Section A -Research paper

A STUDY ON SOME RESULTS OF FUZZY HILBERT SPACES

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Abstract

In this Paper, we discuss about fuzzy inner product space, orthonormal set etc. We establish some fundamental results and Bessel's inequality.

1. Introduction

In studying fuzzy topological vector spaces, Katsaras [1] in 1984, first introduced the notion of fuzzy norm on a linear space. Later on many author viz. Felbin [2], Cheng & Mordeson [3], Bag and Samanta [4] etc. have given different definitions of fuzzy normed linear spaces.

R. Biswas [5] and A. M. El-Abyed & H. M. El-Hamouly [6] first tried to give a meaningful definition of fuzzy inner product space and associated fuzzy norm function. Also those definitions are restricted to the real linear space only. Recently Pinaki Mazumder & S. K. Samanta [7] introduced a definition of fuzzy inner product space whose associated fuzzy norm is of Bag & Samanta [4] type. Where as A.Hasankhani, A. Nazari & M.Saheli [8] have introduced a definition of fuzzy inner product space whose associated norm is of Felbin [2] type. Following the definition of fuzzy inner product space intro- duced by Pinaki Mazumder & S. K. Samanta [7], we study some results. Bessel's inequality and some theorems in fuzzy settings.

2. Some Preliminary Results

In this section, some definitions and preliminary results are given which will be used in this paper.

Definition 2.1 [4]. Let U be a linear space over a field F (field of real / complex numbers).

A fuzzy subset N of $U \times R$ (R is the set of real numbers) is called a fuzzy norm on U if $x, u \in U$ and $c \in F$: following conditions are satisfied:

(N1) $\forall t \in R$ with $t \leq 0$, $N(x,t) = 0$;

(N2) $(\forall t \in R, t > 0,\ N(x,t) = 1)$ iff $x = 0$;

(N3) $\forall t \in R, t > 0, N(cx,t) = N\left(x, \frac{t}{|c|}\right)$ if $c \neq 0$;

(N4) $\forall s, t \in R$, $x, u \in U$; $N(x+u; s+t) \geq \min\{N(x,s), N(u,t)\}$

(N5) $N(x,.)$ is a non-decreasing function of R and $\lim_{t\to\infty} N(x, t) = 1$.

(N6) $\forall t > 0$, $N(x,t) > 0$ *implies* $x = \underline{0}$.

The pair (U,N) will be referred to as a fuzzy normed linear space.

Definition 2.2 [4]. Let (U,N) be a fuzzy normed linear space. Let $\{x_n\}$ be a sequence in U. Then $\{x_n\}$ is said to be convergent if $\exists x \in U$ such that $\lim_{n\to\infty} N(x_n - x, t) = 1\ \forall t > 0$. In that case x is called the limit of the sequence $\{x_n\}$ and we denote it by $\lim x_n$.

Definition 2.3 Let (U, N_1) be a fuzzy normed linear space satisfying (N6). Let $T \in U^*$ and $\left\{\|\ \|_\alpha^1 : \alpha \in (0,1)\right\}$ be the family of α −norms of N_1. We define

$$\|T\|_\alpha^* = \bigvee_{x \in U, x \neq \underline{0}} \frac{|T(x)|}{\|x\|_{1-\alpha}^1} \quad \forall \alpha \in (0,1)$$

Then $\left\{\|\ \|_\alpha^* : \alpha \in (0,1)\right\}$ is an ascending family of norms on U^*.

Definition 2.4 Let (U, N) be a fuzzy normed linear space and $\alpha \in (0,1)$. A sequence $\{x_n\}$ in U is said to be α −convergent in U if $\exists x \in U$ such that $\lim_{n\to\infty} N(x_n - x, t) > \alpha,\ \forall t > 0$ and x is called the limit of $\{x_n\}$.

Definition 2.5 Let (U, N) be a fuzzy normed linear space. A subset F of U is said to be l-fuzzy closed if for each $\alpha \in (0,1)$ and for any sequence $\{x_n\}$ in F and $x \in U$,

$$\left(\lim_{n\to\infty} N(x_n - x, t) \geq \alpha, \forall\, t > 0\right) \Rightarrow x \in F.$$

Definition 2.6 [7]. Let V be a linear space over the field C of complex numbers. Let $\mu: V \times V \times C \to I = [0,1]$ be a mapping such that the following holds.

