Forcing Split and Forcing Non-split Geodetic Number of a Graph

L.S.Chitra[1], Venkanagouda M.Goudar[2]

[1]Research Scholar, Sri Siddhartha Academy of Higher Education

Tumkur, Karnataka, India

Department of Mathematics, Adhichunchanagiri Institute of Technology

Chikkamagaluru, Karnataka, India

[2] Department of Mathematics, Sri Siddhartha Institute of Technology,

Constituent College of Sri Siddhartha Academy of Higher Education

Tumkur, Karnataka, India

chitrals.ait@gmail.com, vmgouda@gmail.com

Abstract

In a connected graph $G = (V, E)$, the set S is a minimum split geodetic set. A subset $T \subseteq S$ is called as the forcing subset for S if S is the unique minimum split geodetic set of G containing T. The minimum cardinality of a forcing subset of S is the forcing split geodetic number $f_s(S)$ of S. The forcing split geodetic number of a connected graph G is $f_s(G) = \min\{f_s(S): S \text{ is a split geodetic set of } G\}$. The set S is a minimum non-split geodetic set in a connected graph $G = (V, E)$ and the subset $T \subseteq S$ is called as forcing subset for S if S is the unique minimum non-split geodetic set of G containing T. The minimum cardinality of a forcing subset of S is the forcing non-split geodetic number $f_{ns}(S)$ of S. The forcing non split geodetic number of a connected graph G is $f_{ns}(G) = \min\{f_{ns}(S): S \text{ is a non} - \text{split geodetic set of } G\}$. Here we determined forcing split and non-split geodetic number of certain classes of graphs. Further, we determine the forcing split and forcing non-split geodetic number of graphs under some binary operations such join, Cartesian product and corona of two graphs.

Keywords: split geodetic set, non-split geodetic set, split geodetic number, non-split geodetic number, Cartesian product, join, corona.

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1. Introduction

In this paper, the graph $G = (V(G), E(G))$ is finite, connected and simple with the vertex set $V(G)$ containing $n \geq 3$ vertices and the edge set $E(G) \subseteq V \times V$. The distance between any two points u, v is $d(u, v)$ is a length of the shortest $u - v$ path in a graph G. An $u - v$ path of length $d(u, v)$ is called an $u - v$ geodesic. This concept was introduced in [1, 4].

The interval $I[u, v]$ containing all the points lying on some $u - v$ geodesic in G and $S \subseteq V(G)$, $I[S] = \bigcup_{u,v \in S} I[u, v]$. A set of vertices S is a geodetic set if $I[S] = V(G)$.

The minimum cardinality of a geodetic set is the geodetic number and is denoted as $g(G)$. This concept was introduced in [2].

A subset T of a minimum geodetic set S is called as the forcing subset for S if S is the unique minimum geodetic set of G containing T. The minimum cardinality among the forcing subsets of S is the forcing geodetic number $f_G(S)$ of a minimum geodetic set S and the forcing geodetic number f(G) of a connected graph G is $f(G) = \min(f_G(S))$. The forcing geodetic sets and the forcing geodetic number of a graph was introduced in [3].

The geodetic set S is said to be a non-split geodetic set in a graph G, if the induced sub graph $< V(G) - S >$ is connected. The cardinality of a non-split geodetic set which is minimum is the non-split geodetic number and is denoted as $g_{ns}(G)$. In [5], the non-split geodetic number was introduced.

The geodetic set S is said to be a split geodetic set in a graph G, if the sub graph $< V(G) - S >$ is disconnected. The cardinality of a split geodetic set which is minimum is the split geodetic number $g_s(G)$. In [6], the split geodetic number was introduced.

Let G and H be any two connected graphs. The join of G, H is denoted by $G + H$ is the graph with vertex set $V(G + H) = V(G) \cup V(H)$ and the edge set $E(G + H) = E(G) \cup E(H) \cup \{uv : u \in V(G), V \in V(H)\}$.

A graph formed by taking a single copy of G with $|V(G)| = n$ copies of H. By joining i^{th} vertex of G with each vertex in the i^{th} copy of H, then it is said to be a corona $G \circ H$ of two graphs G and H.

The cartesian product of any two graphs G, H is a graph with vertex set $V(G) \times V(H) = (u_i, v_j)$ where $u_i \in V(G)$, $v_j \in V(H)$, $1 \leq i \leq m$ and $1 \leq j \leq n$. Any two distinct vertices (u_i, v_j), (u_k, v_l) adjacent in $G \times H$ if and only if either $u_i = u_k$ and $v_j v_l \in E(H)$ or $v_j = v_l$ and $u_i u_k \in E(G)$.

