A CERTAIN SUBCLASS OF MEROMORPHIC FUNCTIONS WITH POSITIVE COEFFICIENTS ASSOCIATED WITH HURWITZ-LERCH ZETA FUNCTION

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Abstract
In this paper, we introduce and study new class $M_n(\wp, \hbar, \gamma, s, b)$ of meromorphic univalent functions defined in $U^* = \{z: z \in C$ and $0 < |z| < 1\} = \mathrm{U} \setminus\{0\}$.We obtain coefficients inequaities, distortion theorems, extreme points, closure theorems, radius of convexity estimates and modified Hadamard products

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1 Introduction

Let σ be denote the class of functions $f(z)$ of the form

$$f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n,\ n \in \mathbb{N} = \{1,2,3,\cdots\} \qquad (1.1)$$

which are analytic in the punctured unit disc $U^* = \{z \in \mathbb{C}: 0 < |z| < 1\} = U\backslash\{0\}$. For functions $f \in \sigma$ given by (1.1) and $g \in \sigma$ given by

$$g(z) = \frac{1}{z} + \sum_{n=1}^{\infty} b_n z^n$$

their Hadamard product (or convolution) is defined by

$$(f * g)(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n b_n z^n.$$

Analytically a function $f \in \sigma$ given by (1.1) is said to be meromorphically starlike of order $\wp$ if it satisfies the following:

$$Re\left\{-\left(\frac{zf'(z)}{f(z)}\right)\right\} > \wp,\ \ (z \in U)$$

for some $\wp(0 \le \wp < 1)$. We say that f is in the class $\sigma^*(\wp)$ of such functions.
Similarly a function $f \in \sigma$ given by (1.1) is said to be meromorphically convex of order $\wp$ if it satisfies the following:

$$Re\left\{-\left(1 + \frac{zf''(z)}{f'(z)}\right)\right\} > \wp,\ \ (z \in U)$$

for some $\wp(0 \le \wp < 1)$. We say that f is in the class $\sigma_k(\wp)$ of such functions.
For a function $f \in \sigma$ given by (1.1) is said to be meromorphically close to convex of order $\hbar$ and type $\wp$ if there exists a function $g \in \sigma^*(\wp)$ such that

$$Re\left\{-\left(\frac{zf'(z)}{g(z)}\right)\right\} > \hbar,\ \ (0 \le \wp < 1,\ 0 \le \hbar < 1,\ z \in U).$$

We say that f is in the class $K(\hbar, \wp)$.
The class $\sigma^*(\wp)$ and various other subclasses of σ have been studied rather extensively by Clunie [3], Miller[9] , Pommerenke [10] , Royster [11] and Akgul [1] .
Recent years, many authors investigated the subclass of meromorphic functions with positive coefficient. In [6], Juneja and Reddy introduced the class of σ_p functions of the form

$$f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n, a_n \ge 0 \qquad (1.2)$$

which are regular and univalent in U. The functions in this class are said to be meromorphic functions with positive coefficient.
The following we recall a general Hurwitz-Lerch Zeta function $\phi(z, s, a)$ defined by (see [14], p. 121)

$$\phi(z, s, a) = \sum_{n=0}^{\infty} \frac{z^n}{(n+a)^s}$$

for $a \in \mathbb{C}\backslash\mathbb{Z}_0^-, s \in \mathbb{C}$ when $|z| < 1$; $\Re(s) > 1$ when $|z| = 1$, where $\mathbb{Z}_0^- = \mathbb{Z}\backslash\{\mathbb{N}\}$, $\mathbb{Z} = \{0, \pm1, \pm2, \cdots\}$, $\mathbb{N} = \{1,2,3,\cdots\}$.
Several interesting properties and characteristics of the Hurwitz-Lerch Zeta function $\phi(z, s, a)$ can be found in the recent investigation by Ferreira and Lopez [4], Lin and Srivastava [7] , Luo and Srivastava [8], Thirupathi Reddy and Venkateswarlu [15] and others.
By making use of Hurwitz-Lerch Zeta function $\phi(z, s, a)$, Srivastava and Attiya recently introduced and investigated the integral operator

$$\mathcal{S}_{s,b}f(z) = z + \sum_{n=2}^{\infty} \left(\frac{1+b}{k+b}\right)^s c_n z^n, (b \in \mathbb{C}\backslash\mathbb{Z}_0^-, s \in \mathbb{C}, z \in U).$$

