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2−ACYCLIC SIMPLE GRAPHOIDAL COVERS ON COMPLETE BIPARTITE GRAPHS

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Abstract
The economy of every nation is centred on its transportation networks, which are also transforming the global economy. To reduce trip times and fuel costs, use graph decomposition techniques to optimize transportation networks. A 2− acyclic simple graphoidal cover of G is a set ψ_G of paths in G such that every edge is in exactly one path in ψ_G and every vertex is an internal vertex of at most two paths in ψ_G and any two paths in ψ_G has at most one vertex in common. The minimum cardinality of the 2−acyclic simple graphoidal cover of G is called the 2−acyclic simple graphoidal covering number of G and is denoted by η_{2as}. In this study, we discuss decomposition of complete bipartite graphs using 2-acyclic simple graphoidal covers.

Keywords: simple graphoidal graphs, 2-acyclic simple graphoidal graphs, complete bipartite graphs.

1. Introduction

A graph's decomposition is a collection of edge-disjoint subgraphs $G_i, i = 1, 2..., n$ of the same graph G, where each edge of the original graph G is contained in exactly one G_i. Several writers to discover several types of graph decomposition, apply different conditions and parameters. Acharya and Sampath Kumar [1, 2] developed the concept of graphoidal cover(g.c). Arumugam and Shahul Hamid developed the concept of a simple graphoidal cover (simple g.c) in their paper [4]. Nagarajan et.al [7, 8] proposed the idea of a 2− graphoidal path cover. Motivation of 2− graphoidal path cover, Venkat narayanan et.al [9] explored 2−acyclic simple graphoidal cover (2−acyclic sgc) on bicyclic graphs. In this paper the authors discuss decompositions of complete bipartite graphs into paths and cycles. Complete bipartite graphs find applications in transportation planning and logistics. For instance, they can represent the relationship between sources and destinations, such as airports and cities, warehouses and stores, or suppliers and customers. By studying the graph structure, transportation planners can optimize routes, allocate resources, and design efficient supply chains.

2. Preliminaries

A finite, simple, non-trivial, connected, and undirected graph is referred to as $G = (V, E)$. The symbols p and q, which stand for the number of elements in V, or order, and the number

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of elements in *E*, or size of G, respectively. For graph theoretic terminology we refer to Harary [6]

Definition 2.1.*[1] A graphoidal cover(g.c) of G is a set ψ_G of (not necessarily open) paths in G satisfying the following conditions.*

(i) *Every path in ψ_G has at least two vertices.*

(ii) *Every vertex of G is an internal vertex of at most one path in ψ_G.*

(iii) *Every edge of G is in exactly one path in ψ_G.*

The minimum cardinality of a graphoidal cover of G is called the graphoidal covering number of G and is denoted by η.

Definition 2.2. *[4] A simple graphoidal cover (simple g.c) of a graph G is a graphoidal cover ψ_G of G such that any two paths in ψ_G have at most one vertex in common. The minimum cardinality of a simple graphoidal cover of G is called simple graphoidal covering number of G and is denoted by η_s.*

Definition 2.3. *[10] A 2-simple graphoidal covering (2-simple g.c) of a graph G is a set ψ_G of paths in G such that every edge is in exactly one path in ψ_G, every vertex is an internal vertex of at most two paths and any two paths in ψ_G have at most one vertex in common. The minimum cardinality of 2-simple graphoidal cover ψ_G of G is known as 2-simple graphoidal covering number of G and is denoted by η_{2s}*

Definition 2.4. *A 2−simple graphoidal cover (2-acyclic sgc) of G is said to be 2−acyclic simple graphoidal cover ψ of G such that every member ψ_G of G is a path. The minimum cardinality of a 2−acyclic simple graphoidal cover of G is called the 2−acylic simple graphoidal covering number of G and is denoted by $\eta_{2as}(G)$.*

Theorem 2.1. *For any (p, q) graph, $\eta_{2as}(G) = q - p - t_2 + t$. where t_2 denotes the number of vertices appears as internal vertex exactly in two paths of ψ_G and t denotes the number of vertices are not internal in ψ_G respectively.*

Corollary 2.1. *There exists a 2−acyclic simple graphoidal cover ψ_G of G in which every vertex is internal vertex in exactly 2 paths in ψ_G of G if and only if $\eta_{2as}(G) = q - 2p$.*

Theorem 2.2. *Let G be a tree with n pendant vertices, then $\eta_{2as}(G) = n - 1 - m$, where*

m is the total number of vertices of degree ≥ 4 in G.

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3. Main Results

Theorem 3.1. *For the complete bipartite graph* $K_{u,v}, u \geq 1, v \geq 1$

$$(i) \quad \eta_{2as}(K_{1,v}) = \begin{cases} 1 & if\ v = 1\ (or)\ v = 2 \\ 2 & if\ v = 3 \\ v-2 & if\ v \geq 4 \end{cases}$$

$$(ii) \quad \eta_{2as}(K_{2,v}) = \begin{cases} 3 & if\ v = 2 \\ 4 & if\ v = 3 (or) v = 4 \\ 2v-5 & if\ v \geq 5 \end{cases}$$

$$(ii) \quad \eta_{2as}(K_{3,v}) = \begin{cases} 5 & if\ v = 3 \\ 6 & if\ v = 4 \\ 8 & if\ v = 5 \\ 9 & if\ v = 6 \\ 3s-9 & if\ v \geq 7 \end{cases}$$

$$(iv) \quad \eta_{2as}(K_{4,v}) = \begin{cases} 9 & if\ v = 4 \\ 11 & if\ v = 5 \\ 13 & if\ v = 6 \\ 15 & if\ v = 7 \\ 4v-14 & if\ v \geq 8 \end{cases}$$

$$(v) \quad \eta_{2as}(K_{5,v}) = \begin{cases} 13 & if\ v = 5 \\ 16 & if\ v = 6 \\ 19 & if\ v = 7 \\ 21 & if\ v = 8 \\ 5v-20 & if\ v \geq 9 \end{cases}$$

$$(vi) \quad \eta_{2as}(K_{6,v}) = \begin{cases} 4v-6 & if\ 6 \leq v \leq 10 \\ 6v-27 & if\ v \geq 11 \end{cases}$$

$$(vii) \quad \eta_{2as}(K_{7,v}) = \begin{cases} 4v-9 & if\ 7 \leq v \leq 13 \\ 7v-35 & if\ v \geq 14 \end{cases}$$

Proof. It is observed that for any 2−acyclic simple graphoidal cover of $K_{u,v}$, any member of ψ_G is a path of length ≤ 2.

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(i) Now let $X=\{r_1\}$ and $Y=\{z_1,z_2,z_3,...,z_v\}$ be the bipartition of $K_{1,v}$ with $p=1+v$, $q=v$.

Case 1. Since $K_{1,1}$ and $K_{1,2}$ are paths. Therefore $\eta_{2as}(G)=1$.

Case 2. When v = 3

Then $K_{1,3}$ is a tree with 3 pendant vertices and no vertex is of degree ≥ 4. Hence by theorem 2.2 , $\eta_{2as}(K_{1,3})=3-1-0=2$.

Case 3. When $v \geq 4$

Then $K_{1,v}$ is a tree with v pendant vertices and one vertex is of degree ≥ 4. Hence by theorem 2.2, $\eta_{2as}(K_{1,v})=v-1-1=v-2$.

(ii) Now let $X=\{r_1,r_2\}$ and $Y=\{z_1,z_2,z_3,...,z_v\}$ be the bipartition of $K_{2,v}$ with $p=2+v, q=2v$.

Case 1. When v = 2

Then $\psi_G=\{(z_1,r_1,z_2),(r_2,z_1),(r_2,z_2)\}$ is a 2−acyclic sgc of $K_{2,2}$ so that $\eta_{2as}(K_{2,2})\leq 3$. For any, 2−acyclic sgc of $K_{2,2}, t_2=0$ and $t_\psi=3$. Since no vertices is of degree ≥ 4. Hence $t_2=0, t\geq 3$ so that $\eta_{2as}(K_{2,2})=q-p-t_2+t\geq 4-4-0+3=3$. Thus $\eta_{2as}(K_{2,2})=3$.

Case 2. When v = 3

Then $\psi_G=\{(z_1,r_1,z_2),(z_2,r_2,z_3),(r_1,z_3),(r_2,z_1)\}$ is a 2−acyclic sgc of $K_{2,3}$ so that $\eta_{2as}(K_{2,3})\leq 4$. For any 2−acyclic sgc of $K_{2,3}, t_2=0$ and $t_\psi=3$. Since no vertices is of degree $\geq$ 4. Hence $t_2=0, t\geq 3$ so that $\eta_{2as}(K_{2,3})=q-p-t_2+t\geq 6-5-0+3=4$. Thus $\eta_{2as}(K_{2,3})=4$.

Case 3. When v = 4

Then $\psi_G=\{(z_1,r_1,z_2),(z_3,r_1,z_4),(z_2,r_2,z_4),(z_1,r_2,z_3)\}$ is a 2−acyclic sgc of $K_{2,4}$ so that $\eta_{2as}(K_{2,4})\leq 4$. For any 2−acyclic sgc of $K_{2,4}, t_\psi=4$ and $t_2(\psi)=2$. Hence $t_2=2, t\geq 4$. so that $\eta_{2as}(K_{2,4})=q-p-t_2+t\geq 8-6-2+4=4$. Thus $\eta_{2as}(K_{2,4})=4$.

Case 4. When $v \geq 5$

Then $\psi_G=\{(z_1,r_1,z_2),(z_3,r_1,z_4),(z_2,r_2,z_3),(z_1,r_2,z_4),(r_1,z_5,r_2)\}$ along with the remaining edges form a 2−acyclic sgc of $K_{2,v}$ so that $|\psi_G|=5+(2v-10)=2v-5$. Hence $\eta_{2as}(K_{2,v})\leq 2v-5$. For any 2−acyclic sgc ψ_G of $K_{2,v}, t_\psi\geq(v-1)$ and $t_2(\psi)=2$. Hence $t_2=2, t\geq(v-1)$ so that $\eta_{2as}(K_{2,v})=q-p-t_2+t\geq 2v-(2+v)-2+(v-1)=2v-5$. Thus $\eta_{2as}(K_{2,v})=2v-5$.