2. Forcing split geodetic number $f_s(G)$ of a graph

The set S is a minimum split geodetic set in a connected graph $G = (V, E)$. A subset $T \subseteq S$ is called as the forcing subset for S if S is the unique minimum split geodetic set of G containing T. The cardinality of a minimum forcing subset of S is the forcing split geodetic number $f_s(S)$ of S. The forcing split geodetic number of a connected graph G is $f_s(G) = \min(f_s(S))$.

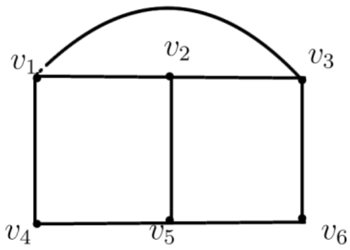

Figure 2.1: General graph.

The graph G given in Figure 2.1, $S_1 = \{v_2, v_4, v_6\}$, $S_2 = \{v_1, v_3, v_5\}$, $S_3 = \{v_1, v_5, v_6\}$ and $S_4 = \{v_3, v_4, v_5\}$ are four minimum split geodetic sets with $g(G) = g_s(G) = 3$. The set S_1 is the only minimum split geodetic set containing the vertex v_2, thus $f_s(S_1) = 1$. The remaining vertices other than v_2 of a graph G belongs to two or more minimum split geodetic sets S_i for $i = 2,3,4$. So that $f_s(S_2) = 2$, $f_s(S_3) = 2$ and $f_s(S_4) = 2$. Therefore the forcing split geodetic number of a connected graph G is $f_s(G) = 1$.

Observation 2.1. Let G be a connected graph of order n then,

- G has only one minimum split geodetic set if and only if $f_s(G) = 0$.
- The graph G has two or more minimum split geodetic sets and a vertex of G belongs to exactly one minimum split geodetic set if and only if $f_s(G) = 1$.
- All vertices of each minimum split geodetic set belongs to at least two minimum split geodetic sets if and only if $f_s(G) \geq 2$.

Theorem 2.2. For a path P_n with $n \geq 5$, the forcing split geodetic number is

$$f_s(P_n) = \begin{cases} 0 & \text{for } n = 5, \\ 1 & \text{otherwise.} \end{cases}$$

Proof. Consider a path of order $n \geq 5$ is $P_n: v_1, v_2, \ldots, v_{n-1}, v_n$.

Suppose that $n = 5$. The unique minimum split geodetic set is $S = \{v_1, v_3, v_5\}$. Hence by Observation 2.1, $f_s(P_n) = 0$.

Suppose that $n > 5$. The set $S_{i-2} = \{v_1, v_n, v_i\}$ where $3 \leq i \leq n-2$ gives different minimum split geodetic sets containing a vertex v_i of P_n, belongs to exactly one minimum split geodetic set. By Observation 2.1, $f_s(P_n) = 1$ for $n > 5$.

Theorem 2.3. Forcing split geodetic number of a cycle C_n with $n \geq 4$ is

$$f_s(C_n) = \begin{cases} 1 & \text{if } n \text{ is even,} \\ 2 & \text{if } n \text{ is odd.} \end{cases}$$

Proof. If n is even, then a cycle $C_{2r}: v_1, v_2, \dots, v_{2r}, v_1$. Clearly, in this cycle the set $S = \{v_i, v_j\}$ containing pair of antipodal vertices are different minimum split geodetic sets. Each vertex containing a unique vertex antipodal to it. By Observation 2.1, $f_s(C_n) = 1$.

If n is odd, then $C_{2r+1}: v_1, v_2, \dots, v_{2r+1}, v_1$ has more than one minimum split geodetic sets with each vertex belongs to two or more distinct minimum split geodetic sets. By Observation 2.1, $f_s(G) \geq 2$. Further, the set $S = \{v_1, v_{r+1}, v_{r+2}\}$ is the unique minimum split geodetic set containing two adjacent vertices v_{r+1}, v_{r+2} in C_n with $d(v_1, v_{r+1}) = d(v_1, v_{r+2})$. Thus $f_s(C_n) = 2$.