Motivated essentially by the above mentioned Srivastava-Atiya operator $\mathcal{S}_{s,b}$, Zhi-Gang Wang and Lei Shi [17] introduced the linear operator

$$\mathcal{S}_{s,b}: \sigma \to \sigma$$

defined in terms of the Hardmard product (or convolution), by

$$\mathcal{S}_{s,b}f(z) = b_{s,b}(z) * f(z), \quad (b \in \mathbb{C}\backslash\mathbb{Z}_0^- \cup \{1\}, s \in \mathbb{C}, f \in \sigma, z \in U^*), \tag{1.3}$$

where for convenience,

$$b_{s,b}(z) = (b-1)^s \left[\phi(z,s,b) - b^{-s} + \frac{1}{z(b-1)^s}\right], \quad (z \in U^*).$$

It can be easily be seen from (1.3) that

$$\mathcal{S}_{s,b}f(z) = \tfrac{1}{z} + \textstyle\sum_{n=1}^{\infty} \Theta\,(n,s,b) a_n z^n, \tag{1.4}$$

$$\text{where } \Theta(n,s,b) = \left(\frac{b-1}{b+n}\right)^s.$$

Indeed, the operator $\mathcal{S}_{s,b}$ can be defined for $b \in \mathbb{C}\backslash\mathbb{Z}_0^- \cup \{1\}$, where

$$\mathcal{S}_{s,0}f(z) = \lim_{b \to 0} \{\mathcal{S}_{s,b}f(z)\}.$$

We observe that

$$\mathcal{S}_{0,b}f(z) = f(z)$$

and

$$\mathcal{S}_{1,\gamma} = \frac{\gamma - 1}{z^\gamma} \int_0^z t^{\gamma-1} f(t)dt \quad (\Re(\gamma) > 1).$$

Furthermore, from the definition (1.4), we find that

$$\mathcal{S}_{s+1,b}f(z) = \frac{b-1}{z^b} \int_0^z t^{b-1}\, \mathcal{S}_{s,b}f(t)dt \quad (\Re(b) > 1). \tag{1.5}$$

Differentiating both sides of (1.5) with respect to z, we get the following useful relationship:

$$z\big(\mathcal{S}_{s+1,b}f\big)'(z) = (b-1)\mathcal{S}_{s,b}f(z) - b\mathcal{S}_{s+1,b}f(z).$$

Definition 1. For $0 \le \wp < 1$, $0 < \hbar \le 1$, $\frac{1}{2} \le \gamma \le 1$, $0 \le s \le 1$, $0 \le b \le 1$, $n \in \mathbb{N}$, we denote by $M_n(\wp, \hbar, \gamma, s, b)$ the subclass of Σ^* consisting of functions of the form (1.1) and satisfying the analytic criterion

$$\left|\frac{z^2\left(\mathcal{S}_{s,b}f(z)\right)' + 1}{(2\gamma-1)z^2\left(\mathcal{S}_{s,b}f(z)\right)' + (2\wp\gamma - 1)}\right| < \hbar. \tag{1.6}$$

2. Coefficient Estimates

Unless otherwise mentioned, we assume throughout this paper that $0 \le \wp < 1$, $0 < \hbar \le 1$, $\frac{1}{2} \le \gamma \le 1$, $0 \le s \le 1$, $0 \le b \le 1$, $n \in \mathbb{N}$ and $z \in U^*$

Theorem 2.1. *The function $f(z) \in M_n(\wp, \hbar, \gamma, s, b)$ if and only if*

$$\sum_{n=1}^{\infty} [n(1 + 2\hbar\gamma - \hbar]\, \Theta(n,s,b) a_n \le 2\hbar\gamma(1 - \wp). \tag{2.1}$$