(iii) Now let $X = \{r_1, r_2, r_3\}$ and $Y = \{z_1, z_2, ..., z_v\}$ be the bipartition of $K_{3,v}$ with $p = 3 + v,\ q = 3v.$

Case 1. When $v = 3$

Then $\psi_G = \{(z_1, r_1, z_2), (z_1, r_2, z_3), (r_1, z_3, r_3), (r_2, z_2, r_3), (r_3, z_1)\}$ is a 2−acyclic sgc of $K_{3,3}$ so that $\eta_{2as}(K_{3,3}) \le 5.$ For any 2−acyclic sgc ψ_G of $K_{3,3}$, $t_2 = 0$ and $t_\psi = 2$. Hence $t_2 = 0$, $t \ge 2$ so that $\eta_{2as}(K_{3,3}) = q - p - t_2 + t \ge 9 - 6 - 0 + 2 = 5.$ Thus $\eta_{2as}(K_{3,3}) = 5.$

Case 2. When $v = 4$

Then $\psi_G = \{(z_1, r_1, z_2), (z_3, r_1, z_4), (z_1, r_2, z_3), (z_2, r_2, z_4), (z_2, r_3, z_3), (z_1, r_3, z_4)\}$ is a 2−acy- clic sgc of $K_{3,4}$ so that $\eta_{2as}(K_{3,4}) \le 6.$ For any 2−acyclic sgc ψ_G of $K_{3,4}, t_2 = 3$ and $t_\psi = 4.$ Hence $t_2 = 3, t \ge 4$ so that $\eta_{2s}(K_{3,4}) = q - p - t_2 + t \ge 12 - 7 - 3 + 4 = 6.$ Thus $\eta_{2as}(K_{3,4}) = 6.$

Case 3. When $v = 5$

Then $\psi_G = \{(z_1, r_1, z_2), (z_3, r_1, z_4), (z_1, r_2, z_3), (z_2, r_2, z_4), (z_2, r_3, z_3), (z_1, r_3, z_4), (r_1, z_5, r_2), (r_3, z_5)\}$ is a 2−acyclic sgc of $K_{3,5}$ so that $\eta_{2as}(K_{3,5}) \le 8.$ For any 2−acyclic sgc ψ_G of $K_{3,5}$, $t_2 = 3$ and $t_\psi = 4.$ Hence $t_2 = 3, t \ge 4$ so that $\eta_{2as}(K_{3,5}) \ge 15 - 8 - 3 + 4 = 8.$ Thus $\eta_{2as}(K_{3,5}) = 8.$

Case 4. When $v = 6$

Then $\psi_G = \{(z_1, r_1, z_2), (z_3, r_1, z_4), (z_1, r_2, z_3), (z_2, r_2, z_4), (z_2, r_3, z_3), (z_1, r_3, z_4), (r_1, z_5, r_2), (r_2, z_6, r_3), (r_3, z_5), (r_1, z_6)\}$ is a 2−acyclic sgc of $K_{3,6}$ so that $\eta_{2as}(K_{3,6}) \le 10.$ For any 2−acyclic sgc ψ_G of $K_{3,6}, t_2 = 3$ and $t_\psi = 4.$ Hence $t_2 = 3, t \ge 4$ so that $\eta_{2as}(K_{3,5}) \ge 18 - 9 - 3 + 4 = 10.$ Thus $\eta_{2as}(K_{3,6}) = 10.$

Case 5. When $v \ge 7$

Then $\psi_G = \{(z_1, r_1, z_2), (z_3, r_1, z_4), (z_1, r_2, z_3), (z_2, r_2, z_4), (z_2, r_3, z_3), (z_1, r_3, z_4), (r_1, z_5, r_2), (r_2, z_6, r_3), (r_1, z_7, r_3)\}$ along with the remaining edges form a 2−acyclic sgc of $K_{3,v}$ so that $|\psi_G| = 9 + (3v - 18) = 3v - 9.$ Hence $\eta_{2as}(K_{3,v}) \le 3v - 9.$ For any 2−acyclic sgc ψ_G of $K_{3,v}$, $t_2 = 3$ and $t_\psi \ge v - 3.$ Hence $t_2 = 3,\ t \ge v - 3$ so that $\eta_{2as}(K_{3,v}) \ge 3v - (3 + v) - 3 + (v - 3)$ $= 3v - 9.$ Thus $\eta_{2as}(K_{3,v}) = 3v - 9.$

(iv) Now let $X = \{r_1, r_2, r_3, r_4\}$ and $Y = \{z_1, z_2, ..., z_v\}$ be the bipartition of $K_{4,v}$ with

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$p = 4 + v$, $q = 4v$.

Case 1. When $v = 4$

Then $\psi_G = \{(z_1, r_1, z_2), (z_2, r_2, z_3), (z_2, r_3, z_4), (z_3, r_4, z_4), (r_1, z_4, r_2), (r_1, z_3, r_3), (r_2, z_1, r_4), (r_3, z_1), (r_4, z_2)\}$ is a 2−acyclic sgc of $K_{4,4}$ so that $\eta_{2as}(K_{4,4}) \le 9.$ For any 2−acyclic sgc of $K_{4,4}$, if $|\psi_G| < 8$ is not possible, since any member of ψ is a path of length ≤ 2. If $|\psi_G| = 8,$ then ψ_G contains exactly eight paths, which is a contradiction, since any two paths in ψ_G contains more than one vertex common. Therefore $|\psi_G| \ge 9.$ Hence $\eta_{2as}(K_{4,4}) \ge 9.$ Thus $\eta_{2as}(K_{4,4}) = 9.$

Case 2. When $v = 5$

Then $\psi_G = \{(z_1, r_1, z_2), (z_3, r_1, z_5), (z_1, r_2, z_3), (z_2, r_2, z_4), (z_1, r_3, z_5), (z_2, r_3, z_3), (z_1, r_4, z_4), (z_2, r_4, z_5), (r_1, z_4, r_3), (r_2, z_5), (r_4, z_3)\}$ is a 2−acyclic sgc of $K_{4,5}$ so that $\eta_{2as}(K_{4,5}) \le 11.$ For any 2−acyclic sgc ψ_G of $K_{4,5}, t_2 = 4$ *and* $t_\psi = 4$ .Hence $t_2 = 4, t \ge 4$ so that $\eta_{2as}(K_{3,5}) \ge 20 - 9 - 4 + 4 = 11.$ Thus $\eta_{2as}(K_{4,5}) = 11.$

Case 3. When $v = 6$

Then $\psi_G = \{(z_1, r_1, z_2), (z_4, r_1, z_6), (z_1, r_2, z_5), (z_2, r_2, z_4), (z_1, r_3, z_6), (z_3, r_3, z_4), (z_1, r_4, z_3), (z_5, r_4, z_6), (r_1, z_3, r_2), (r_1, z_5, r_3), (r_3, z_2, r_4), (r_4, z_4), (r_2, z_6)\}$ is a 2−acyclic sgc of $K_{4,6}$ so that $\eta_{2as}(K_{4,6}) \le 13.$ For any 2−acyclic sgc ψ_G of $K_{4,6}$, $t_2 =$ 4 and $t_\psi =$ 3. Hence $t_2 = 4, t \ge 3$ so that $\eta_{2s}(K_{4,6}) \ge 24 - 10 - 4 + 3 = 13.$ Thus $\eta_{2as}(K_{4,6}) = 13..$

Case 4. When $v = 7$

Then $\psi_G = \{(z_1, r_1, z_2), (z_4, r_1, z_6), (z_1, r_2, z_5), (z_2, r_2, z_4), (z_1, r_3, z_6), (z_3, r_3, z_4), (z_1, r_4, z_3), (z_5, r_4, z_6), (r_1, z_3, r_2), (r_1, z_5, r_3), (r_3, z_2, r_4), (r_1, z_7, r_4), (r_2, z_7, r_3), (r_4, z_4), (r_2, z_6)\}$ is a 2−acyclic sgc of $K_{4,7}$ so that $\eta_{2as}(K_{4,7}) \le 15.$ For any 2−acyclic sgc ψ_G of $K_{4,7}$, $t_2 = 5$ and $t_\psi \ge 3.$ Hence $t_2 = 5,$ $t \ge 3$ so that $\eta_{2s}(K_{4,7}) \ge 24 - 10 - 5 + 3 = 15.$ Thus $\eta_{2as}(K_{4,7}) = 15.$

Case 5. When $v \ge 8$

Then $\psi_G = \{(z_1, r_1, z_2), (z_4, r_1, z_6), (z_1, r_2, z_5), (z_2, r_2, z_4), (z_1, r_3, z_6), (z_3, r_3, z_4), (z_1, r_4, z_3), (z_5, r_4, z_6), (r_1, z_3, r_2), (r_1, z_5, r_3), (r_3, z_2, r_4), (r_1, z_7, r_4), (r_2, z_7, r_3), (r_2, z_8, r_4)\}$ along with the remaining edges form a 2−acyclic sgc of $K_{4,v}$ so that $|\psi_G| = 14 + 4v - 28 = 4v - 14.$ Hence $\eta_{2as}(K_{4,v}) \le 4v - 14.$ For any 2−acyclic sgc ψ_G of $K_{4,v}$ $, t_2 = 5$ *and* $t_\psi = v - 5.$ Hence $t_2 \le 5, t \ge v - 5$ so that $\eta_{2as}(K_{4,v}) \ge 4v - (4 + v) - 5 + (v - 5) = 4v - 14.$ Thus $\eta_{2as}(K_{4,v}) = 4v - 14.$

(v) Now let $X = \{r_1, r_2, r_3, r_4, r_5\}$ and $Y = \{z_1, z_2, z_3, ..., z_v\}$ be the bipartition of $K_{5,v}$ with $p = 5 + v, q = 5v.$

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Case 1. When $v = 5$

Then $\psi_G = \{(z_2,r_1,z_3),(z_3,r_2,z_4),(z_1,r_3,z_3),(z_1,r_4,z_4),(z_1,r_5,z_5),(z_2,r_3,z_4),(r_1,z_1,r_2),(r_2,z_2,r_4),(r_4,z_3,r_5),(r_1,z_4,r_5),(r_2,z_5,r_3),(r_1,z_5,r_4),(r_5,z_2)\}$ is a 2−acyclic sgc of $K_{5,5}$ so that $\eta_{2as}(K_{5,5}) \le 13$. For any 2−acyclic sgc ψ_G of $K_{5,5}, t_2 = 2$ *and* $t_\psi = 0$. Hence $t_2 \le 2,\ t \ge 0$ so that $\eta_{2as}(K_{5,5}) \ge 25-10-2+0 = 13$. Thus $\eta_{2as}(K_{5,5}) = 13$.