Theorem 2.4. The forcing split geodetic number of a wheel W_n with $n \geq 5$ is

$$f_s(W_n) = \begin{cases} 2 & \text{if n is even,} \\ 1 & \text{if n is odd.} \end{cases}$$

Proof. Suppose that n is even. The graph $G = W_n = C_{n-1} + K_1$ has $(n-1)$ different minimum split geodetic sets and each vertex belongs to two or more distinct minimum split geodetic sets. By Observation 2.1, $f_s(G) \geq 2$. Any two adjacent vertices in W_n forms a unique subset in minimum split geodetic sets. Thus $f_s(W_n) = 2$.

Suppose that n is odd, $n \geq 5$. The graph $G = W_n$ has two minimum split geodetic sets containing different vertices with common universal vertex in W_n. Thus $f_s(W_n) = 1$.

Theorem 2.5. The forcing split geodetic number of a tree T containing at least four internal vertices and k end vertices is $f_s(T) = 1$.

Proof. Consider, the set $S = \{v_1, v_2, \dots, v_k\}$ of end vertices and the set of internal vertices $\{u_1, u_2, \dots, u_x\} \subset V - S$ in a tree T. Clearly, the set S is a geodetic set and $< V(T) - S >$ is connected. Therefore consider $S' = S \cup \{u_i\}$, where u_i is an arbitrary internal vertex which is not an end vertex in $V - S$ forms a split geodetic set. The arbitrary internal vertex belongs to exactly one minimum split geodetic set. Thus by Observation 2.1, $f_s(T) = 1$.

Observation 2.6. There is no forcing split geodetic number in star.

Theorem 2.7. If $G = K_{m,n}$, the forcing split geodetic number for positive integers m and $n > 1$ is

$$f_s(K_{m,n}) = \begin{cases} 1 & \text{for } m = n, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. The graph $G = K_{m,n}$ is formed by the partite sets $A = \{a_i\}$ where $1 \leq i \leq m$ and $B = \{b_j\}$ where $1 \leq j \leq n$ with $m < n$. The vertex set of G is $A \cup B$.

For $m = n$, the graph $G = K_{m,n}$ having partite sets with same number of vertices forms two different minimum split geodetic sets $S_1 = \{a_i\}$ and $S_2 = \{b_j\}$ where $1 \le i, j \le m$ containing different vertices. Thus by Observation 2.1, $f_s(K_{m,n}) = 1$.

Suppose that $m \ne n$ and $m < n$. The unique minimum split geodetic set is S=A. Thus by Observation 2.1, $f_s(K_{m,n}) = 0$.

Theorem 2.8. The connected graph G' formed by joining a leaf vertex v to the cycle $C_n = G$, $n \ge 4$. The forcing split geodetic number of G' is $f_s(G') = 1$.

Proof. The graph G' formed by adding the leaf vertex $v \notin G$ to $C_n = G$ where $n \ge 4$. In G', there exists two or more minimum split geodetic sets of the form $S = \{v, u_i, u_j\}$. The set S containing a leaf vertex v and the pair of antipodal vertices u_i, u_j of a graph $C_n = G$.

If n is even, then the set S with different antipodal vertices and each vertex containing a unique vertex antipodal to it. Therefore $f_s(G') = 1$.

If n is odd, then the antipodal vertex u_i is non adjacent to the vertex v with $d(u_i, v) = 2$ belongs to exactly one minimum split geodetic set. Thus $f_s(G') = 1$.

Theorem 2.9. The graph $G' = C_n \circ K_1$ is formed by adding leaf vertex $v \notin G$ to every vertex u_i where $1 \le i \le n$ in a cycle $C_n = G$, $n \ge 4$. The forcing split geodetic number of a graph G' is

$$f_s(G') = \begin{cases} 1 & \text{for } n = 4, \\ 2 & \text{for } n > 4. \end{cases}$$

Proof. The connected graph $G' = C_n \circ K_1$ is formed by adding leaf vertex v to every vertex u_i where $1 \le i \le n$ in a cycle $C_n = G$, $n \ge 4$.

Case 1: Suppose that $n = 4$. In a graph $G' = C_4 \circ K_1$, the set X containing four end vertices forms a geodetic set and $< V(G') - X >$ is connected. Clearly, the set of end vertices X is non-split geodetic set. Any two antipodal vertices with the set X forms two minimum split geodetic sets contains different antipodal vertices. Thus $f_s(G') = 1$.

Case 2: Suppose that $n > 4$. In the graph $G' = C_n \circ K_1$, the different minimum split geodetic sets are $S_k = X \cup \{v_i, v_j\}$ where X is a set of end vertices in G' and $\{v_i, v_j\}$ are k possible nonadjacent vertices in G. Each non adjacent vertices belongs two or more minimum split geodetic sets, so $f_s(G') \ge 2$. Any two non adjacent vertices makes a unique subset in minimum split geodetic set S_k in a graph G'. Thus $f_s(G') = 2$.

