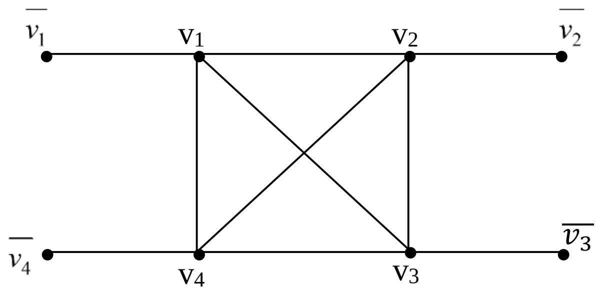


**Figure 3**

**Case (i)** Suppose $r(K_n\overline{K_n}) = 1$. By theorem 1.3, $K_n\overline{K_n}$ is a regular graph. But $K_n\overline{K_n}$ is not a regular graph. Which is a contradiction. Hence $r(K_n\overline{K_n}) \neq 1$.

**Case (ii)** Suppose $r(K_n\overline{K_n}) > 2$. By theorem 1.4, $K_n\overline{K_n}$ is a complete bipartite graph $K_{3,n}$ where n ≥ 1. From figure 3, $K_n\overline{K_n}$ is not a complete bipartite graph. Which is a contradiction. Hence $r\left(K_n\overline{K_n}\right)$ ≯ 2. Hence in both the cases $r\left(K_n\overline{K_n}\right)\neq 1$ and $r\left(K_n\overline{K_n}\right)$ ≯ 2.Thus $r\left(K_n\overline{K_n}\right)$ = 2.

**Theorem 2.9.** Let G=$K_{m,n}$ be a complete bipartite graph on m+n vertices. Then $G\overline{G}$ is non-Ferrrers for every $m,n\geq 2$

**Proof.** Consider a complete bi-partite graph $K_{m,n}$ with m, n (≥2) vertices. Let $V_1$ & $V_2$ be the two partitions of G = $K_{m,n}$. Let $x_1,x_2,x_3....,x_m$ be the vertices of $V_1$ and $y_1,y_2,y_3....,y_n$ be the vertices of $V_2$. Also $\overline{x_1},\overline{x_2},\overline{x_3}....,\overline{x_m}$ and $\overline{y_1},\overline{y_2},\overline{y_3}....,\overline{y_n}$ be the vertices of $\overline{K}_{m,n}$. Consider the four vertices $\overline{x_1}$, $\overline{x_m}$, $\overline{y_1}$, $\overline{y_n}$ in $K_{m,n}\overline{K_{m,n}}$. Clearly $d\left(\overline{x_1},\overline{y_n}\right)=4$ and $d\left(\overline{x_m},\overline{y_1}\right)=3$ (or) $d\left(\overline{x_1},\overline{y_1}\right)=3$ and $d\left(\overline{x_m},\overline{y_n}\right)=3$. In both the cases, $d\left(\overline{x_1},\overline{y_n}\right)+d\left(\overline{x_m},\overline{y_1}\right)>4$.Therefore by Theoren 1.1, $G\overline{G}$ is non -Ferrers.

**Theorem 2.10** For a complete bipartite graph $K_{m,n}$ , $r\left(K_{m,n}\,\overline{K_{m,n}}\right)=3$

**Proof.** To prove, $r\left(K_{m,n}\,\overline{K_{m,n}}\right)=3$. Suppose $r\left(K_{m,n}\,\overline{K_{m,n}}\right)\neq 3$.

**Case (i).** Suppose $r\left(K_{m,n}\,\overline{K_{m,n}}\right)=1$, then $K_{m,n}\overline{K_{m,n}}$ is a regular graph (or) $P_2$ (or) $K_n$. But $K_{m,n}\overline{K_{m,n}}$ is not a regular graph. Hence $r\left(K_{m,n}\,\overline{K_{m,n}}\right)\neq 1$.