Case 2. When $v = 6$

Then $\psi_G = \{(z_1,r_1,z_2),(z_3,r_1,z_4),(z_1,r_2,z_3),(z_2,r_2,z_5),(z_2,r_3,z_3),(z_1,r_3,z_4),(z_2,r_4,z_4),(z_3,r_4,z_5),(z_2,r_5,z_6),(z_4,r_5,z_5),(r_1,z_6,r_4),(r_2,z_6,r_3),(r_4,z_1,r_5),(r_1,z_5,r_3),(r_2,z_4),(r_5,z_3)\}$ is a 2−acyclic sgc of $K_{5,6}$ so that $\eta_{2as}(K_{5,6}) \le 16$. For any 2−acyclic sgc ψ_G of $K_{5,6}$, $t_2 =$ 6 and $t_\psi = 3$. Hence $t_2 = 6, t \ge 3$ so that $\eta_{2as}(K_{5,6}) \ge 30-11-6+3 = 16$. Thus $\eta_{2as}(K_{5,6}) = 16$

Case 3. When $v = 7$

Then $\psi_G = \{(z_1,r_1,z_2),(z_3,r_1,z_4),(z_1,r_2,z_3),(z_2,r_2,z_5),(z_2,r_3,z_3),(z_1,r_3,z_4),(z_3,r_4,z_5),(z_2,r_4,z_4),(z_4,r_5,z_5),(z_2,r_5,z_6),(r_1,z_6,r_4),(r_2,z_6,r_3),(r_4,z_1,r_5),(r_1,z_5,r_3),(r_1,z_7,r_2),(r_3,z_7,r_5),(r_2,z_4),(r_5,z_3),(r_4,z_7)\}$ is a 2−acyclic sgc of $K_{5,7}$ so that $\eta_{2as}(K_{5,7}) \le 19$. For any 2−acyclic sgc ψ_G of $K_{5,7}$, $t_2 = 7$ *and* $t_\psi = 3$. Hence $t_2 = 7, t \ge 3$ so that $\eta_{2as}(K_{5,7}) \ge 35-12-7+3 = 19$. Thus $\eta_{2as}(K_{5,7}) = 19$.

Case 4. When $v = 8$

Then $\psi_G = \{(z_2,r_1,z_3),(z_3,r_2,z_4),(z_1,r_3,z_3),(z_1,r_4,z_4),(z_1,r_5,z_5),(z_7,r_1,z_8),(z_6,r_2,z_8),(z_2,r_3,z_4),(z_6,r_4,z_7),(z_2,r_5,z_6),(r_1,z_1,r_2),(r_2,z_2,r_4),(r_4,z_3,r_5),(r_1,z_4,r_5),(r_1,z_5,r_4),(r_2,z_5,r_3),(r_1,z_6,r_3),(r_2,z_7,r_5),(r_3,z_8,r_5),(r_3,z_7),(r_4,z_8)\}$ is a 2−acyclic sgc of $K_{5,8}$ so that $\eta_{2as}(K_{5,8}) \le 21$. For any 2−acyclic sgc ψ_G of $K_{5,8}$, $t_2 = 6$ *and* $t_\psi = 0$. Hence $t_2 = 6, t \ge 0$ so that $\eta_{2as}(K_{5,8}) \ge 42-13-6+0 = 21$. Thus $\eta_{2as}(K_{5,8}) = 21$.

Case 4. When $v \ge 9$

Then $\psi_G = \{(z_2,r_1,z_3),(z_3,r_2,z_4),(z_1,r_3,z_3),(z_1,r_4,z_4),(z_1,r_5,z_5),(z_7,r_1,z_8),(z_6,r_2,z_8),(z_2,r_3,z_4),(z_6,r_4,z_7),(z_2,r_5,z_6),(r_1,z_1,r_2),(r_2,z_2,r_4),(r_4,z_3,r_5),(r_1,z_4,r_5),(r_1,z_5,r_4),(r_2,z_5,r_3),(r_1,z_6,r_3),(r_2,z_7,r_5),(r_3,z_8,r_5),(r_3,z_9,r_4)\}$ along with the remaining edges form a 2−acyclic sgc of $K_{5,v}$ so that $|\psi_G| = 20+5v-40 = 5v-20$. Hence $\eta_{2as}(K_{5,v}) \le 5v-20$. For any 2−acyclic sgc ψ_G of $K_{5,v}$, $t_2 = 6$ and $t_\psi \le v-9$. Hence $t_2 = 6, t \ge v-9$ so that $\eta_{2as}(K_{5,8}) \ge 5v-(5+v)-6+(v-9) = 4v-20$. Thus $\eta_{2as}(K_{5,v}) = 5v-20$.

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(vi) Now let $X=\{r_1,r_2,r_3,r_4,r_5,r_6\}$ and $Y=\{z_1,z_2,z_3,...,z_v\}$ be the bipartition of $K_{6,v}$ with $p=6+v, q=6v.$ Then there are two cases.

Case 1. When $6\le v\le 10$

Then the collection of paths are $P_1=(z_1,r_1,z_3), P_2=(z_2,r_2,z_4), P_3=(z_3,r_3,z_5), P_4=(z_4,r_4,z_6), P_5=(z_1,r_5,z_5), P_6=(z_2,r_6,z_6), P_7=(z_5,r_1,z_6), P_8=(z_1,r_2,z_6), P_9=(z_1,r_3,z_2), P_{10}=(z_2,r_4,z_3), P_{11}=(z_3,r_5,z_4), P_{12}=(z_4,r_6,z_5), P_{13}=(r_4,z_1,r_6), P_{14}=(r_1,z_2,r_5), P_{15}=(r_2,z_3,r_6), P_{16}=(r_1,z_4,r_3), P_{17}=(r_2,z_5,r_4), P_{18}=(r_3,z_6,r_5), Q_j=(r_k,z_{6+j},r_l)$ *and* $R_j=(r_p,z_{6+j},r_q): k\ne l\ne p\ne q$

$, j:1,2,3,4$ *and* $6+j\le v$. Then $\psi=\{P_j: j=1,2,..,18\}\cup\{Q_j: j=1,2,3,4\}\cup\{R_j: j=1,2,3,4\}$ along with the remaining edges form a minimum 2−acyclic sgc of $K_{6,v}$ so that $|\psi|=(2v+6)+(6v-2(2v+6))=4v-6.$ Hence $\eta_{2as}(K_{6,v})\le 4v-6.$ For any 2−acyclic sgc ψ_G of $K_{6,v}, t_2=v$ and $t_\psi=0.$ Hence $t_2=v,\ t\ge 0$ so that $\eta_{2as}(K_{6,v})\ge 6v-(6+v)-v=4v-6.$ Thus $\eta_{2as}(K_{6,v})=4v-6.$

Case 2. When $v\ge 11$

Then the collection of paths are $P_1=(z_1,r_1,z_3), P_2=(z_2,r_2,z_4), P_3=(z_3,r_3,z_5), P_4=(z_4,r_4,z_6), P_5=(z_1,r_5,z_5), P_6=(z_2,r_6,z_6), P_7=(z_5,r_1,z_6), P_8=(z_1,r_2,z_6), P_9=(z_1,r_3,z_2), P_{10}=(z_2,r_4,z_3), P_{11}=(z_3,r_5,z_4), P_{12}=(z_4,r_6,z_5), P_{13}=(r_4,z_1,r_6), P_{14}=(r_1,z_2,r_5), P_{15}=(r_2,z_3,r_6), P_{16}=(r_1,z_4,r_3), P_{17}=(r_2,z_5,r_4), P_{18}=(r_3,z_6,r_5), P_{19}=(r_1,z_7,r_2), P_{20}=(r_3,z_7,r_4), P_{21}=(r_2,z_8,r_3), P_{22}=(r_5,z_8,r_6), P_{23}=(r_1,z_9,r_4), P_{24}=(r_2,z_9,r_5), P_{25}=(r_1,z_{10},r_6), P_{26}=(r_4,z_{10},r_5),$ *and* $P_{27}=(r_3,z_{11},r_6).$ Then $\psi=\{P_j: j=1,2,...,27\}$ together with remaining edges form a 2−acyclic sgc of $K_{6,v}$ so that $|\psi|=27+6v-54=6v-27.$ Hence $\eta_{2as}(K_{6,v})\le 6v-27.$ For any 2−acyclic sgc ψ_G of $K_{6,v}$, $t_2=10$ *and* $t_\psi=v-11.$ Hence $t_2=10, t\ge v-11$ so that $\eta_{2as}(K_{6,v})\ge 6v-(6+v)-10+(v-11)=6v-27.$ Thus $\eta_{2as}(K_{6,v})=6v-27.$

(vii) Now let $X=\{r_1,r_2,r_3,r_4,r_5,r_6,r_7\}$ and $Y=\{z_1,z_2,z_3,...,z_v\}$ be the bipartition of $K_{7,v}$ with $p=7+v, q=7v.$ Then there are two cases.

Case 1. When $7\le v\le 13$

Then the collection of paths are $P_1=(z_1,r_1,z_3), P_2=(z_2,r_1,z_6), P_3=(z_2,r_2,z_4), P_4=(z_1,r_2,z_5), P_5=(z_3,r_3,z_5), P_6=(z_1,r_3,z_4), P_7=(z_4,r_4,z_6), P_8=(z_2,r_4,z_5), P_9=(z_5,r_5,z_7), P_{10}=(z_1,r_5,z_2), P_{11}=(z_2,r_6,z_7), P_{12}=(z_3,r_6,z_6), P_{13}=(z_1,r_7,z_6), P_{14}=(z_4,r_7,z_7), P_{15}=(r_4,z_1,r_6), P_{16}=$

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$(r_3,z_2,r_7), P_{17}=(r_2,z_3,r_4), P_{18}=(r_5,z_3,r_7), P_{19}=(r_1,z_4,r_5), P_{20}=(r_6,z_5,r_7), P_{21}=(r_2,z_6,r_3), P_{22}$ $=(r_1,z_7,r_2), P_{23}=(r_3,z_7,r_4), Q_j=(r_k,z_{7+j},r_l)$ *and* $R_j=(r_p,z_{7+j},r_q): k\neq l\neq p\neq q, j:1,2,...6$ and $7+j\leq v.$ Then $\psi=\{P_j: j=1,2,...,23\}\cup\{Q_j: j=1,2,...,6\}\cup\{R_j: j=1,2,...,6\}$ together with remaining edges form a 2−acyclic sgc of $K_{7,v}$ so that $|\psi|=2v+9+(7v-2(2v+9))$ $=5v-9.$ Hence $\eta_{2as}(K_{7,v})\leq 5v-9.$ For any 2-acyclic sgc ψ_G of $K_{7,v}$, $t_2=v+2$ and $t_\psi=0.$ Hence $t_2=(v+2), t\geq 0$ so that $\eta_{2as}(K_{7,v})\geq 7v-(7+v)-(v+2)=5v-9.$ Thus $\eta_{2as}(K_{7,v})=$ $5v-9.$