Theorem 2.10. Let G' be the graph formed by adding k-leaf vertices $v_i \notin G$ where $1 \le i \le k$ to any vertex $u \in G = C_n$, where C_n is a cycle with $n \ge 4$. The forcing split geodetic number of G' is $f_s(G') = 1$.

Proof. The graph G' formed by adding the k-leaf vertices $v_i \notin G$ where $1 \le i \le k$ to the vertex $u \in G = C_n$. The different minimum split geodetic sets of the form $\{v_i, u_j, u_l\}$ containing a leaf vertices v_i, $1 \le i \le k$ with a pair of antipodal vertices u_j, u_l.

If n is even, then the pair of antipodal vertices u_j, u_l each vertex containing a unique vertex antipodal to it. Therefore $f_s(G') = 1$.

If n is odd, then the vertex $u_j \in G'$ is non adjacent to the vertex v_i where $1 \le i \le k$ with $d(v_i, u_j) = 2$ belongs to exactly one minimum split geodetic set. Thus $f_s(G') = 1$.

Theorem 2.11. The forcing split geodetic number of join of two paths, P_{n_1} and P_{n_2} with $n_1, n_2 \ge 4$ is

$$f_s\left(P_{n_1} + P_{n_2}\right) = \begin{cases} 2 & \text{for } n_1 = n_2, \\ 1 & \text{otherwise.} \end{cases}$$

Proof. Consider $V = \{v_1, v_2, \ldots, v_{n_1}\}$ and $U = \{u_1, u_2, \ldots, u_{n_2}\}$ are the vertex sets of a path P_{n_1} and P_{n_2} respectively. From the definition, the join of these two graphs is $G = (P_{n_1} + P_{n_2})$ with the vertex set $W = V \cup U$ with $n_1, n_2 \ge 4$.

Case 1: Suppose that $n_1 = n_2$. The different possible minimum split geodetic sets with cardinality $n_1 + 1$ are $S_k = V \cup u_k$ where u_k is any internal vertex in a set U and $S_l = U \cup v_l$ where v_l is any internal vertex in V . Each vertex in these minimum split geodetic sets belongs to at least two sets. So $f_s(G') \ge 2$. Since the end vertex in V and any internal vertex in U form the unique subset in minimum split geodetic set of cardinality two in S_k, similarly in S_l, the end vertex in *U* and any internal vertex in V form the unique subset in minimum split geodetic set of cardinality two. Thus $f_s(G') = 2$ for $n_1 = n_2$.

Case 2: Suppose that $n_1 \ne n_2$ and $|V| < |U|$. Every possible minimum split geodetic sets contains all vertices of set V and one internal vertex in U. Since the internal vertex in U belongs to exactly one split geodetic set which is minimum. So that $f_s(G') = 1$.

Similarly, if $|U| < |V|$, minimum split geodetic sets contains all vertices of set U and one internal vertex in V . Since the internal vertex in V belongs to exactly one split geodetic set which is minimum. So $f_s(G') = 1$.

Theorem 2.12. The forcing split geodetic number of corona product of two paths

P_{n_1} and P_{n_2} with $n_1 \geq 2$, $n_2 \geq 3$ is

$$f_s(P_{n_1} \circ P_{n_2}) = \begin{cases} 1 & \text{if } n_2 \text{ is odd} \\ n_1 + 1 & \text{otherwise.} \end{cases}$$

Proof. Consider $V = \{v_1, v_2, \dots, v_{n_1}\}$ and $U = \{u_1, u_2, \dots, u_{n_2}\}$ are the two vertex sets of P_{n_1} and P_{n_2} respectively. From the definition, the corona product of these two paths $G = (P_{n_1} \circ P_{n_2})$ having $n_1 + n_1 n_2$ vertices is formed by taking a copy of P_{n_1} and $|V| = n_1$ copies of P_{n_2} also by joining i^{th} vertex of P_{n_1} with each vertex in the i^{th} copy of P_{n_2}.

Case 1: Suppose that n_2 is odd. In a graph $G = (P_{n_1} \circ P_{n_2})$, the geodetic set X having $(n_1 \left\lceil \frac{n_2}{2} \right\rceil)$ vertices. The minimum different split geodetic sets are $S_i = X \cup v_i$, $1 \leq i \leq n_1$ where $v_i \in V$. Since the vertices $v_i \in V$ belongs to exactly one split geodetic set which is minimum and this is the unique subset of minimum split geodetic set. So that $f_s(S_i) = 1$, $1 \leq i \leq n_1$. Therefore $f_s(G) = 1$.