Proof. Suppose (2.1) holds, so

$$\left|z^2\left(\mathcal{S}_{s,b}f(z)\right)' + 1\right| - \hbar\left|(2\gamma-1)z^2\left(\mathcal{S}_{s,b}f(z)\right)' + (2\wp\gamma - 1)\right|$$

$$= \left|\sum_{n=1}^{\infty} n\, \Theta(n,s,b) a_n z^{n+1}\right| - \hbar \left|2\gamma(\wp - 1) + \sum_{n=1}^{\infty} n\,(2\gamma - 1)\Theta(n,s,b) a_n z^{n+1}\right|$$

$$\leq \sum_{n=1}^{\infty} n\, \Theta(n,s,b) a_n r^{n+1}| - \hbar \left\{2\gamma(\wp - 1) + \sum_{n=1}^{\infty} n\,(2\gamma - 1)\Theta(n,s,b) a_n r^{n+1}\right\}$$

$$= \sum_{n=1}^{\infty} n\,(1 + 2\hbar\gamma - \hbar)\Theta(n,s,b) a_n r^{n+1} - 2\hbar\gamma(1 - \wp)$$

Since the above inequality holds for all $r, 0 < r < 1$,

letting $r \to 1^-$, we have

$$\sum_{n=1}^{\infty} n\,(1 + 2\hbar\gamma - \hbar)\Theta(n,s,b) a_n - 2\hbar\gamma(1 - \wp) \leq 0$$

by (2.1), hence $f(z) \in M_n(\wp, \hbar, \gamma, s, b)$.

Conversely, suppose that $f(z)$ is in the class $M_n(\wp, \hbar, \gamma, s, b)$, then

$$\left|\frac{z^2\left(\mathcal{S}_{s,b} f(z)\right)' + 1}{(2\gamma - 1)z^2\left(\mathcal{S}_{s,b} f(z)\right)' + (2\wp\gamma - 1)}\right| = \left|\frac{\sum_{n=1}^{\infty} n\, \Theta(n,s,b) a_n z^{n+1}}{2\gamma(1 - \wp) - \sum_{n=1}^{\infty} n\,(2\gamma - 1)\Theta(n,s,b) a_n z^{n+1}}\right| \leq \hbar.$$

Using the fact that $Re(z) \leq |z|$ for all z, we have

$$\left|\frac{z^2\left(\mathcal{S}_{s,b} f(z)\right)' + 1}{(2\gamma - 1)z^2\left(\mathcal{S}_{s,b} f(z)\right)' + (2\wp\gamma - 1)}\right| \leq$$

$$\left\{\frac{\sum_{n=1}^{\infty} n\, \Theta(n,s,b) a_n z^{n+1}}{2\gamma(1 - \wp) - \sum_{n=1}^{\infty} n\,(2\gamma - 1)\Theta(n,s,b) a_n z^{n+1}}\right\} \leq \hbar \qquad (2.2).$$

If we choose z to be real so that $z^2\left(\mathcal{S}_{s,b} f(z)\right)'$ is real. Upon cleaning the denominator in (2.2) and letting $z \to 1^-$ through positive values, we obtain

$$\sum_{n=1}^{\infty} n\,[1 + 2\hbar\gamma - \hbar]\Theta(n,s,b) a_n \leq 2\hbar\gamma(1 - \wp).$$

This completes the proof of the theorem.

Corollary 2.1. *Let the function $f(z)$ denoted by (1.1) be in the class $M_n(\wp, \hbar, \gamma, s, b)$, then* $a_n \leq \frac{2\hbar\gamma(1-\wp)}{n[1+2\hbar\gamma-\hbar]\Theta(n,s,b)} \quad (n \geq 1)$,

with equality for the function

$$f(z) = \frac{1}{z} + \frac{2\hbar\gamma(1-\wp)}{n[1+2\hbar\gamma-\hbar]\Theta(n,s,b)} z^n \qquad (2.3)$$

3.Distortion Theorems

Theorem 3.1. Let the function $f(z) \in M_n(\wp, \hbar, \gamma, s, b)$, then for $0 < |z| = \gamma < 1$, *we have*

$$\frac{1}{r} - \frac{2\hbar\gamma(1-\wp)}{(1+2\hbar\gamma-\hbar)\Theta(1,s,b)} r \leq |f(z)| \leq \frac{1}{r} - \frac{2\hbar\gamma(1-\wp)}{(1+2\hbar\gamma-\hbar)\Theta(1,s,b)} r \qquad (3.1)$$

with equality for the function

$$f(z)=\frac{1}{z}+\frac{2\hbar\gamma(1-\wp)}{(1+2\hbar\gamma-\hbar)\Theta(1,s,b)}z^n \qquad (3.2)$$