Case 2. When $v\geq 14$

Then the collection of paths are $P_1=(z_1,r_1,z_3), P_2=(z_2,r_1,z_6), P_3=(z_2,r_2,z_4), P_4=(z_1,$ $r_2,z_5), P_5=(z_3,r_3,z_5), P_6=(z_1,r_3,z_4), P_7=(z_4,r_4,z_6), P_8=(z_2,r_4,z_5), P_9=(z_5,r_5,z_7), P_{10}=(z_1,r_5,$ $z_2), P_{11}=(z_2,r_6,z_7), P_{12}=(z_3,r_6,z_6), P_{13}=(z_1,r_7,z_6), P_{14}=(z_4,r_7,z_7), P_{15}=(r_4,z_1,r_6), P_{16}=(r_3,z_2$ $,r_7), P_{17}=(r_2,z_3,r_4), P_{18}=(r_5,z_3,r_7), P_{19}=(r_1,z_4,r_5), P_{20}=(r_6,z_5,r_7), P_{21}=(r_2,z_6,r_3), P_{22}=(r_1,z_7$ $,r_2), P_{23}=(r_3,z_7,r_4), P_{24}=(r_1,z_8,r_3), P_{25}=(r_5,z_8,r_6), P_{26}=(r_1,z_9,r_4), P_{27}=(r_2,z_9,r_5), P_{28}=(r_1,z_{10}$ $,r_6), P_{29}=(r_3,z_{10},r_5), P_{30}=(r_1,z_{11},r_7), P_{31}=(r_3,z_{11},r_6), P_{32}=(r_2,z_{12},r_6), P_{33}=(r_4,z_{12},r_7), P_{34}=$ (r_2,z_{13},r_7) & $P_{35}=(r_4,z_{13},r_5).$ Then $\psi=\{P_j: j=1,2,...,35\}$ together with the remaining edges form a 2−acyclic sgc of $K_{7,v}$ so that $|\psi|=35+7v-70=7v-35.$ Hence $\eta_{2as}(K_{7,v})\leq$ $7v-35.$ For any 2−acyclic sgc ψ_G of $K_{7,v}$, $t_2=15$ *and* $t_\psi=v-13.$ Hence $t_2=15, t\geq v-13,$ so that $\eta_{2as}(K_{7,v})\geq 7v-7+v-15+v-13=7v-35.$ Thus $\eta_{2as}(K_{7,v})=7v-35.$ □

Theorem 3.2. *For a complete bipartite graph* $K_{u,v},(u\geq 8)$ *and u is even, then*

$$\eta_{2as}(K_{u,v})=\begin{cases} uv-2u-2v & \text{if } u\leq v\leq\lfloor u^2-u/4\rfloor \\ \left\lfloor\dfrac{2uv-u^2-3u}{2}\right\rfloor & \text{if } v>\lfloor u^2-u/4\rfloor \end{cases}$$

Proof. Now let $X=\{r_1,r_2,r_3,......,r_u\}$ and $Y=\{z_1,z_2,z_3,...,z_v\}$ be the bipartition of $K_{u,v}$ $(u\geq 8)$, u *even* with $p=8+v, q=8v.$ Then there are two cases.

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Case 1. When $u \le v \le \left\lfloor \frac{u^2-u}{4} \right\rfloor$

Then there are three subcases.

Subcase 1.1. When $u = 8$

Then the collection of paths are $P_1 = (z_1, r_1, z_3), P_2 = (z_2, r_2, z_4), P_3 = (z_3, r_3, z_5), P_4 = (z_4\ r_4, z_6), P_5 = (z_5, r_5, z_7), P_6 = (z_6, r_6, z_8), P_7 = (z_1, r_7, z_7), P_8 = (z_2, r_8, z_8), P_9 = (z_4, r_1, z_8), P_{10} = (z_3, r_2\ z_7), P_{11} = (z_2, r_3, z_6), P_{12} = (z_1, r_4, z_5), P_{13} = (z_1, r_5, z_8), P_{14} = (z_2, r_6, z_5), P_{15} = (z_3, r_7, z_6), P_{16} = (z_4, r_8, z_7), P_{17} = (r_2, z_1, r_6), P_{18} = (r_1, z_2, r_7), P_{19} = (r_4, z_3, r_8), P_{20} = (r_5, z_4, r_7), P_{21} = (r_1, z_5, r_8), P_{22} = (r_2, z_6, r_8),$ $P_{23} = (r_1, z_7, r_3), P_{24} = (r_2, z_8, r_4), P_{25} = (r_3, z_1, r_8), P_{26} = (r_4, z_2, r_5), P_{27} = (r_5, z_3, r_6), P_{28} = (r_3, z_4, r_6),$ $P_{29} = (r_2, z_5, r_7), P_{30} = (r_1, z_6, r_5), P_{31} = (r_4, z_7, r_6), P_{32} = (r_3, z_8, r_7), Q_j = (r_k, z_{8+j}, r_l)$ *and* $R_j = (r_p\ z_{8+j}, r_q) : k \ne l \ne p \ne q, j : 1,2,..,6$ *and* $8 + j \le v.$ Then $\psi = \{P_j : j = 1,..,32\} \cup \{Q_j : j = 1,..,6\} \cup \{R_j : j = 1,2,...,6\}$ along with the remaining edges form a minimum 2−acyclic sgc of $K_{8,v}$ in which every vertices made internal twice. By corollary 2.1, $\eta_{2as}(K_{8,v}) = q - 2p = uv - 2u$ $-2v.$

Subcase 1.2. When $u = 10$

Then the collection of paths are

$M_j = (z_j, r_j, z_{j+2}) : j = 1,2,...,8$

$Nj = (z_{(2-j)}, r_{(9+j)}, z_{(v-j)}) : j = 0,1$

$P_j = (z_{(6-j)}, r_{(j+1)}, z_{(10-j)}) : j : 0,1.,5$

$Q_j = (z_j, r_{(6+j)}, z_{(j+3)}) : j = 1,2,3,4$

$R_j = (r_{(4-j)}, z_{(j+1)}, r_{(8-j)}) : j = 0,1,2$

$S_j = (r_{(5-j)}, z_{(4+j)}, r_{(6+j)}) : j = 0,1,2,3,4$

$T_j = (r_j, z_{(8+j)}, r_{(8+j)}) : j = 1,2$

$U_j = (r_{(5-j)}, z_{(j+1)}, r_{(9+j)}) : j = 0,1$

$V = (r_5, z_3, r_7)$

$W_j = (r_{(8+j)}, z_{(4+j)}, r_{(9+j)}) : j = 0,1$

$X_j = (r_{(2+j)}, z_{(6+j)}, r_{(7-j)}) : j = 0,1$

$Y_j = \left(r_{(4+j)}, z_{(8+j)}, r_{(9-j)}\right) : j = 0,1,2$

$A_j = (r_k, z_{(10+j)}, r_l)$

$B_j = \left(r_p, z_{(10+j)}, r_q\right)$ $: k \neq l \neq p \neq q,\ j : 1,2,...,12$ and $10 + j \leq v.$

Then $\psi = \{M_j : j = 1,2,...,8\} \cup \{N_j : j = 0,1\} \cup \{P_j : j = 0,1,...,5\} \cup \{Q_j : j = 1,2,3,4\} \cup \{R_j : j = 0,1,2\} \cup \{S_j : j = 0,1,...,4\} \cup \{T_j : j = 1,2\} \cup \{U_j : j = 0,1\} \cup \left\{V\right\} \cup \left\{W_j : j = 0,1\right\} \cup \{X_j : j = 0,1\} \cup \left\{Y_j : j = 0,1,2\right\} \cup \{A_j : j = 1,2,...,12\} \cup \{B_j : j = 1,2,...,12\}$ along with the remaining edges form a minimum 2−acyclic sgc ψ_G of $K_{10,v}$ in which every vertices made internal twice. By corollary **2.1,** $\eta_{2as}\left(K_{10,v}\right) = q - 2p = uv - 2u - 2v.$

Subcase 1.3. When $u \geq 12$

Then the collection of paths are

$P_j = (z_j, r_j, z_{(j+2)}) : j = 1,2...(v-2)$

$Q_j = (z_{(j+1)}, r_{(v-1+j)}, z_{(v-j)}) : j = 0,1$

$R_j = (z_{(j+1)}, r_{(v-j)}, z_{(5+j)}) : j = 0,1,...(v-5)$

$S_j = (z_{(4-j)}, r_{(j+1)}, z_{(v-j)}) : j = 0,1,2,3$

$T_j = (r_{(j+1)}, z_{(6-j)}, r_{(v-j)}) : j = 0,1,....5$

$U_j = (r_j, z_{(6+j)}, r_{(10+j)}) : j = 1,2,...(v-10)$

$V_j = (r_{(6-j)}, z_{((v-3)-j)}, r_{(v-j)}) : j = 0,1,2,3$

$W_j = (r_{(j+1)}, z_{((v-4)-j)}, r_{(j+3)}) : j = 0,1,...(v-5)$

$X_j = (r_{(j+1)}, z_{((v-3)+j)}, r_{((v-2)-j)}) : j = 0,1,2,3$

$Y_j = (r_k, z_{(u+j)}, r_l)$

$Z_j = (r_p, z_{(u+j)}, r_q) : k \neq l \neq p \neq q,\ j = 1,2,......,\left\lfloor \left(u^2 - 5u\right)/4 \right\rfloor \,\&\, u + j \leq v$

Then $\psi = \{P_j : j = 1,2....,(v-2)\} \cup \{Q_j : j = 0,1\} \cup \{R_j : j = 0,1,...,(v-5)\} \cup \{S_j : j = 0,1,2,3\} \cup \{T_j : j = 0,1,....5\} \cup \{U_j : j = 1,2,....(v-10)\} \cup \{V_j : j = 0,1,2,3\} \cup \{W_j : j = 0,1,....,(v-5)\} \cup \{X_j : j = 0,1,2,3\} \cup \{Y_j : j = 1,2, ...\left\lfloor \left(u^2 - 5u\right)/4 \right\rfloor\} \cup \{Z_j : j = 1,2,......,\left\lfloor \left(u^2 - 5u\right)/4 \right\rfloor\}$ along with the remaining edges form a minimum 2−acyclic sgc of $K_{u,v}$ in which every vertices made internal twice. By corollary **2.1,** $\eta_{2as}\left(K_{u,v}\right) = q - 2p = uv - 2u - 2v.$

Case 2. When $v > \left\lfloor \dfrac{u^2 - u}{4} \right\rfloor$

Then there are three subcases.