Case 2: Suppose that n_2 is not an odd number. The number of possible geodetic sets are $n_1(\frac{n_2}{2})^2$ and each minimum geodetic set X containing $n_1(\frac{n_2}{2} + 1)$ vertices. The different split geodetic sets are $S_i = X \cup v_i$, $1 \leq i \leq n_1$ where $v_i \in V$. Every vertex in these sets belongs two or more minimum split geodetic sets. By Observation 2.1, $f_s(S_i) \geq 2$. Since any internal vertex of P_{n_2} in each $|V| = n_1$ copy with one vertex in P_{n_1} form the unique subset in a minimum split geodetic set of cardinality $n_1 + 1$. Therefore $f_s(S_i) = n_1 + 1$ and for the graph G, $f_s(G) = n_1 + 1$.

Theorem 2.13. For the cartesian product of two paths P_{n_1} and P_{n_2} with $n_1 \geq 2$ and $n_2 \geq 5$, the forcing split geodetic number

$$f_s(P_{n_1} \times P_{n_2}) = \begin{cases} 2 & \text{if } n_1 = 3 \\ 1 & \text{otherwise.} \end{cases}$$

Proof. Consider the vertex set $V = \{v_1, v_2, \dots, v_{n_1}\}$ of P_{n_1} and $W = \{w_1, w_2, \dots, w_{n_2}\}$ of P_{n_2}. Let $(P_{n_1} \times P_{n_2}) = G$ be a cartesian product of two paths P_{n_1} and P_{n_2} with $n_1 \geq 2$, $n_2 \geq 5$.

Case 1: If $n_1 = 3$, then the set $X = \{(v_1, w_1)(v_3, w_{n_2})\}$ is a minimum geodetic sets in $G = (P_{n_1} \times P_{n_2})$. Since the induced sub graph $< V(G) - X >$ is connected, the set X is a non-split geodetic set. There exist many split geodetic sets in G of cardinality four and every vertex in these sets belongs to two or more minimum split geodetic sets. By Observation 2.1, $f_s(G) \geq 2$. The set $S = X \cup \{(v_2, w_1)(v_3, w_2)\}$. is the minimum unique split geodetic set containing the subset $T = \{(v_2, w_1)(v_3, w_2)\}$. Also any subset U of S with $|U| < |T|$ is not a forcing subset of split geodetic set. Therefore $f_s(G) = 2$.

Case 2: If $n_1 \neq 3$, then the minimum geodetic set $X = \{(v_1, w_1)(v_{n_1}, w_{n_2})\}$ is not a split geodetic set. Since $< V(G) - X >$ is connected, the set $S = X \cup \{(v_{n_1-1}, w_1)(v_{n_1}, w_2)\}$ form split geodetic set. There exists many split geodetic sets in G of cardinality four but some vertex in these sets belongs to exactly one split geodetic set. The set S is the unique minimum split geodetic set containing the subset $T = \{(v_{n_1}, w_2)\}$. This implies that $f_s(S) = 1$ also $f_s(G) = 1$.

3. Forcing non-split geodetic number $f_{ns}(G)$ of a graph

The set S is a minimum non-split geodetic set in a connected graph $G = (V, E)$ and the subset $T \subseteq S$ is called as the forcing subset for S if S is the unique minimum non-split geodetic set of G containing T. The minimum cardinality of a forcing subset of S is the forcing non-split geodetic number $f_{ns}(S)$ of S. The forcing non-split geodetic number of a connected graph G is $f_{ns}(G) = \min(f_{ns}(S))$.

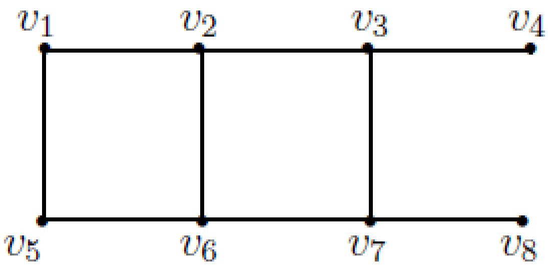

Figure 3.1: General graph.