Proof. Suppose that f is in $M_n(\wp,\hbar,\gamma,s,b)$. In view of Theorem 2.1, we have

$$(1+2\hbar\gamma-\hbar)(1-s)^2(b+1)(b+2)\sum_{n=1}^{\infty}a_n \le \sum_{n=1}^{\infty}n\,[1+2\hbar\gamma-\hbar]\Theta(n,s,b)a_n \le 2\hbar\gamma(1-\wp)$$

. Then

$$\sum_{n=1}^{\infty}a_n \le \frac{2\hbar\gamma(1-\wp)}{(1+2\hbar\gamma-\hbar)\Theta(1,s,b)}. \qquad (3.3)$$

Consequently, we obtain

$$\begin{aligned}|f(z)| &= \left|\frac{1}{z}+\sum_{n=1}^{\infty}a_n z^n\right| \le \frac{1}{|z|}+\sum_{n=1}^{\infty}a_n|z|^n \\ &\le \frac{1}{r}+r\sum_{n=1}^{\infty}a_n \\ &\le \frac{1}{r}+\frac{2\hbar\gamma(1-\wp)}{(1+2\hbar\gamma-\hbar)\Theta(1,s,b)}r \qquad (3.4)\end{aligned}$$

Also,

$$\begin{aligned}|f(z)| &= \left|\frac{1}{z}+\sum_{n=1}^{\infty}a_n z^n\right| \ge \frac{1}{|z|}-\sum_{n=1}^{\infty}a_n|z|^n \\ &\ge \frac{1}{r}-r\sum_{n=1}^{\infty}a_n \\ &\ge \frac{1}{r}-\frac{2\hbar\gamma(1-\wp)}{(1+2\hbar\gamma-\hbar)\Theta(1,s,b)}r. \qquad (3.5)\end{aligned}$$

Hence, (3.1) follows.

Theorem 3.2. Let the function $f\in M_n(\wp,\hbar,\gamma,s,b)$, then for $0<|z|=r<1$, we have

$$\frac{1}{r^2}-\frac{2\hbar\gamma(1-\wp)}{(1+2\hbar\gamma-\hbar)\Theta(1,s,b))} \le |f'(z)| \le \frac{1}{r^2}+\frac{2\hbar\gamma(1-\wp)}{(1+2\hbar\gamma-\hbar)\Theta(1,s,b))} \qquad (3.6)$$

with equality for the function $f(z)$ *given by (3.2).*

Proof. From theorem 2.1, and (3.3), we have,

$$\sum_{n=1}^{\infty}n\,a_n \le \frac{2\hbar\gamma(1-\wp)}{(1+2\hbar\gamma-\hbar)\Theta(1,s,b)}. \qquad (3.7)$$

The remaining part of the proof is similar to the proof of Theorem 3.1, so we omit the details.

4. Closure Theorems

Let the functions $f_j(z)$ be defined for j = 1,2,...,m by

$$f_j(z)=\frac{1}{z}+\sum_{n=1}^{\infty}a_{n,j}\,z^n, \quad \left(a_{n,j}\ge 0\right) \qquad (4.1)$$

Theorem 4.1.. *Let* $f_j(z)\in M_n(\wp,\hbar,\gamma,s,b),(j=1,2,\ldots.m)$. Then the function

$$h(z)=\frac{1}{z}+\sum_{n=1}^{\infty}\left(\frac{1}{m}\sum_{j=1}^{\infty}a_{n,j}\right)z^n \qquad (4.2)$$

is in $M_n(\wp,\hbar,\gamma,s,b)$.

Proof. Since$f_j(z)\in M_n(\wp,\hbar,\gamma,s,b),(j=1,2,\ldots.m)$, it follows from Theorem 2.1, that

$$\sum_{n=1}^{\infty} n\,[1+2\hbar\gamma-\hbar]\Theta(n,s,b)a_{n,j} \leq 2\hbar\gamma(1-\wp),$$

for every j = 1,2,....,m. Hence

$$\sum_{n=1}^{\infty} n\,[1+2\hbar\gamma-\hbar]\Theta(n,s,b)\left(\frac{1}{m}\sum_{j=1}^{\infty} a_{n,j}\right)$$

$$=\frac{1}{m}\sum_{j=1}^{\infty}\left[\sum_{n=1}^{\infty} n\,[1+2\hbar\gamma-\hbar]\Theta(n,s,b)a_{n,j}\right] \leq 2\hbar\gamma(1-\wp).$$

From Theorem 2.1, it follows that $h(z) \in M_n(\wp,\hbar,\gamma,s,b)$
This completes the proof.