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Subcase 2.1. When $u=8$

Then the collection of paths are $P_1=(z_1,r_1,z_3), P_2=(z_2,r_2,z_4), P_3=(z_3,r_3,z_5), P_4=(z_4, r_4, z_6), P_5=(z_5,r_5,z_7), P_6=(z_6,r_6,z_8), P_7=(z_1,r_7,z_7), P_8=(z_2,r_8,z_8), P_9=(z_4,r_1,z_8), P_{10}=(z_3, ,r_2,z_7), P_{11}=(z_2,r_3,z_6), P_{12}=(z_1,r_4,z_5), P_{13}=(z_1,r_5,z_8), P_{14}=(z_2,r_6,z_5), P_{15}=(z_3,r_7,z_6), P_{16}= (z_4,r_8,z_7), P_{17}=(r_2,z_1,r_6), P_{18}=(r_1,z_2,r_7), P_{19}=(r_4,z_3,r_8), P_{20}=(r_5,z_4,r_7), P_{21}=(r_1,z_5,r_8), P_{22}=$

$(r_2,z_6,r_8), P_{23}=(r_1,z_7,r_3), P_{24}=(r_2,z_8,r_4), P_{25}=(r_3,z_1,r_8), P_{26}=(r_4,z_2,r_5), P_{27}=(r_5,z_3,r_6), P_{28}=$ $(r_3,z_4,r_6), P_{29}=(r_2,z_5,r_7), P_{30}=(r_1,z_6,r_5), P_{31}=(r_4,z_7,r_6), P_{32}=(r_3,z_8,r_7), P_{33}=(r_1,z_9,r_8), P_{34}=$ $(r_3,z_9,r_4), P_{35}=(r_1,z_{10},r_4), P_{36}=(r_5,z_{10},r_8), P_{37}=(r_1,z_{11},r_6), P_{38}=(r_7,z_{11},r_8), P_{39}=(r, z_2, r_{12})$
$P_{40}=(r_4,z_{12},r_7), P_{41}=(r_2,z_{13},r_5), P_{42}=(r_6,z_{13},r_7), P_{43}=(r_3,z_{14},r_5), P_{44}=(r_6,z_{14},r_8)$. Then $\psi=\{P_j : j=1,2\ldots,44\}$ along with the remaining edges form 2-acyclic sgc of $K_{8,v}$ in which $\{y_j : j=15,16,\ldots\}$ are not made internal. Thus $|\psi|=44+8v-88=8v-44=\left\lfloor \frac{2uv-u^2-3u}{2} \right\rfloor$.

$\left\lfloor \frac{2uv-u^2-3u}{2} \right\rfloor$. Hence $\eta_{2as}\le\left\lfloor \frac{2uv-u^2-3u}{2} \right\rfloor$. For any 2− acyclic sgc ψ_G of $K_{8,v}$ $t_2(\psi)=$ $\left\lfloor \frac{u^2+3u}{4} \right\rfloor, t_\psi=\left\lfloor \frac{4v-u^2+u}{4} \right\rfloor$. Hence $t_2\le\left\lfloor \frac{u^2+3u}{4} \right\rfloor$, $t\ge\left\lfloor \frac{4v-u^2+u}{4} \right\rfloor$ so that $\eta_{2as}(K_{8,v})\ge$ $\left\lfloor \frac{2uv-u^2-3u}{2} \right\rfloor$. Thus $\eta_{2as}(K_{8,v})=\left\lfloor \frac{2uv-u^2-3u}{2} \right\rfloor$.

Subcase 2.2. When $u=10$

Then the collection of paths are $P_1=(z_1,r_1,z_3), P_2=(z_2,r_2,z_4), P_3=(z_3,r_3,z_5), P_4=$ $(z_4,r_4,z_6), P_5=(z_5,r_5,z_7), P_6=(z_6,r_6,z_8), P_7=(z_7,r_7,z_9), P_8=(z_8,r_8,z_{10}), P_9=(z_2,r_9,z_{10}),$ $P_{10}=(z_1,r_{10},z_9), P_{11}=(z_6,r_1,z_{10}), P_{12}=(z_5,r_2,z_9), P_{13}=(z_4,r_3,z_8), P_{14}=(z_3,r_4,z_7), P_{15}=(z_2, r_5,z_6), P_{16}=(z_1,r_6,z_5), P_{17}=(z_1,r_7,z_4), P_{18}=(z_2,r_8,z_5), P_{19}=(z_3,r_9,z_6), P_{20}=(z_4,r_{10},z_7), P_{21} =(r_4,z_1,r_8), P_{22}=(r_3,z_2,r_7), P_{23}=(r_2,z_3,r_6), P_{24}=(r_5,z_4,r_6), P_{25}=(r_4,z_5,r_7), P_{26}=(r_3,z_6,r_8),$ $P_{27}=(r_2,z_7,r_9), P_{28}=(r_1,z_8,r_{10}), P_{29}=(r_1,z_9,r_9), P_{30}=(r_2,z_{10},r_{10}), P_{31}=(r_5,z_1,r_9), P_{32}=(r_4,z_2,$

$r_{10}), P_{33}=(r_5,z_3,r_7), P_{34}=(r_8,z_4,r_9), P_{35}=(r_9,z_5,r_{10}), P_{36}=(r_2,z_6,r_7), P_{37}=(r_3,z_7,r_6), P_{38}=(r_4, ,z_8,r_9), P_{39}=(r_5,z_9,r_8), P_{40}=(r_6,z_{10},r_7), P_{41}=(r_1,z_{11},r_2), P_{42}=(r_4,z_{11},r_5), P_{43}=(r_1,z_{12},r_3), P_{44}=$

$(r_4,z_{12},r_6), P_{45}=(r_1,z_{13},r_4), P_{46}=(r_3,z_{13},r_5), P_{47}=(r_1,z_{14},r_5), P_{48}=(r_6,z_{14},r_8), P_{49}=(r_1,z_{15},r_6),$

$P_{50}=(r_3,z_{15},r_9), P_{51}=(r_1,z_{16},r_7), P_{52}=(r_8,z_{16},r_{10}), P_{53}=(r_1,z_{17},r_8), P_{54}=(r_3,z_{17},r_{10}), P_{55}=(r_8,$

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$z_{18}, r_3), P_{56} = (r_6, z_{18}, r_{10}), P_{57} = (r_2, z_{19}, r_4), P_{58} = (r_7, z_{19}, r_8), P_{59} = (r_2, z_{20}, r_5), P_{60} = (r_7, z_{20}, r_9),$
$P_{61} = (r_2, z_{21}, r_8), P_{62} = (r_6, z_{21}, r_9), P_{63} = (r_3, z_{22}, r_4), P_{64} = (r_7, z_{22}, r_{10})$ *and* $P_{65} = (r_5, z_{23}, r_{10})$.
Then $\psi = \{P_j : j = 1, 2, ..., 65\}$ along with the remaining edges form a minimum 2-acyclic sgc of $K_{10,v}$ in which vertices $\{v_j : j = 24, ...\}$ are not internal. Thus $|\psi| = 65 + (10v - 130) = 10v - 65 = \left\lfloor \frac{2uv - u^2 - 3u}{2} \right\rfloor$. Hence $\eta_{2as} \leq \left\lfloor \frac{2uv - u^2 - 3u}{2} \right\rfloor$. For any 2−acyclic sgc ψ_G of $K_{10,v}$, $t_2(\psi) = \left\lfloor \frac{u^2 + 3u - 2}{4} \right\rfloor, t_\psi = \left\lfloor \frac{4v - u^2 + u - 2}{4} \right\rfloor$. Hence $t_2 \leq \left\lfloor \frac{u^2 + 3u - 2}{4} \right\rfloor$, $t \geq \left\lfloor \frac{4v - u^2 + u - 2}{4} \right\rfloor$ so that $\eta_{2as}(K_{8,v}) \geq \left\lfloor \frac{2uv - u^2 - 3u}{2} \right\rfloor$. Thus $\eta_{2as}(K_{8,v}) = \left\lfloor \frac{2uv - u^2 - 3u}{2} \right\rfloor$.

Subcase 2.3. When $u \geq 12$, Then there are two subcases.

Subcase 2.3.1. When $u \equiv 0\ (mod\ 4)$

Then the collection of paths are

$P_j = (z_j, r_j, z_{(j+2)}) : j = 1, 2 ... (v-2)$

$Q_j = (z_{(j+1)}, r_{(v-1+j)}, z_{(v-j)}) : j = 0, 1$

$R_j = (z_{(j+1)}, r_{(v-j)}, z_{(5+j)}) : j = 0, 1, ... (v-5)$

$S_j = (z_{(4-j)}, r_{(j+1)}, z_{(v-j)}) : j = 0, 1, 2, 3$

$T_j = (r_{(j+1)}, z_{(6-j)}, r_{(v-j)}) : j = 0, 1, ...5$

$U_j = (r_j, z_{(6+j)}, r_{(10+j)}) : j = 0, 1, ..., (v-10)$

$V_j = (r_{(6-j)}, z_{((v-3)-j)}, r_{(v-j)}) : j = 0, 1, 2, 3$

$W_j = (r_{(j+1)}, z_{((v-4)-j)}, r_{(j+3)}) : j = 0, 1, ... (v-5)$

$X_j = (r_{(j+1)}, z_{((v-3)+j)}, r_{((v-2)-j)}) : j = 0, 1, 2, 3$

$Y_j = (r_k, z_{(u+j)}, r_l)$

$Z_j = (r_p, z_{(u+j)}, r_q) : k \neq l \neq p \neq q,\ j = 1, 2, ..., \left\lfloor (u^2 - 5u)/4 \right\rfloor \ \& u + j \leq v$

Then $\psi = \{P_j : j = 1, 2 ...., (v-2)\} \cup \{Q_j : j = 0, 1\} \cup \{R_j : j = 0, 1, ..., (v-5)\} \cup \{S_j : j = 0, 1, 2, 3\} \cup \{T_j : j = 0, 1, ..., 5\} \cup \{U_j : j = 1, 2, ... (v-10)\} \cup \{V_j : j = 0, 1, 2, 3\} \cup \{W_j : j = 0, 1, ..., (v-5)\} \cup \{X_j : j = 0, 1, 2, 3\} \cup \{Y_j : j = 1, 2, .., \left\lfloor (u^2 - 5u)/4) \right\rfloor\} \cup \{Z_j : j = j : 1, 2, .., \left\lfloor (u^2 - 5u)/4) \right\rfloor\}$ along with the remaining edges form a 2−acyclic sgc with $|\psi| = \left\lfloor \frac{u^2 + 3u}{4} \right\rfloor + \left\lfloor \frac{2uv - u^2 - 3u}{2} \right\rfloor = \left\lfloor \frac{2uv - u^2 - 3u}{2} \right\rfloor$. Hence $\eta_{2as} \leq \left\lfloor \frac{2uv - u^2 - 3u}{2} \right\rfloor$. For any 2−acyclic sgc ψ_G of $K_{u,v}, t_2(\psi) =$

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$\left\lfloor \frac{u^2+3u}{4} \right\rfloor, t_\psi = \left\lfloor \frac{4v-u^2+u}{4} \right\rfloor$. Hence $t_2 \le \left\lfloor \frac{u^2+3u}{4} \right\rfloor$, $t \ge \left\lfloor \frac{4v-u^2+u}{4} \right\rfloor$ so that

$\eta_{2as}\left(K_{u,v}\right) \ge \left\lfloor \frac{2uv-u^2-3u}{2} \right\rfloor$. Thus $\eta_{2as}\left(K_{8,v}\right) = \left\lfloor \frac{2uv-u^2-3u}{2} \right\rfloor$.