The two minimum non-split geodetic sets in G shown in Figure 3.1, are $S_1 = \{v_1, v_4, v_8\}$ and $S_2 = \{v_4, v_5, v_8\}$. Clearly, set S_1 is the only minimum non-split geodetic set containing v_1 and the vertex v_5 of G belongs to the set S_2. Thus $f_{ns}(S_1) = f_{ns}(S_2) = 1$. Therefore for the graph G, $f_{ns}(G) = 1$.

Observation 3.1. Let G be a connected graph then,

- G has only one minimum non-split geodetic set if and only if $f_{ns}(G) = 0$.
- The graph G has two or more minimum non-split geodetic sets and a vertex of G belongs to exactly one minimum non-split geodetic set if and only if $f_{ns}(G) = 1$.
- All vertices of each minimum non-split geodetic set belongs to two or more minimum non-split geodetic sets if and only if $f_{ns}(G) \geq 2$.

Theorem 3.2. The forcing non-split geodetic number of a path P_n with $n \geq 5$, a tree T and the star $K_{1,n}$ is zero.

Proof. In a path P_n with $n \geq 5$, a tree T and the star $K_{1,n}$, the set X of end vertices forms the minimum non-split geodetic set which is unique. By Observation 3.1, the forcing non-split geodetic number is zero.

Theorem 3.3. The forcing non-split geodetic number of a cycle $G = C_n$ with $n \geq 4$ is

$$f_{ns}(C_n) = \begin{cases} 3 & \text{for n is even,} \\ 1 + \left\lceil \frac{n}{2} \right\rceil & \text{for n is odd.} \end{cases}$$

Proof. Case 1: Suppose n is even. Consider a cycle C_{2r} containing $2r$ vertices $\{v_1, v_2, \dots, v_{2r}, v_1\}$. The pair of antipodal vertices in a set $X = \{v_i, v_j\}$ forms different geodetic sets. Since $< V(G) - X >$ is not connected, the set X is not a non-split geodetic set and the set $S = \{v_1, v_2, \dots, v_r, v_{r+1}\}$ of cardinality $(\frac{n}{2} + 1)$ forms a non-split geodetic set. There exists n number of minimum non-split geodetic sets and all vertices in n minimum non-split geodetic set belongs to two or more minimum non-split geodetic sets so that $f_{ns}(G) \geq 2$. The unique minimum non-split geodetic set containing the subset $T = \{v_1, v_2, v_{r+1}\}$. This implies that $f_{ns}(C_n) = 3$.

Case 2: Suppose n is odd. Consider a cycle C_{2r+1} containing $2r + 1$ vertices $\{v_1, v_2, \dots, v_{2r+1}, v_1\}$. The set $X = \{v_i, v_j, v_k\}$ forms minimum geodetic sets where $d(v_i, v_j) = d(v_i, v_k)$ and v_j,v_k are adjacent vertices. Here $< V(G) - X >$ is not connected. The n number of minimum non-split geodetic sets S_i where $1 \leq i \leq n$ of cardinality $1 + \left\lceil \frac{n}{2} \right\rceil$ is obtained by taking all the internal vertices from v_i to v_j or v_i to v_k with X. Every vertex of each minimum n number of non-split geodetic set belongs to two or more minimum non-split geodetic sets so that $f_{ns}(G) \geq 2$. Also, any subset T of S_i with $|T| < |S_i|$ is not a forcing subset of non-split geodetic sets S_i, $1 \leq i \leq n$. Therefore $f_{ns}(C_n) = 1 + \left\lceil \frac{n}{2} \right\rceil$.

Theorem 3.4. The forcing non-split geodetic number of a wheel W_n with $n \geq 5$ is

$$f_{ns}(W_n) = \begin{cases} 2 & \text{if n is even,} \\ 1 & \text{if n is odd.} \end{cases}$$

Proof. The proof is similar to the proof of Theorem 2.4.

Theorem 3.5. The forcing non-split geodetic number of $K_{m,n}$ for any positive integers $m, n \geq 2$ is

$$f_{ns}(K_{m,n}) = \begin{cases} m+n-1 & \text{for } m = 2, \\ 4 & \text{for } m \geq 3. \end{cases}$$

Proof. The graph $G = K_{m,n}$ with $A = \{a_i\}$ where $1 \leq i \leq m$ and $B = \{b_j\}$ where $1 \leq j \leq n$ are partite sets also $m \leq n$. $A \cup B$ is a vertex set of a graph G.