(FIP1) For $s, t \in C, \mu(x + y, z, |t| + |s|) \geq \min\{\mu(x, z, |t|), \mu(y, z, |s|)\}$;
(FIP2) For $s, t \in C,\ \mu(x, y, |st|) \leq \min\{\mu(x, x, |s|^2), \mu(y, y, |t|^2)\}$;
(FIP3) For $t \in C, \mu(x, y, t) = \mu(y, x, \bar{t})$.
(FIP4) $\mu(\alpha x, y, t) = \mu\left(x, y, \frac{t}{|\alpha|}\right),\ \alpha(\neq 0) \in C, t \in C$;
(FIP5) $\mu(x, x, t) = 0\ \forall t \in C \backslash R^+$;
(FIP6) $(\mu(x, x, t) = 1\ \forall t > 0)\ iff\ x = \underline{0}$;
(FIP7) $\mu(x, x, .): R \to I (= [0,1])$ is a monotonic non-decreasing function of R and $\lim_{t\to\infty} \mu(x, x, t) = 1$

We shall call μ to be the fuzzy inner product (FIP in short) function on V and (V,μ) is called a fuzzy inner product space (FIP space).

Theorem 1. [7]. *Let V be a linear space over C. Let μ be a FIP on V. Then*

$$N(x,t)=\begin{cases}\mu(x,x,t^2), & t\in R \quad if\ t>0\\ 0 & \quad\quad\ \ if\ t\le 0\end{cases}$$

is a fuzzy norm on V. Now if μ satisfies the following conditions:

(FIP8) $(\mu(x,x,t^2)>0,\ \forall t>0)\Longrightarrow x=0$ and

(FIP9) For all $x,y\in V$ and $p,q\in R$,
$\mu(x+y,\ x+y,\ 2q^2)\wedge\mu(x-y,\ x-y,\ 2p^2)\ge\mu(x,x,p^2)\wedge\mu(x,x,q^2)$ then $\|x_n\|_\alpha=\wedge\{t>0{:}\,N(x,t)\ge\alpha\}$ $\alpha\in(0,1)$ is an ordinary norm ($\alpha-norm$) on V satisfying parallelogram law.
Then using Polarization identity we get ordinary inner product, called the α −inner product, as follows:

$$<x,y>_\alpha=\frac{1}{4}\left(\|x+y\|_\alpha^2-\|x-y\|_\alpha^2\right)+\frac{1}{4}i\left(\|x+iy\|_\alpha^2-\|x-iy\|_\alpha^2\right),\ \forall\alpha\in(0,1)$$

Definition 2.7 [7]. Let $(V,\ \mu)$ be a FIP space satisfying (FIP8). V is said to be level complete if for any $\alpha\in(0,1)$, every sequence converges in V w.r.t. $\|\ \ \|_\alpha$ (the α-norm generated by the fuzzy norm N which is induced by fuzzy inner product μ).

Definition 2.8 [7]. Let $(V,\ \mu)$ be a FIP space. V is said to be a fuzzy Hilbert space, if it is level complete.

3. Some Results on Fuzzy Inner Product Spaces

Here we study some results on fuzzy inner product spaces.

Theorem 2. *Let (V,N) be a fuzzy normed linear space. Assume that for $x,y\in V$ and $s,t\in C$,*

$$\min\{N(x,|st|),N(y,|st|)\}\ge\min\{N(x,|s|^2),N(y,|t|^2)\}.$$

Define $\mu'{:}\,V\times V\times C\to[0,1]$ as $\mu'(x,\ y,\ s+t)=0$ if $x=y$ and $s+t\in C-R^+$ and elsewhere as $\mu'(x,\ y,\ s+t)=N(x,\ |s|)\vee N(y,\ |t|)$. Then μ' is a fuzzy inner product on V.

Proof of Theorem 2

(FIP1) For $s, t \in C$ and $x, y, z \in V$ we have

$$\begin{aligned}
\mu'(x+y, z, |s|+|t|) &= \mu'(x+y, z, |s|+|t|) \\
&= \mu'(x+y, z, |s|+|t|+0) \\
&= N(x+y, |s|+|t|) \vee N(z, 0) \\
&= N(x+y, |s|+|t|) \\
&\geq \min\{N(x, |s|), N(y, |t|)\} \\
&= \min\{\mu'(x, z, |s|), \mu'(y, z, |t|)\}
\end{aligned}$$

(FIP2) $\mu'(x, y, |st|) = N(x, |st|) = N(y, |st|)$

$$\begin{aligned}
&= \min\{N(x, |st|), N(y, |st|)\} \\
&\geq \min\{N(x, |s|^2), N(y, |t|^2)\} \ [By\ condition] \\
&= \min\{\mu'(x, x, |s|^2), \mu'(y, y, |t|^2)\}.
\end{aligned}$$

(FIP3) $\mu'(x, y, t) = N(x, |t|) = N(x, |\bar{t}|)$

$$\begin{aligned}
&= \mu'(x, y, \bar{t}) = N(y, |\bar{t}|) \\
&= \mu'(y, x, \bar{t}).
\end{aligned}$$