Subcase 2.3.2. When $u \equiv 2(mod\ 4)$

Then the collection of paths are

$P_j = (z_j, r_j, z_{(j+2)}) : j = 1, 2...(v-2)$

$Q_j = (z_{(j+1)}, r_{(v-1+j)}, z_{(v-j)}) : j = 0,1$

$R_j = (z_{(j+1)}, r_{(v-j)}, z_{(5+j)}) : j = 0,1,...(v-5)$

$S_j = (z_{(4-j)}, r_{(j+1)}, z_{(v-j)}) : j = 0,1,2,3$

$T_j = (r_{(j+1)}, z_{(6-j)}, r_{(v-j)}) : j = 0,1,....5$

$U_j = (r_j, z_{(6+j)}, r_{(10+j)}) : j = 1,2,...(v-10)$

$V_j = (r_{(6-j)}, z_{((v-3)-j)}, r_{(v-j)}) : j = 0,1,2,3$

$W_j = (r_{(j+1)}, z_{((v-4)-j)}, r_{(j+3)}) : j = 0,1,...(v-5)$

$X_j = (r_{(j+1)}, z_{((v-3)+j)}, r_{((v-2)-j)}) : j = 0,1,2,3$

$Y_j = (r_k, z_{(u+j)}, r_l)$

$Z_j = (r_p, z_{(u+j)}, r_q) : k \ne l \ne p \ne q,\ j = 1,2,......, \left\lfloor \left(u^2-5u\right)/4 \right\rfloor \ \& u+j \le v$

$M = (r_u, z_{\left\lfloor \frac{u^2-u+4}{2} \right\rfloor}, r_v)\ \ k \ne l \ne p \ne q \ne u \ne v$

Then $\psi = \{P_j : j = 1,2....,(v-2)\} \cup \{Q_j : j = 0,1\} \cup \{R_j : j = 0,1,...,(v-5)\} \cup \{S_j : j = 0,1,2,3\}$ $\cup \{T_j : j = 0,1,...,5\} \cup \{U_j : j = 1,2,...(v-10)\} \cup \{V_j : j = 0,1,2,3\} \cup \{W_j : j = 0,1,...,(v-5)\} \cup$ $\{X_j : j = 0,1,2,3\} \cup \{Y_j : j = 1,2,.., \lfloor (u^2-5u)/4) \rfloor\} \cup \{Z_j : j = j : 1,2,.., \lfloor (u^2-5u)/4) \rfloor\} \cup \{M\}$ along with the remaining edges form a 2−acyclic sgc with $|\psi| = \left\lfloor \frac{u^2+3u}{4} \right\rfloor + \left\lfloor \frac{2uv-u^2-3u}{2} \right\rfloor = \left\lfloor \frac{2uv-u^2-3u}{2} \right\rfloor$. Hence $\eta_{2as} \le \left\lfloor \frac{2uv-u^2-3u}{2} \right\rfloor$. For any 2−acyclic sgc ψ_G of $K_{u,v}, t_2(\psi) = \left\lfloor \frac{u^2+3u-2}{4} \right\rfloor, t_\psi = \left\lfloor \frac{4v-u^2+u-2}{4} \right\rfloor$. Hence $t_2 \le$, $\left\lfloor \frac{u^2+3u-2}{4} \right\rfloor$ $t \ge \left\lfloor \frac{4v-u^2+u-2}{4} \right\rfloor$ so that $\eta_{2as}\left(K_{u,v}\right) \ge \left\lfloor \frac{2uv-u^2-3u}{2} \right\rfloor$. Thus $\eta_{2as}\left(K_{8,v}\right) = \left\lfloor \frac{2uv-u^2-3u}{2} \right\rfloor$.

Theorem 3.3. *For a complete bipartite graph* $K_{u,v}, (u \ge 9)$ *and u is odd, then*

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$$\eta_{2as} = \begin{cases} uv-2u-2v & \text{if } \left(u \equiv 1(\text{mod } 4) \text{ and } u \le v \le \left\lfloor u^2 - u/4 \right\rfloor \right) \\ & (or) \\ & \left(u \equiv 3(\text{mod } 4) \text{ and } u \le v \le \left\lfloor \left(u^2 - u - 2\right)/4 \right\rfloor \right) \\ \left\lfloor \dfrac{2uv-u^2-3u}{2} \right\rfloor & \text{if } \left(u \equiv 1(\text{mod } 4) \text{ and } v > \left\lfloor u^2 - u/4 \right\rfloor \right) \\ & (or) \\ & \left(u \equiv 3(\text{mod } 4) \text{ and } v > \left\lfloor \left(u^2 - u - 2\right)/4 \right\rfloor \right) \end{cases}$$

Proof. Now let $X = \{r_1, r_2, r_3, ..., r_u\}$ and $Y = \{z_1, z_2, z_3, ..., z_v\}$ be the bipartition of $K_{u,v}$ $(u \ge 9)$, $u\, odd$ with $p = u+v, q = uv$. Then there are four cases.

Case 1. When $u \equiv 1 \text{ (mod 4)}$ and $u \le v \le \left\lfloor \dfrac{u^2+u}{4} \right\rfloor$, Then there are two cases.

Subcase 1.1. When $u = 9$

Then the collection of paths are $P_1 = (z_1, r_1, z_4), P_2 = (z_2, r_2, z_5), P_3 = (z_3, r_3, z_6), P_4 = (z_4, r_4, z_7), P_5 = (z_5, r_5, z_8), P_6 = (z_6, r_6, z_9), P_7 = (z_3, r_7, z_9), P_8 = (z_2, r_8, z_8), P_9 = (z_1, r_9, z_7), P_{10} = (z_2, r_1, z_6), P_{11} = (z_3, r_2, z_7), P_{12} = (z_4, r_3, z_8), P_{13} = (z_5, r_4, z_9), P_{14} = (z_2, r_5, z_7), P_{15} = (z_1, r_6, z_3), P_{16} = (z_2, r_7, z_4), P_{17} = (z_3, r_8, z_5), P_{18} = (z_4, r_9, z_6), P_{19} = (r_2, z_1, r_8), P_{20} = (r_3, z_2, r_9), P_{21} = (r_4, z_3, r_9)$, $P_{22} = (r_5, z_4, r_8), P_{23} = (r_3, z_5, r_6), P_{24} = (r_2, z_6, r_7), P_{25} = (r_1, z_7, r_8), P_{26} = (r_1, z_8, r_4), P_{27} = (r_2, z_9, r_5)$, $P_{28} = (r_5, z_1, r_7), P_{29} = (r_4, z_2, r_6), P_{30} = (r_1, z_3, r_5), P_{31} = (r_2, z_4, r_6), P_{32} = (r_1, z_5, r_9), P_{33} = (r_4, z_6, r_8)$, $P_{34} = (r_3, z_7, r_7), P_{35} = (r_2, z_8, r_9), P_{36} = (r_3, z_9, r_8), Q_j = (r_k, z_{9+j}, r_l)$ and $R_j = (r_p, z_{9+j}, r_q)$.

Then $\psi = \{P_j : j = 1,2..,36\} \cup \{Q_j : j = 1,2,..,9\} \cup \{R_j : j = 1,2,...,9\}$ along with the remaining edges form a minimum 2−acyclic sgc of $K_{9,v}$ in which every vertices made internal twice. By corollary 2.1, $\eta_{2as}(K_{9,v}) = q - 2p = uv - 2u - 2v$

Subcase 1.1. When $u \ge 13$

Then the collection of paths are

$M_j = (z_j, r_j, z(4+j)) : j = 1,2,...,(v-4)$

$N_j = \left(z_{(j+1)}, r_{(v-j)}, z_{((v-3)+j)}\right) : j = 0,1...,3$

$P_j = \left(z_{(6-j)}, r_{(j+1)}, z_{((v-1)-j)}\right) : j : 0,1,...,5$

$Q = (z_2, r_7, z_3)$

$R_j = \left(z_{(j+1)}, r_{(8+j)}, z_{(4+j)}\right) : j = 0,1,...,(v-8)$

$S_j = \left(r_{(4-j)}, z_{(j+1)}, r_{((v-3)+j)}\right) : j = 0,1...,3$

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$T_j = \left(r_{(8+j)}, z_{(5+j)}, r_{(10+j)}\right): j = 0,1,...,(v-10)$

$U_j = \left(r_{j+1}, z_{(v-4)+j)}, r_{(3+j)}\right): j = 0,1,...,4$

$V_j = (r_{(5-j)}, z_{(j+1)}, r_{((v-1)+j)}): j = 0,1$

$W_j = \left(r_{j+1}, z_{(3+j)}, r_{(5+j)}\right): j = 0,1,...,(v-5)$

$X_j = (r_{(2+j)}, z_{((v-1)+j)}, r_{(v-j)}): j = 0,1$

$Y_j = \left(r_k, z_{(u+j)}, r_l\right); Z_j = (r_p, z_{(u+j)}, r_q),\ k \neq l \neq p \neq q,\ \ j = 1,2,....\left\lfloor \frac{u^2-5u}{4} \right\rfloor, u + j \leq v$

Then $\psi = \{M_j : j = 1,2,...,(v-4)\} \cup \{N_j : j = 0,1,2,3\} \cup \{P_j : j = 0,1,...,5\} \cup \{Q\} \cup \{R_j\ : j = 0,1,....,(v-8)\} \cup \{S_j\ : j = 0,1...,3\} \cup \{T_j\ : j : j = 0,1,..,(v-10)\} \cup \{U_j\ : j = 0,1..,4\} \cup \{V_j : j = , 0,1\} \cup \{W_j : j = 0,1,...,(v-5)\} \cup \left\{X_j : j = 0,1\right\} \cup \{Y_j : j = 1,...\lfloor (u^2-5u)/4 \rfloor\} \cup \{Z_j : j = 1,... \lfloor (u^2-5u)/4 \rfloor\}$ along with the remaining edges form a minimum 2−acyclic sgc of $K_{13,v}$ in which every vertices made internal twice. Therefore $\eta_{2as}\left(K_{13,v}\right) = q - 2p = uv - 2u - 2v$.

Case 2. When $u \equiv 3 (\text{mod } 4)$ *and* $u \leq v \leq \lfloor (u^2-u-2)/4 \rfloor$, then there are two subcases.