**Case (ii).** Suppose $r\left(K_{m,n}\,\overline{K_{m,n}}\right)=2$, then $K_{m,n}\overline{K_{m,n}}$ is either a path containing atleast 3 vertices (or) $p_2\overline{p_2}$ or $C_3\overline{C_3}$ (or) $W_4\overline{W_4}$ (or) $K_n\overline{K_n}$ . In all the cases the graph is Ferrers. Which is a contradiction. Hence $r\left(K_{m,n}\,\overline{K_{m,n}}\right)\neq 2$

**Case (iii).** Suppose that $K_{m,n}\overline{K_{m,n}}$ > 3, then $K_{m,n}\overline{K_{m,n}}$ is either a wheel graph with atleast 6 vertices (or) $K_{3,3}$. In all the cases the graph is a Ferrers graph. Which is a contradiction. Hence $r\left(K_{m,n}\,\overline{K_{m,n}}\right)=3$

**Theorem 2.11.** For a tree G, $G\overline{G}$ is non Ferrers except path $P_2$ (n = 2).

**Proof.**

We Prove the result by the following two cases.

**Case (i)** Suppose G is a Ferrers tree.

To prove that $G\overline{G}$ is a non Ferrers graph. Since G is a Ferrers tree. By Theorem 1.6. G has 2 internal vertices. Let us assume that $x_1, x_2$ be the internal vertices and $x_3,x_5,x_7,.....,x_{n-1}$ be the vertices adjacent with $x_1$ and $x_2,x_4,x_6,....,x_n$ be the vertices adjacent with $x_2$. Also, $\overline{x_1},\overline{x_2},\overline{x_3},....,\overline{x_{n-1}},\overline{x_n}$ are the vertices in $\overline{G}$. Consider the four vertices $x_{n-1},\overline{x_{n-1}},x_2,x_n$ in $G\overline{G}$. Clearly $d(x_n,x_{n-1})=3$ and $d(x_2,\overline{x_{n-1}})=2$. Then we find $d(x_n,x_{n-1})+d(x_2,\overline{x_{n-1}})>4$. By Theorem 1.1, $G\overline{G}$ is non Ferrers.

**Case (ii)** Suppose G is a non-Ferrers tree.

To prove that $G\overline{G}$ is a non-Ferrers graph. Since G is a non-Ferrers tree. Let $x_1,x_2,x_3....,x_n$ be the vertices of G, and $\overline{x_1},\overline{x_2},\overline{x_3}....,\overline{x_n}$ be the vertices of $\overline{G}$. Consider the four vertices $x_{n-1},x_n,x_1,\overline{x_1}$ in $G\overline{G}$. Clearly $d(x_{n-1},x_1)=3$ and

$d(x_n, \overline{x_1}) = 3.$ In both the cases $d(x_{n-1}, x_1) + d(x_n, \overline{x_1}) > 4.$ By Theorem 1.1, $G\overline{G}$ is non Ferrers.

**Theorem 2.12.** Let G be any graph, $G\overline{G}$ is Ferrers, then diam ($G\overline{G}$) = 3 and rad ($G\overline{G}$) = 2.

**Proof.** Let G be any graph and $G\overline{G}$ is a Ferrers graph. To prove that, diam ($G\overline{G}$) = 3 and

rad $(G\overline{G}) = 2$ since $G\overline{G}$ is a Ferrers g raph, by Theorem 1.5, d (u,v) ≤ 3 for all u, v ϵ V($G\overline{G}$). Then $G\overline{G}$ graph attains the upper bound value. Hence diam ($G\overline{G}$) = 3. Now to prove that

rad $(G\overline{G}) = 2$. It is enough to prove that rad $(G\overline{G}) \neq 3$ and rad $(G\overline{G}) \neq 1$.

**Case (i)** Suppose $rad(G\overline{G}) = 3$, then diam $(G\overline{G}) \geq 3.$ If diam $(G\overline{G}) > 3$, then by Theorem 2.12, $G\overline{G}$ is not a Ferrers graph. Which is a contradiction. Hence $r(G\overline{G}) \neq 3$

**Case (ii)** Suppose $rad(G\overline{G}) = 1$, then $G\overline{G}$ is a regular graph. By Theorem 2.12, $G\overline{G}$ is not a regular graph. Hence $rad(G\overline{G}) \neq 1$. Hence in both the cases $rad(G\overline{G}) \neq 3$ and $rad(G\overline{G}) \neq 1$. Thus $rad(G\overline{G}) = 2$.

## 3. Conclusion.

In this paper we proved that the regular number of the complementary prism of a Ferrers graph is 2. Also we have seen that the diameter and radius of the complementary prism of a Ferrers graph.

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