(FIP4) $\mu'(\alpha x, y, t) = N(\alpha x, |t|)$

$$\begin{aligned}
&= N\left(x, \frac{|t|}{|\alpha|}\right) \ [\alpha \neq 0] \\
&= \mu'\left(y, x, \frac{t}{|\alpha|}\right).
\end{aligned}$$

(FIP5) $\mu'(x, x, t) = 0 \ \forall t \in C - R^{+} \ [By\ condition]$

(FIP6) $\mu'(x,x,t) \quad = 1 \ \forall t > 0$

$\Leftrightarrow N(x,t) \ = 1 \, \forall t > 0$

$\Leftrightarrow x \ = 0.$

(FIP7) Since $\mu'(x,x,.) = N(x,.)$ and $N(x,.)$ is a monotonic non-decreasing function of R and $\lim_{t\to\infty} N(x,t) = 1$

$\Rightarrow \mu'$ has also the property. Thus μ' is a fuzzy inner product on V.

The proof of Theorem 2 is complete.

4. Orthonormal set, Bessel's Inequality

In this section orthonormal set, sequence are defined and Bessel's inequality, are established in fuzzy Hilbert spaces.

Definition 4.1 Let (V,μ) be a fuzzy inner product space satisfying (FIP8) and (FIP9) and $\alpha \in (0,1)$. An α-fuzzy orthogonal set M in V is said to be α-fuzzy orthonormal if the elements have α-norm 1 that is $\forall x,y \in M$,

$$< x,y >_\alpha = \begin{cases} 1 & if \quad x = y \\ 0 & if \quad x \neq y \end{cases}$$

where $< \ , \ >_\alpha$ is the induced inner product by μ.

Definition 4.2 Let (V,μ) be a fuzzy inner product space satisfying (FIP8) and (FIP9). A fuzzy orthogonal set M in V is said to be fuzzy orthonormal if the elements have α-norm 1 $\forall \alpha \in (0,1)$ that is $\forall x,y \in M$,

$$< x,y >_\alpha = \begin{cases} 1 & if \quad x = y \\ 0 & if \quad x \neq y \end{cases} \qquad \forall \alpha \in (0,1)$$

where $< \ , \ >_\alpha$ is induced inner product by μ.

Definition 4.3. Let (V,μ) be a fuzzy inner product space satisfying (FIP8) and (FIP9). A fuzzy orthonormal set $M \subset V$ is called complete fuzzy orthonormal set if there is no α-fuzzy orthonormal set $(\alpha \in (0,1))$ of which M is a proper subset. If M is countable then we call M is a complete fuzzy orthonormal sequence.

Theorem 3. (Bessel's inequality) Let (V,u) be a fuzzy Hilbert space satisfying (FIP8) and (FIP9) and $(\alpha \in (0,1))$ and $\{e_k\}$ be an α-fuzzy orthonormal sequence in V. Then for every $x \in V$,

$$\sum_{k=1}^{\infty} |< x, e_k >_\alpha|^2 \le \|x\|_\alpha^2$$

Proof of the theorem 3

Since α-fuzzy orthonormal sequence is orthonormal sequence in $(V, < \ , \ >_\alpha)$, so by Bessel's inequality in crisp inner product we have

$$\sum_{k=1}^{\infty} |< x, e_k >_\alpha|^2 \le \|x\|_\alpha^2$$

Theorem 4. Let (V,μ) be a Hilbert space satisfying (FIP9) and $\{e_i\}$ is fuzzy orthonormal sequence in V. Then the following satements are equivalent.

(i) $\{e_i\}$ is complete fuzzy orthonormal.

(ii) $x \perp e_i$ for $i = 1,2,\ldots \Rightarrow x = \underline{0}$.

(iii) For every $x \in V$, $x = \sum_{k=1}^{\infty} < x, e_i >_\alpha \ \forall \alpha \in (0,1)$ and hence $< x, e_k >_\alpha = < x, e_k >_\beta \quad \forall \alpha, \beta \in (0,1)$
i.e. x is independent on α.

(iv) For every $x \in V$,

$$\|x\|_\alpha^2 = \sum_{k=1}^{\infty} |< x, e_i >_\alpha| \ \forall \alpha \in (0,1)$$

and hence

$$||x||_{\alpha}^{2} = ||x||_{\beta}^{2} \quad \forall \alpha, \beta \in (0,1).$$

Proof of the theorem 4

(a) Suppose (i) holds.
Let $\{e_i\}$ be a complete fuzzy orthonormal sequence and $x \perp e_i$ for $i = 1,2, \ldots$
$\Rightarrow x \perp_{\alpha} e_i \ \forall \alpha \in (0,1)$ and $i = 1,2, \ldots$
$\Rightarrow < x, e_i >_{\alpha} = 0 \ \forall \alpha \in (0,1)$ and $i = 1,2, \ldots$
Set for a fixed α_0, $e^{\alpha_0} = \frac{x}{||x||_{\alpha_0}}$
Then $||e^{\alpha_0}|| = < e^{\alpha_0}, e^{\alpha_0} >_{\alpha_0} = 1$ and $< e^{\alpha_0}, e_i >_{\alpha_0} = 0$ for $i = 1,2, \ldots$
Therefore we get an α_0-fuzzy orthonormal sequence $\{e^{\alpha_0}, e_1, e_2, \ldots\}$ of which $\{e_1, e_2, \ldots\}$ is proper subset a contraction to completeness.
There for $e^{\alpha_0} = \underline{0}$.
$\Rightarrow x = 0.$
So $(i) \Rightarrow (ii)$.