Case 1: For $m = 2$, the different minimum non-split geodetic sets are S_i where $1 \leq i \leq m + n$ and the cardinality of each set S_i is $m + n - 1$. Every vertex in S_i belongs to two or more minimum non-split geodetic sets. Also, any subset T of S_i with $|T| < |S_i|$ is not a forcing subset of non-split geodetic sets S_i, $1 \leq i \leq m + n$. Therefore $f_{ns}(K_{m,n}) = m + n - 1$.

Case 2: For $m \geq 3$, different minimum non-split geodetic sets in $G = K_{m,n}$ are S_i where $1 \leq i \leq \binom{m}{r} \times \binom{n}{r}$ with cardinality four. Also no subset T of S_i with $|T| < |S_i|$ is a forcing subset of non-split geodetic sets. Therefore $f_{ns}(K_{m,n}) = 4$.

Theorem 3.6. If the graph G' formed by joining the leaf vertex $v \notin G$ to a cycle $C_n = G$ where $n \geq 3$. Then $f_{ns}(G') = 0$.

Proof. The graph G' formed by adding the leaf vertex $v \notin G$ to a cycle $C_n = G$ where $n \geq 3$.

If n is even, then the non-split geodetic set in G' is $S = \{v, u\}$ which is minimum and unique, containing antipodal vertices $u \in G$ and $v \notin G$. By Observation 3.1, the forcing non-split geodetic number $f_{ns}(G') = 0$.

If n is odd, then the unique minimum non-split geodetic set in G' is $S = \{v, u_i, u_j\}$ where v is the leaf vertex and $u_i, u_j \in G$ are any two adjacent vertices such that $d(v, u_i) = d(v, u_j)$. By Observation 3.1, the forcing non-split geodetic number $f_{ns}(G') = 0$.

Theorem 3.7. If the graph G' is formed by joining a leaf vertex $v \notin G$ to each vertex u_i where $1 \leq i \leq n$ in $C_n = G$, $n \geq 3$. Then $f_{ns}(G') = 0$.

Proof. The graph $G' = C_n \circ K_1$ formed by adding the leaf vertex v to each vertex u_i where $1 \leq i \leq n$ in $C_n = G$, $n \geq 3$. The set S of leaf vertices in G' is a minimum non-split geodetic set. By Observation 3.1, the forcing non-split geodetic number $f_{ns}(G') = 0$.

Theorem 3.8. The graph G' is formed by joining k-leaf vertices $v_i \notin G$ where $1 \leq i \leq k$ to the vertex $u \in G = C_n$ where C_n is a cycle with $n \geq 3$. Then the forcing non-split geodetic number of G' is $f_{ns}(G') = 0$.

Proof. The graph G' formed by adding the k-leaf vertices $v_i \notin G$ where $1 \leq i \leq k$ to the vertex of $u \in C_n$ where $n \geq 3$.

If n is even, then the unique minimum non-split geodetic set in G' is $S = \{v_i, u_l\}$, $1 \le i \le k$ containing the pendant vertices $v_i \in G'$ and the vertex $u_l \in C_n$ such that $d(v_i, u_l) = diam(G')$. By Observation 3.1, the forcing non-split geodetic number $f_{ns}(G') = 0$.
If n is odd in $C_n = G$, then the non-split geodetic set is $S = \{v_i, u_l, u_j\}$ where $v_i \in G'$ are pendant vertices and u_l, u_j are any two adjacent vertices such that $d(v_i, u_l) = d(v_i, u_j)$. By Observation 3.1, the forcing non-split geodetic number $f_{ns}(G') = 0$.

Theorem 3.9. The forcing non-split geodetic number of join of two graphs P_{n_1} and P_{n_2} where $n_1, n_2 \ge 3$, $n_2 \ge n_1$ is

$$f_{ns}(P_{n_1} + P_{n_2}) = \begin{cases} 1 & \text{for } n_1 = n_2 \text{ or } n_1 \text{ is even,} \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Consider $V = \{v_1, v_2, \dots, v_{n_1}\}$ and $U = \{u_1, u_2, \dots, u_{n_2}\}$ are the two vertex sets of a path P_{n_1} and P_{n_2} respectively. The join of these two graphs is $G = (P_{n_1} + P_{n_2})$ with the vertex set $W = V \cup U$ with $n_1, n_2 \ge 3$.
Case 1: If $n_1 = n_2$ and n_1 is odd, then there exists only two non-split geodetic sets which is minimum in a connected graph $G = (P_{n_1} + P_{n_2})$ from two vertex sets V and U. These two non-split geodetic sets with different vertices of cardinality $\left\lceil \frac{n_1}{2} \right\rceil$. By Observation 3.1, the forcing non-split geodetic number $f_{ns}(G) = 1$. If $n_1 \le n_2$ and n_1 is even two or more non-split geodetic sets of cardinality $\left\lceil \frac{n_1+1}{2} \right\rceil$ having different vertices. The internal vertex which is adjacent to the end points in P_{n_1} belongs to exactly one minimum non-split geodetic set. By Observation 3.1, the forcing non-split geodetic number $f_{ns}(G) = 1$.
Case 2: If $n_1 < n_2$ and n_1 is not an even. Then the graph G has only one non-split geodetic set. By Observation 3.1, $f_{ns}(G) = 0$.