Subcase 2.1. When $u = 11$

Then the collection of paths are

$M_j = \left(z_j, r_j, z_{3+j}\right): j = 1,2,...,8$

$N_j = \left(z_{(j+1)}, r_{(v-j)}, z_{(9+j)}\right): j = 0,1,2$

$P_j = \left(z_{(j+1)}, r_{(6-j)}, z_{(8+j)}\right): j : 0,1,2,3$

$Q_j = (z_{(6+j)}, r_{(j+1)}, z_{(10+j)}): j = 0,1$

$R = \left(z_1, r_7, z_{11}\right)$

$S_j = \left(z_{(j+1)}, r_{(8+j)}, z_{(5+j)}\right): j = 0,1,2,3$

$T_j = \left(r_{(3+j)}, z_{(j+1)}, r_{(5+j)}\right): j = 0,1,...,6$

$U_j = \left(r_{(j+1)}, z_{(8+j)}, r_{(7+j)}\right): j = 0,1,2,3$

$V_j = (r_{(j+1)}, z_{(2-j)}, r_{(v-j)}): j = 0,1$

$W = \left(r_6, z_3, r_{11}\right)$

$X_j = \left(r_{(2+j)}, z_{(4+j)}, r_{(5+j)}\right): j = 0,1,2,3$

$Y_j = (r_{(j+1)}, z_{(9-j)}, r_{(3+j)}): j = 0,1$

$Z_j = (r_{(1+j)}, z_{(v-j)}, r_{(5+j)}): j = 0,1$

$A_j = \left(r_k, z_{(11+j)}, r_l\right); B_j = (r_p, z_{(11+j)}, r_q)\ \ k \neq l \neq p \neq q,\ j = 1,2....16$ *and* $11 + j \leq v$

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Then $\psi=\left\{M_j: j=1,2,\ldots,8\right\}\cup\left\{N_j: j=0,1,2,3\right\}\cup\left\{P_j: j=0,1,\ldots,2\right\}\cup\{Q_j: j=0,1\}\cup\left\{R\right\}$ $\cup\left\{S_j : j=0,1,2,3\right\}\cup\left\{T_j : j=0,1,\ldots,6\right\}\cup\left\{U_j : j=0,1,2,3\right\}\cup\{V_j : j=0,1\}\cup\{W\}\cup\{X_j : j=$ $0,1,2,3\}\cup\left\{Y_j : j=0,1\right\}\cup\left\{Z_j : j=0,1\right\}\cup\{A_j : j=0,1,\ldots,16\}\cup\{B_j : j=0,1,\ldots,16\}$ along with the remaining edges form a minimum 2−acyclic sgc of $K_{13,v}$ in which every vertices made internal twice. Therefore $\eta_{2as}\left(K_{11,v}\right)=q-2p=uv-2u-2v$.

Subcase 2.1. When $u\geq 15$

$M_j=\left(z_j, r_j, z_{(4+j)}\right): j=1,2,\ldots,(v-4)$

$N_j=\left(z_{(j+1)}, r_{(v-j)}, z_{((v-3)+j)}\right): j=0,1,2,3$

$P_j=\left(z_{(6-j)}, r_{(j+1)}, z_{((v-1)-j)}\right): j:0,1,\ldots,5$

$Q=\left(z_2, r_7, z_3\right)$

$R_j=\left(z_{(j+1)}, r_{(8+j)}, z_{(4+j)}\right): j=0,1,\ldots,(v-8)$

$S_j=\left(r_{(4-j)}, z_{(j+1)}, r_{((v-3)+j)}\right): j=0,1,2,3$

$T_j=\left(r_{(8+j)}, z_{(5+j)}, r_{(10+j)}\right): j=0,1,\ldots,(v-10)$

$U_j=\left(r_{j+1}, z_{(v-4)+j)}, r_{(3+j)}\right): j=0,1,\ldots,4$

$V_j=(r_{(5-j)}, z_{(j+1)}, r_{((v-1)+j)}): j=0,1$

$W_j=\left(r_{j+1}, z_{(3+j)}, r_{(5+j)}\right): j=0,1,\ldots,(v-5)$

$Xj=(r_{(2+j)}, z_{((v-1)+j)}, r_{(v-j)}): j=0,1$

$Y_j=\left(r_k, z_{(u+j)}, r_l\right);\ Z_j=\left(r_p, z_{(u+j)}, r_q\right)\ k\neq l\neq p\neq q,\ \ j=1,2,\ldots.\left\lfloor\frac{u^2-5u}{4}\right\rfloor, u+j\leq v$

Then $\psi=\{M_j: j=1,2,..,(v-4)\}\cup\{N_j: j=0,1,2,3\}\cup\{P_j: j=0,1,..,5\}\cup\{Q\}\cup\{R_j : j=0,$ $\ldots,(v-8)\}\cup\{S_j : j=0,1,2,3\}\cup\{T_j : j=0,1,\ldots,(v-10)\}\cup\{U_j : j=0,1\ldots,4\}\cup\left\{V_j: j=0,1\right\}\cup$ $\{W_j: j=0,1,.,(v-5)\}\cup\left\{X_j: j=0,1\right\}\cup\{Y_j: j=1,..\left\lfloor(u^2-5u)/4\right\rfloor\}\cup\{Z_j: j=1,..\left\lfloor(u^2-5u)/4\right\rfloor\}$

along with the remaining edges form a minimum 2−acyclic sgc of $K_{15,v}$ in which every vertices made internal twice. Therefore $\eta_{2as}\left(K_{15,v}\right)=q-2p=uv-2u-2v$.

Case 3. When $u\equiv 1(mod\,4)$ *and* $v>\left\lfloor\frac{u^2-u}{4}\right\rfloor$, Then there are two subcases.

Subcase 3.1. When $u=9$

Then the collection of paths are $P_1=\left(z_1, r_1, z_4\right), P_2=\left(z_2, r_2, z_5\right), P_3=\left(z_3, r_3, z_6\right), P_4=$ $\left(z_4, r_4, z_7\right), P_5=\left(z_5, r_5, z_8\right), P_6=\left(z_6, r_6, z_9\right), P_7=\left(z_3, r_7, z_9\right), P_8=\left(z_2, r_8, z_8\right), P_9=\left(z_1, r_9, z_7\right), P_{10}=$

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$(z_2,r_1,z_6),P_{11}=(z_3,r_2,z_7),P_{12}=(z_4,r_3,z_8),P_{13}=(z_5,r_4,z_9),P_{14}=(z_2,r_5,z_7),P_{15}=(z_1,r_6,z_3),P_{16}$ $=(z_2,r_7,z_4),P_{17}=(z_3,r_8,z_5),P_{18}=(z_4,r_9,z_6),P_{19}=(r_2,z_1,r_8),P_{20}=(r_3,z_2,r_9),P_{21}=(r_4,z_3,r_9),$ $P_{22}=(r_5,z_4,r_8),P_{23}=(r_3,z_5,r_6),P_{24}=(r_2,z_6,r_7),P_{25}=(r_1,z_7,r_8),P_{26}=(r_1,z_8,r_4),P_{27}=(r_2,z_9,r_5),$ $P_{28}=(r_5,z_1,r_7),P_{29}=(r_4,z_2,r_6),P_{30}=(r_1,z_3,r_5),P_{31}=(r_2,z_4,r_6),P_{32}=(r_1,z_5,r_9),P_{33}=(r_4,z_6,r_8),$ $P_{34}=(r_3,z_7,r_7),P_{35}=(r_2,z_8,r_9),P_{36}=(r_3,z_9,r_8),Q_j=(r_k,z_{9+j},r_l)$ and $R_j=(r_p,z_{9+j},r_q)$.

Then $\psi=\{P_j : j=1,2..,36\}\cup\{Q_j\}\cup\{R_j\},\ k\neq l\neq p\neq q,\ j=1,2,...,9\ and\ 9+j\leq v$ so that $|\psi|=54+(9v-108)=9v-54=\left\lfloor\frac{2uv-u^2-3u}{2}\right\rfloor$. Hence $\eta_{2as}(K_{9,v})\leq\left\lfloor\frac{2uv-u^2-3u}{2}\right\rfloor$. For any 2−acyclic sgc ψ_G of $K_{9,v}, t_2(\psi)=\left\lfloor\frac{u^2+3u}{4}\right\rfloor, t_\psi=\left\lfloor\frac{4v-u^2+u}{4}\right\rfloor$. Hence $t_2\leq\left\lfloor\frac{u^2+3u}{4}\right\rfloor$ $t\geq\left\lfloor\frac{4v-u^2+u}{4}\right\rfloor$ so that $\eta_{2as}(K_{9,v})\geq\left\lfloor\frac{2uv-u^2-3u}{2}\right\rfloor$. Thus $\eta_{2as}(K_{9,v})=\left\lfloor\frac{2uv-u^2-3u}{2}\right\rfloor$.

Subcase 3.2. When $u\geq 13$

Then the collection of paths are

$M_j=(z_j,r_j,z(4+j)) : j=1,2,...,(v-4)$

$N_j=\left(z_{(j+1)},r_{(v-j)},z_{((v-3)+j)}\right) : j=0,1...,3$

$P_j=\left(z_{(6-j)},r_{(j+1)},z_{((v-1)-j)}\right) : j:0,1,...,5$

$Q=(z_2,r_7,z_3)$

$R_j=\left(z_{(j+1)},r_{(8+j)},z_{(4+j)}\right) : j=0,1,...,(v-8)$

$S_j=\left(r_{(4-j)},z_{(j+1)},r_{((v-3)+j)}\right) : j=0,1...,3$

$T_j=\left(r_{(8+j)},z_{(5+j)},r_{(10+j)}\right) : j=0,1,...,(v-10)$

$U_j=\left(r_{j+1},z_{(v-4)+j)},r_{(3+j)}\right) : j=0,1,...,4$

$V_j=(r_{(5-j)},z_{(j+1)},r_{((v-1)+j)}) : j=0,1$

$W_j=\left(r_{j+1},z_{(3+j)},r_{(5+j)}\right) : j=0,1,...,(v-5) : j=0,1,...,(v-8)$

$X_j=(r_{(2+j)},z_{((v-1)+j)},r_{(v-j)}) : j=0,1$

$Y_j=\left(r_k,z_{(u+j)},r_l\right);Z_j=(r_p,z_{(u+j)},r_q),\ k\neq l\neq p\neq q,\ \ j=1,2,....\left\lfloor\frac{u^2-5u}{4}\right\rfloor,\ u+j\leq v$

Then $\psi=\{M_j : j=1,2,...,(v-4)\}\cup\{N_j : j=0,1,2,3\}\cup\{P_j : j=0,1,...,5\}\cup\{Q\}\cup\{R_j : j=$ $0,1,....,(v-8)\}\cup\{S_j : j=0,1...,3\}\cup\{T_j : j : j=0,1,..,(v-10)\}\cup\{U_j : j=0,1..,4\}\cup\{V_j : j=,$ $0,1\}\cup\{W_j : j=0,1,...,(v-5)\}\cup\{X_j : j=0,1\}\cup\{Y_j : j=1,...\lfloor(u^2-5u)/4\rfloor\}\cup\{Z_j : j=1,...$

$\lfloor (u^2-5u)/4 \rfloor\}$ so that $|\psi| = \left\lfloor \frac{2uv-u^2-3u}{2} \right\rfloor$. Hence $\eta_{2as}(K_{13,v}) \le \left\lfloor \frac{2uv-u^2-3u}{2} \right\rfloor$. For any 2−acyclic sgc ψ_G of $K_{13,v}, t_2(\psi) = \left\lfloor \frac{u^2+3u}{4} \right\rfloor, t_\psi = \left\lfloor \frac{4v-u^2+u}{4} \right\rfloor$. Hence $t_2 \le \left\lfloor \frac{u^2+3u}{4} \right\rfloor$ $t \ge \left\lfloor \frac{4v-u^2+u}{4} \right\rfloor$ so that $\eta_{2as}(K_{13,v}) \ge \left\lfloor \frac{2uv-u^2-3u}{2} \right\rfloor$. Thus $\eta_{2as}(K_{13,v}) = \left\lfloor \frac{2uv-u^2-3u}{2} \right\rfloor$.