(b) Suppose (ii) holds.
Let $x \perp_{\alpha} e_i$ for $i = 1,2, \ldots$ implies $x = 0$.

$$\Rightarrow x - \sum_{i=1}^{\infty} < x, e_i >_{\alpha} e_i \perp_{\alpha} e_j \quad j = 1,2, \ldots \quad and \quad \forall \alpha \in (0,1)$$

$$\Rightarrow x - \sum_{i=1}^{\infty} < x, e_i >_{\alpha} e_i \perp_{\alpha} e_j \quad j = 1,2, \ldots \quad and \ \forall \alpha \in (0,1)$$

$$\Rightarrow x - \sum_{i=1}^{\infty} < x, e_i >_{\alpha} e_i = 0 \ \ \forall \alpha \in (0,1)$$

$$\Rightarrow x = \sum_{i=1}^{\infty} < x, e_i >_{\alpha} e_i = \sum_{i=1}^{\infty} < x, e_i >_{\beta} e_i \quad \forall \alpha, \beta \in (0,1)$$

$$\Rightarrow \sum_{i=1}^{\infty} < x, e_i >_{\alpha} e_i - \sum_{i=1}^{\infty} < x, e_i >_{\beta} e_i = 0 \quad \forall \alpha, \beta \in (0,1)$$

Since $\{e_i\}$ is linearly independent, therefore

$$\Rightarrow < x, e_i >_{\alpha} - < x, e_i >_{\beta} = 0 \qquad i = 1,2, \ldots \ \text{and} \ \ \forall \alpha, \beta \in (0,1)$$

$$\Rightarrow < x, e_i >_{\alpha} = < x, e_i >_{\beta} \qquad i = 1,2, \ldots \ \text{and} \ \ \forall \alpha, \beta \in (0,1)$$

Thus $(ii) \Longrightarrow (iii)$.

(c) Suppose (iii) holds.
Let

$$x = \sum_{i=1}^{\infty} < x, e_i >_{\alpha} e_i \ \ \forall \alpha \in (0,1)$$

Now $\quad ||x||_{\alpha}^{2} = < x, x >_{\alpha}$

$$=< \sum_{i=1}^{\infty} < x, e_i >_\alpha e_i, \sum_{i=1}^{\infty} < x, e_i >_\alpha e_i >_\alpha$$

$$=< \lim_{n\to\infty} \sum_{i=1}^{n} < x, e_i >_\alpha e_i , \lim_{n\to\infty} \sum_{i=1}^{n} < x, e_i >_\alpha e_i >_\alpha$$

$$= \lim_{n\to\infty} < \sum_{i=1}^{n} < x, e_i >_\alpha e_i, \sum_{j=1}^{n} < x, e_j >_\alpha e_j >_\alpha$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} < x, e_i >_\alpha \overline{< x, e_i >_\alpha}$$

$$= \sum_{i=1}^{n} |< x, e_i >_\alpha|^2 \quad \forall \alpha \in (0,1)$$

Now from (iii) we have $< x, e_i >_\alpha =< x, e_i >_\beta \quad i = 1,2, \ldots$ and $\forall \alpha, \beta \in (0,1)$

So $(iii) \Longrightarrow (iv)$.

(d) Suppose (iv) holds and $\{e_i\}$ is not complete. Then we get for an $\alpha \in (0,1)$

$\{e^\alpha, e_1, e_2, \ldots\}$ of which $\{e_1, e_2, \ldots\}$ is aproper subset and $\left\| e^\alpha \right\|_\alpha = 1$ and $< e^\alpha, e_i >_\alpha = 0 \; \forall i = 1,2, \ldots$

Now

$$\left\| e^\alpha \right\|_\alpha^2 = \sum_{i=1}^{\infty} |< e^\alpha, e_i >_\alpha|^2 = 0$$

$$\Rightarrow e^\alpha = \overline{0}.$$

Thus $(iv) \Longrightarrow (i)$.

6. Conclusion

In this paper, we have discussed fuzzy inner product space α −fuzzy orthonormal set, complete fuzzy orthonormal set etc. have been introduced. We establish Bessel's inequality.

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