Theorem 4.2. The class $M_n(\wp,\hbar,\gamma,s,b)$ is closed under convex linear combinations.
Proof. Let $f_j(z),(j=1,2)$ defined by (4.1) be in the class $M_n(\wp,\hbar,\gamma,s,b)$, then it is sufficient to show that

$$h(z)=\xi f_1(z)+(1-\xi)f_2(z), \quad (0 \leq \xi \leq 1) \tag{4.3}$$

is in the class $M_n(\wp,\hbar,\gamma,s,b)$. Since

$$h(z)=\frac{1}{z}+\sum_{n=1}^{\infty}\left[\xi a_{n,1}+(1-\xi)a_{n,2}\right] z^n, \tag{4.4}$$

then, we have from Theorem 2.1, that

$$\sum_{n=1}^{\infty} n\,[1+2\hbar\gamma-\hbar]\Theta(n,s,b)\left[\xi a_{n,1}+(1-\xi)a_{n,2}\right]$$

$$\leq 2\xi\hbar\gamma(1-\wp)+2\hbar\gamma(1-\xi)(1-\wp)=2\hbar\gamma(1-\wp)$$

So, $h(z) \in M_n(\wp,\hbar,\gamma,s,b)$.

Theorem 4.3. *Let* $0 \leq \rho < 1$, *then*
$M_n(\wp,\hbar,\gamma,s,b) \leq M_n(\wp,\hbar,1,s,b)=M_n(\wp,\hbar,s,b)$
Where

$$\rho = 1-\frac{\gamma(1+\hbar)(1-\wp)}{(1+2\hbar\gamma-\hbar)}. \tag{4.5}$$

Proof. *Let* $f(z) \in M_n(\wp,\hbar,\gamma,s,b)$, then

$$\sum_{n=1}^{\infty}\frac{n[1+2\hbar\gamma-\hbar]\Theta(n,s,b)}{2\hbar\gamma(1-\wp)}a_n \leq 1. \tag{4.6}$$

We need to find the value of ρ such that

$$\sum_{n=1}^{\infty}\frac{n(1+\hbar)}{2\hbar(1-\rho)}\Theta(n,s,b)a_n \leq 1. \tag{4.7}$$

In view of equations (4.6) and (4.7), we have

$$\frac{n[1+\hbar]}{2\hbar(1-\rho)}\Theta(n,s,b) \leq \frac{n[1+2\hbar\gamma-\hbar]\Theta(n,s,b)}{2\hbar\gamma(1-\wp)}.$$

That is

$$\rho \leq 1-\frac{\gamma(1+\hbar)(1-\wp)}{(1+2\hbar\gamma-\hbar)}$$

. Which completes the proof of theorem.

Theorem 4.4. *Let* $f_0(z)=\frac{1}{z}$ *and*

$$f_n(z)=\frac{1}{z}+\frac{2\hbar\gamma(1-\wp)}{n[1+2\hbar\gamma-\hbar]\Theta(n,s,b)}z^n, \quad n \geq 1. \tag{4.8}$$

Then $f(z)$ is in the class $M_n(\wp, \hbar, \gamma, s, b)$ if and only if, it can be expressed in the form

$$f(z) = \sum_{n=0}^{\infty} s_n f_n(z) \quad (4.9)$$

where $s_n \geq 0$ *and* $\sum_{n=0}^{\infty} s_n = 1$.

Proof. Assume that

$$f(z) = \sum_{n=0}^{\infty} s_n f_n(z) = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2\hbar\gamma(1-\wp)}{n[1+2\hbar\gamma-\hbar]\Theta(n,s,b)} s_n z^n. \quad (4.10)$$

Then it follows that

$$\sum_{n=1}^{\infty} \frac{2\hbar\gamma(1-\wp)}{n[1+2\hbar\gamma-\hbar]\Theta(n,s,b)} s_n . \frac{n[1+2\hbar\gamma-\hbar]\Theta(n,s,b)}{2\hbar\gamma(1-\wp)}$$

$= \sum_{n=1}^{\infty} s_n = 1 - s_0 \leq 1$,

which implies that $f(z) \in M_n(\wp, \hbar, \gamma, s, b)$.