Case 4. When $u \equiv 3$ (mod 4) *and* $v > \lfloor (u^2-u-2)/4 \rfloor$, then there are two subcases.

Subcase 4.1. When $u = 11$

Then the collection of paths are

$M_j = (z_j, r_j, z_{3+j}) : j = 1,2,...,8$

$N_j = (z_{(j+1)}, r_{(v-j)}, z_{(9+j)}) : j = 0,1,2$

$P_j = (z_{(j+1)}, r_{(6-j)}, z_{(8+j)}) : j : 0,1,2,3$

$Q_j = (z_{(6+j)}, r_{(j+1)}, z_{(10+j)}) : j = 0,1$

$R = (z_1, r_7, z_{11})$

$S_j = (z_{(j+1)}, r_{(8+j)}, z_{(5+j)}) : j = 0,1,2,3$

$T_j = (r_{(3+j)}, z_{(j+1)}, r_{(5+j)}) : j = 0,1,...,6$

$U_j = (r_{(j+1)}, z_{(8+j)}, r_{(7+j)}) : j = 0,1,2,3$

$V_j = (r_{(j+1)}, z_{(2-j)}, r_{(v-j)}) : j = 0,1$

$W = (r_6, z_3, r_{11})$

$X_j = (r_{(2+j)}, z_{(4+j)}, r_{(5+j)}) : j = 0,1,2,3$

$Y_j = (r_{(j+1)}, z_{(9-j)}, r_{(3+j)}) : j = 0,1$

$Z_j = (r_{(1+j)}, z_{(v-j)}, r_{(5+j)}) : j = 0,1$

$A_j = (r_k, z_{(11+j)}, r_l)$; $B_j = (r_p, z_{(11+j)}, r_q)$ $k \ne l \ne p \ne q,\ j = 1,2....16$ *and* $11 + j \le v$

$M = \left(r_u, z_{\left\lfloor \frac{u^2-u+2}{4} \right\rfloor}, r_v \right)$ $k \ne l \ne p \ne q \ne u \ne v$

Then $\psi = \{M_j : j = 1,2,...,8\} \cup \{N_j : j = 0,1,2,3\} \cup \{P_j : j = 0,1,...,2\} \cup \{Q_j : j = 0,1\} \cup \{R\}$ $\cup \{S_j : j = 0,1,2,3\} \cup \{T_j : j = 0,1,...,6\} \cup \{U_j : j = 0,1,2,3\} \cup \{V_j : j = 0,1\} \cup \{W\} \cup \{X_j : j = 0,1,2,3\} \cup \{Y_j : j = 0,1\} \cup \{Z_j : j = 0,1\} \cup \{A_j : j = 0,1,...,16\} \cup \{B_j : j = 0,1,...,16\} \cup \{M\}$ so that $|\psi| = \left\lfloor \frac{2uv-u^2-3u}{2} \right\rfloor$. Hence $\eta_{2as}(K_{13,v}) \le \left\lfloor \frac{2uv-u^2-3u}{2} \right\rfloor$. For any 2−acyclic sgc ψ_G

of $K_{13,v}, t_2(\psi) = \left\lfloor \frac{u^2+3u-2}{4} \right\rfloor, t_\psi = \left\lfloor \frac{4v-u^2+u-2}{4} \right\rfloor$. Hence $t_2 \le \left\lfloor \frac{u^2+3u-2}{4} \right\rfloor$

$t \ge \left\lfloor \frac{4v-u^2+u-2}{4} \right\rfloor$ so that $\eta_{2as}(K_{11,v}) \ge \left\lfloor \frac{2uv-u^2-3u}{2} \right\rfloor$. Thus $\eta_{2as}(K_{13,v}) = \left\lfloor \frac{2uv-u^2-3u}{2} \right\rfloor$.

Subcase 4.2. When $u \ge 15$

The collection of paths are

$M_j = (z_j, r_j, z_{(4+j)}) : j = 1,2,...,(v-4)$

$N_j = (z_{(j+1)}, r_{(v-j)}, z_{((v-3)+j)}) : j = 0,1,2,3$

$P_j = (z_{(6-j)}, r_{(j+1)}, z_{((v-1)-j)}) : j : 0,1,...,5$

$Q = (z_2, r_7, z_3)$

$R_j = (z_{(j+1)}, r_{(8+j)}, z_{(4+j)}) : j = 0,1,...,(v-8)$

$S_j = (r_{(4-j)}, z_{(j+1)}, r_{((v-3)+j)}) : j = 0,1,2,3$

$T_j = (r_{(8+j)}, z_{(5+j)}, r_{(10+j)}) : j = 0,1,...,(v-10)$

$U_j = (r_{j+1}, z_{(v-4)+j)}, r_{(3+j)}) : j = 0,1,...,4$

$V_j = (r_{(5-j)}, z_{(j+1)}, r_{((v-1)+j)}) : j = 0,1$

$W_j = (r_{j+1}, z_{(3+j)}, r_{(5+j)}) : j = 0,1,...,(v-5)$

$Xj = (r_{(2+j)}, z_{((v-1)+j)}, r_{(v-j)}) : j = 0,1$

$Y_j = (r_k, z_{(u+j)}, r_l);\ Z_j = (r_p, z_{(u+j)}, r_q)\ k \ne l \ne p \ne q,\ \ j = 1,2,....\left\lfloor \frac{u^2-5u}{4} \right\rfloor, u + j \le v$

$M = \left(r_u, z_{\left\lfloor \frac{u^2-u+2}{4} \right\rfloor}, r_v \right) k \ne l \ne p \ne q \ne u \ne v$

Then $\psi = \{M_j : j = 1,2,..,(v-4)\} \cup \{N_j : j = 0,1,2,3\} \cup \{P_j : j = 0,1,..,5\} \cup \{Q\} \cup \{R_j : j = 0, ...,(v-8)\} \cup \{S_j : j = 0,1,2,3\} \cup \{T_j : j = 0,1,...,(v-10)\} \cup \{U_j : j = 0,1...,4\} \cup \{V_j : j = 0,1\} \cup \{W_j : j = 0,1,.,(v-5)\} \cup \{X_j : j = 0,1\} \cup \{Y_j : j = 1,..\lfloor (u^2-5u)/4 \rfloor\} \cup \{Z_j : j = 1,..\lfloor (u^2-5u)/4 \rfloor\}$

$\cup \{M\}$ so that $|\psi| = \left\lfloor \frac{2uv-u^2-3u}{2} \right\rfloor$. Hence $\eta_{2as}(K_{13,v}) \le \left\lfloor \frac{2uv-u^2-3u}{2} \right\rfloor$. For any $2-$ acyclic sgc ψ_G of $K_{13,v}, t_2(\psi) = \left\lfloor \frac{u^2+3u-2}{4} \right\rfloor, t_\psi = \left\lfloor \frac{4v-u^2+u-2}{4} \right\rfloor$. Hence $t_2 \le \left\lfloor \frac{u^2+3u-2}{4} \right\rfloor$

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$t \geq \left\lfloor \frac{4v-u^2+u-2}{4} \right\rfloor$ so that $\eta_{2as}\left(K_{11,v}\right) \geq \left\lfloor \frac{2uv-u^2-3u}{2} \right\rfloor$. Thus $\eta_{2as}\left(K_{u,v}\right) =$ $\left\lfloor \frac{2uv-u^2-3u}{2} \right\rfloor$.

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References

[1] B Acharya and E Sampathkumar. Graphoidal covers and graphoidal covering number of a graph. *Indian J. pure appl. Math*, 18(10):882–890, 1987.

[2] BD Acharya. Further results on the graphoidal covering number of a graph. *Graph Theory News Lett.*, 17(4):1–3, 1988.

[3] S Arumugam, B Devadas Acharya, and E Sampathkumar. Graphoidalre covers of a graph: a creative review. In *Proc. Natl. Work. Graph Theory its Appl. Manonmaniam Sundaranar Univ. Tirunelveli, Eds. S. Arumugam, BD Acharya E. Sampathkumar, Tata McGraw Hill*, pages 1–28, 1996.

[4] I Sahul Arumugam, S and Hamid. Simple graphoidal covers in a graph. *J. Comb. Math. Comb. Comput.*, 64:79–95, 2008.

[5] P. K. Das and K. Ratan Singh. On 2-graphoidal covering number of a graph. *Int. J. Pure Appl. Math.*, 72(2):125–135, 2011.

[6] Frank Harary. *Graph theory*. Addison-Wesley, Reading, MA, 1969.

[7] K Nagarajan, A Nagarajan, and S Somasundaram. 2-Graphoidal Path Covers. *Int. J. Appl. Math.*, 21(4):615–628, 2008.

[8] K Nagarajan, A Nagarajan, and S Somasundram. m-graphoidal path covers of a graph. In *Proceedings of the Fifth International Conference on Number Theory and Smarandache Notions*, pages 58–67, 2009.

[9] Venkat Narayanan, Suresh Suseela, et al. On 2 acyclic simple graphoidal covering of bicyclic graphs. *J. Math. Comput. Sci.*, 10(3):606–632, 2020.

[10] Venkat Narayanan, Suresh Suseela, and Rukhmoni Kala. On 2−simple graphoidal cover of a graph. *AIP Conference Proceedings*, 2277(4):100010–1–100010–10, 2020.

[11] Venkat Narayanan, Saravanan. Decomposition of complete bipartite graphs into paths and cycles using 2-simple graphoidal covers, *European Chemical Bulletin*, 12(8) : 4372-4388, 2023

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ISSN 2063-5346

[12] E Sampathkumar. Semigraphs and their applications. Technical report, 2000.