Theorem 3.10. The forcing non-split geodetic number of corona of two graphs P_{n_1} and P_{n_2} with $n_2 \ge n_1$, $n_2 \ge 3$ is

$$f_{ns}(P_{n_1} \circ P_{n_2}) = \begin{cases} 0 & \text{for } n_2 \text{ is odd,} \\ n_1 & \text{otherwise.} \end{cases}$$

Proof. Consider $V = \{v_1, v_2, \dots, v_{n_1}\}$ and $U = \{u_1, u_2, \dots, u_{n_2}\}$ are the two vertex sets of P_{n_1} and P_{n_2} respectively. The corona of these two paths is $G = (P_{n_1} \circ P_{n_2})$ having $n_1 + n_1 n_2$ vertices formed by taking one copy of P_{n_1}, $|V| = n_1$ copies of P_{n_2} and joining i^{th} vertex of P_{n_1} with each vertex in the i^{th} copy of P_{n_2}.

Case 1: If n_2 is odd, then the graph $G = (P_{n_1} \circ P_{n_2})$ has unique non-split geodetic set X having $n_1 \left\lceil \frac{n_2}{2} \right\rceil$ vertices. By Observation 3.1, $f_{ns}(G) = 0$.

Case 2: If n_2 is not an odd, then the graph $G = (P_{n_1} \circ P_{n_2})$ contains $(\frac{n_2}{2})^{n_1}$ number of non-split geodetic sets of cardinality $n_1(\frac{n_2}{2} + 1)$ All vertices of each minimum non-split geodetic set belongs to two or more minimum non-split geodetic sets so $f_{ns}(G) \geq 2$. Any internal vertex which is adjacent to the end points of P_{n_2} in each $|V| = n_1$ copy of G form the unique subset in minimum non-split geodetic sets S_i $1 \leq i \leq \{(\frac{n_2}{2})^{n_1}\}$. The cardinality of this subset is n_1. Also, any subset T of S_i with $|T| < |S_i|$ is not a forcing subset of non-split geodetic sets S_i, $1 \leq i \leq \{(\frac{n_2}{2})^{n_1}\}$. Therefore $f_{ns}(S_i) = n_1$. Thus $f_{ns}(G) = n_1$.

Theorem 3.11. The forcing non-split geodetic number in cartesian product of two graphs P_{n_1} and P_{n_2} with $n_1, n_2 \geq 2$ is

$$f_{ns}(P_{n_1} \times P_{n_2}) = \begin{cases} 3 & \text{for } n_1 = n_2 = 2, \\ 1 & \text{otherwise.} \end{cases}$$

Proof. Consider $V = \{v_1, v_2, \dots, v_{n_1}\}$ and $W = \{w_1, w_2, \dots, w_{n_2}\}$ are the two vertex sets of P_{n_1} and P_{n_2} respectively. Let $(P_{n_1} \times P_{n_2}) = G$ be a cartesian product of two paths P_{n_1} and P_{n_2}.

Case 1: If $n_1 = n_2 = 2$, then the graph $(P_{n_1} \times P_{n_2}) = G$ is a cycle C_4. By Theorem 3.3, $f_{ns}(G) = 3$.

Case 2: If $n_1 = n_2 \neq 2$, then $S_1 = \{(v_1, w_1), (v_{n_1}, w_{n_2})\}$ and $S_2 = \{(v_{n_1}, w_1), (v_1, w_{n_2})\}$ are the only two non-split geodetic sets with different vertices. Thus $f_{ns}(S_i) = 1$ where $i = 1,2$. Therefore $f_{ns}(G) = 1$.

4. Conclusion

In this paper we studied the forcing split and forcing non-split geodetic number of a graph and obtained some results on the join, corona and cartesian product of two graphs. Further studied the results on adding a leaf vertex to a cycle C_n.

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