Conversely, assume that the function $f(z)$ defined by (1.1) be in the class $M_n(\wp, \hbar, \gamma, s, b)$.

Then

$$a_n \leq \frac{2\hbar\gamma(1-\wp)}{n[1+2\hbar\gamma-\hbar]\Theta(n,s,b)}.$$

Setting

$$s_n = \frac{n[1+2\hbar\gamma-\hbar]\Theta(n,s,b)}{2\hbar\gamma(1-\wp)}, \quad n \geq 1$$

and

$$s_0 = 1 - \sum_{n=1}^{\infty} s_n,$$

we can see that $f(z)$ can be expressed in the form (4.9).

This completes the proof of the theorem.

5. Integral Operators

Theorem 5.1. *Let the function* $f(z) \in M_n(\wp, \hbar, \gamma, s, b)$. *Then the integral operator*

$$F_c(z) = c\int_0^1 u^c f(u,z)dz, \quad (0 < u \leq 1; c > 0) \quad (5.1)$$

is in the class $M_0(\xi)$, *where*

$$\xi = 1 - \frac{2\hbar\gamma c(1-\wp)}{1+2\hbar\gamma-\hbar)(c+2)}. \quad (5.2)$$

The result is sharp for the function $f(z)$ given by *(*3.2*)*

Proof. Let $f(z) \in M_0(\xi)$, then

$$F_c(z) = c\int_0^1 u^c f(u,z)dz = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{c}{n+c+1} a_n z^n \quad (5.3)$$

It is sufficient to show that

$$\sum_{n=1}^{\infty} \frac{nc}{(n+c+1)(1-\xi)} a_n \leq 1 \quad (5.4)$$

Since $f(z) \in M_n(\wp, \hbar, \gamma, s, b)$, then

$$\sum_{n=1}^{\infty} \frac{n(1+2\hbar\gamma-\hbar)\Theta(n,s,b)}{2\hbar\gamma(1-\wp)} a_n \leq 1 \quad (5.5)$$

From (5.3) and (5.5), we have

$$\frac{nc}{(n+c+1)(1-\xi)} \leq \frac{n(1+2\hbar\gamma-\hbar)\Theta(n,s,b)}{2\hbar\gamma(1-\wp)},$$

Then

$$\xi \leq 1-\frac{2\hbar\gamma c(1-\wp)}{n(1+2\hbar\gamma-\hbar)(n+c+1)}.$$

Since

$$H(n)=1-\frac{2\hbar\gamma c(1-\wp)}{n(1+2\hbar\gamma-\hbar)(n+c+1)}$$

is an increasing function of n $(n \geq 1)$, we obtain

$$\xi \leq H(1)=1-\frac{2\hbar\gamma c(1-\wp)}{n(1+2\hbar\gamma-\hbar)(c+2)}$$

and hence the proof of theorem 5.1 is completed.

6. Radius of Convexity

Theorem 6.1. *Let the function $f(z) \in M_n(\wp,\hbar,\gamma,s,b)$. Then $f(z)$ is meromorphically convex of order δ $(0 \leq \delta < 1)$ in $0<|z|<r$, where*

$$r \leq \left\{\frac{(1+2\hbar\gamma-\hbar)(1-\delta)\Theta(n,s,b)}{2\hbar\gamma(n+2-\delta)(1-\wp)}\right\}^{\frac{1}{n}+1} \qquad (6.1)$$

The result is sharp.

Proof. We must show that

$$\left|2+\frac{zf''(z)}{f'(z)}\right| \leq 1-\delta \quad \text{for } 0<|z|<r, \qquad (6.2)$$

where r is given by (6.1). Indeed, we find from (6.2) that

$$\left|2+\frac{zf''(z)}{f'(z)}\right| \leq \sum_{n=1}^{\infty}\frac{n(n+1)a_n|z|^{n+1}}{1-\sum_{n=1}^{\infty} n\,a_n|z|^{n+1}}$$

This will be bounded by $1-\delta$, if

$$\sum_{n=1}^{\infty}\frac{n(n+2-\delta)}{1-\delta}a_n r^{n+1} \leq 1. \qquad (6.3)$$

But by using Theorem 2.1, (6.3) will be true, if

$$\frac{n(n+2-\delta)}{1-\delta}r^{n+1} \leq \frac{n[1+2\hbar\gamma-\hbar]\Theta(n,s,b)}{2\hbar\gamma(1-\wp)}$$

. Then

$$r \leq \left\{\frac{(1+2\hbar\gamma-\hbar)(1-\delta)\Theta(n,s,b)}{2\hbar\gamma(n+2-\delta)(1-\wp)}\right\}^{1/n+1}$$

This completes the proof of theorem.

7. Modified Hadamard Product

For $f_j(z)$ $(j=1,2)$ defined by (4.1), the modified Hadamard product of $f_1(z)$ and $f_2(z)$ defined by

$$(f_1 * f_2)(z)=\frac{1}{z}+\sum_{n=1}^{\infty} a_{n,1}\,a_{n,2}z^n=(f_2 * f_1)(z) \qquad (7.1)$$

Theorem 12. *Let $f_j(z) \in M_n(\wp,\hbar,\gamma,s,b)$ $(j=1,2)$. Then $(f_1 * f_2)(z) \in M_n(\phi,\hbar,\gamma,s,b)$, where*

$$\phi = 1-\frac{2\hbar\gamma(1-\wp)^2}{(1+2\hbar\gamma-\hbar)\Theta(1,s,b)} \qquad (7.2)$$

The result is sharp for the functions $f_j(z),(j=1,2)$ given by $f_j(z)=\frac{1}{z}+\frac{2\hbar\gamma(1-\wp)}{(1+2\hbar\gamma-\hbar)(1-s)^2(b+1)(b+2)}$. (7.3)

Proof. Using the technique for Schild and Silverman [12] , we need to find the largest ϕ such that

$$\sum_{n=1}^{\infty} \frac{n[1+2\hbar\gamma-\hbar]\Theta(n,s,b)}{2\hbar\gamma(1-\phi)} a_{n,1}a_{n,2} \leq 1 \qquad (7.4)$$

Since $f_j(z) \in M_n(\wp,\hbar,\gamma,s,b)$, $(j=1,2)$, we readily see that

$$\sum_{n=1}^{\infty} \frac{n[1+2\hbar\gamma-\hbar]\Theta(n,s,b)}{2\hbar\gamma(1-\wp)} a_{n,1} \leq 1 \qquad (7.5)$$

and

$$\sum_{n=1}^{\infty} \frac{n[1+2\hbar\gamma-\hbar]\Theta(n,s,b)}{2\hbar\gamma(1-\wp)} a_{n,2} \leq 1. \qquad (7.6)$$

By the Cauchy Schwarz inequality, we have

$$\sum_{n=1}^{\infty} \frac{n[1+2\hbar\gamma-\hbar]}{2\hbar\gamma(1-\wp)} \sqrt{a_{n,1}a_{n,2}} \leq 1. \qquad (7.7)$$

Thus it is sufficient to show that

$$\frac{n[1+2\hbar\gamma-\hbar]\Theta(n,s,b)}{2\hbar\gamma(1-\phi)} a_{n,1}a_{n,2} \leq \frac{n[1+2\hbar\gamma-\hbar]\Theta(n,s,b)}{2\hbar\gamma(1-\wp)} \sqrt{a_{n,1}a_{n,2}} \qquad (7.8)$$

or equivalently that

$$\sqrt{a_{n,1}a_{n,2}} \leq \frac{1-\phi)}{(1-\wp)}$$

Connecting with (7.7), it is sufficient to prove that

$$\frac{2\hbar\gamma(1-\wp)}{n[1+2\hbar\gamma-\hbar]\Theta(n,s,b)} \leq \frac{(1-\phi)}{(1-\wp)}. \qquad (7.9)$$

It follows from (7.9) that

$$\phi \leq 1 - \frac{2\hbar\gamma(1-\wp)^2}{n[1+2\hbar\gamma-\hbar]\Theta(n,s,b)}.$$

Now defining the function $G(n)$ by

$$G(n) = 1 - \frac{2\hbar\gamma(1-\wp)^2}{n[1+2\hbar\gamma-\hbar]\Theta(n,s,b)}.$$

We see that G(n) is an increasing function of $n (n \geq 1)$.
Therefore, we conclude that

$$\phi \leq G(1) = 1 - \frac{2\hbar\gamma(1-\wp)^2}{[1+2\hbar\gamma-\hbar]\Theta(1,s,b)}.$$

which evidently completes the proof of the theorem. □

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