TOTAL EDGE IRREGULARITY STRENGTH OF SOME POLYTOPE STRUCTURES R_m , Q_m , B_m

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Abstract

Given a graph G with vertex set and edge set, a function defined from vertex set and edge set to 1, 2, ..., *k* is called an edge irregular total *k*-labeling if for every pair of distinct edges, the weight of the edges are all distinct. The minimum *k* for which *G* has an edge irregular total *k*-labeling is called the total edge irregularity strength of *G*. The total edge irregularity strength of *G* is denoted by *tes*(*G*). In our present study we have considered some graphs of the family of convex polytopes and have obtained their total edge irregularity strength.
Keywords: edge irregular total *k*-labeling, total edge irregularity strength, the graphs of convex polytopes, prism graph, antiprism graph.

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1. Introduction

Graph theory has always been an exciting area of research opening up to many avenues. One of the key branches in it is graph labeling. Most graph labeling techniques trace their origin to one introduced by Rosa [7]. Rosa identified three types of labelings, which he called α-labeling, β-labeling and ρ-labeling [7]. The β-labeling were later renamed as graceful by Golomb and since then graceful labeling has been well studied [7]. For a dynamic survey of various graph labelings along with an extensive bibliography, one may refer to Gallian [7].

Labeled graphs serve as useful models for a broad range of applications such as coding theory, X-ray crystallography, radar, astronomy, circuit design, communication network, addressing, database management, secret sharing schemes, models for constraint programming over finite domains and network passwords. According to Wang, Rao and Rao, graph labelings are used for incorporating redundancy in disks, designing drilling machines, creating layouts for circuit boards and configuring resistor networks [7].

Bača, Jendrol', Miller and Ryan [8] introduced the total edge irregularity strength of a graph. Total edge irregularity strength has been well studied for honeycomb mesh networks [5], hexagonal networks [6], butterfly networks [1,3], benes networks [1] and series compositions of uniform theta graphs [4], generalized uniform theta graph and the lower bound has been determined [10]. Umer et al., [14] applied the technique of 3-total edge product cordial labeling on some families of convex polytopes. Syed Ahtshma Ul Haq Bokhary et al., [13] proved the total irregularity strength of convex polytope graphs S_n, T_n, U_n, Some Polytope Structures [11] and Plane graphs of Convex Polytopes [12].

We now begin with some known results on *tes*(G) and basic definitions.

Theorem 1.1. [8] Let G be a graph with m edges. Then $tes(G) \geq \lceil (m+2)/3 \rceil$.

Theorem 1.2. [8] Let G be a graph with maximum degree Δ. Then $tes(G) \geq \lceil [\Delta + 2]/3 \rceil$.

In our study, we have considered some families of convex polytopes. Our results on edge irregular total k-labeling applied to these graphs are presented in this paper. Further we have proved that a bound on *tes* is sharp as given in Theorem 1.1.

Given a graph $G = (V, E)$ a labeling $\partial : V \cup E \rightarrow \{1,2, ..., k\}$ is called an edge irregular total k-labeling if for every pair of distinct edges uv and xy, the edge sums are $\partial(u) + \partial(uv) + \partial(v) \neq \partial(x) + \partial(xy) + \partial(y)$. The minimum k for which G has an edge irregular total k-labeling is called the total edge irregularity strength of G. The total edge irregularity strength of G is denoted by *tes*(G).

In our paper we call the weight of the edges as edge sums.

Definition 1.2. [12] A prism graph Y_m is Cartesian product graph $C_m \times P_2$, where C_m is a cycle graph of order m and P_2 is a path graph of order 2.

Definition 1.3. [12] A m-sided anti-prism graph $A_m$is a polyhedron composed of two parallel copies of some particular m-sided polygon connected by alternating band of triangles.

Definition 1.4. [12] The convex polytope S_m is composed of two parallel copies of prism graphs connected by alternating band of triangles. It consists of three-sided faces, four-sided faces and m-sided face. The graph S_m has two m-sided faces. One is the unbounded external face. We consider only the inner m-sided face.

2. Main Results

2.1 Convex Polytope R_m

For $m \geq 5$, a combination of prism graph Y_m and antiprism graph A_m is known as convex polytope graph R_m. It consists of the inner cycle vertices u_i, $1 \leq i \leq m$, the middle cycle vertices v_i, $1 \leq i \leq m$ and the outer cycle vertices w_i, $1 \leq i \leq m$. The graph R_m has two m-sided faces. One is the unbounded external face. We consider only the inner m-sided face.

Notation:

The vertex set and edge set of R_m are defined as follows:

$V(R_m) = \{u_i , v_i , w_i , 1 \leq i \leq m\}$ and

$E(R_m) = \{u_i u_{i+1} , 1 \leq i \leq m - 1\} \cup \{v_i u_{i+1}, 1 \leq i \leq m - 1\} \cup \{w_i u_{i+1} , 1 \leq i \leq m - 1\} \cup \{u_i v_i , 1 \leq i \leq m\} \cup \{u_{i+1} v_i , 1 \leq i \leq m - 1\} \cup \{v_i w_i , 1 \leq i \leq m\} \cup \{u_m u_1 \} \cup \{v_m v_1\} \cup \{w_m w_i\} \cup \{u_1 v_m\}$. See Figure 1.

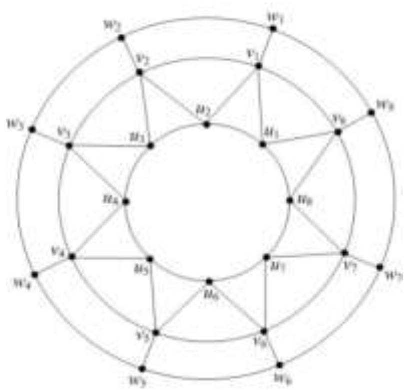

Figure 1: Convex Polytope R_m

Theorem 1:

For every $m \geq 5$ the total edge irregularity strength of the convex polytope R_m is $tes(R_m) = \lceil (6m + 2)/3 \rceil = 2m + 1$.

Proof:

The number of vertices of R_m is $3m$ and the number of edges of R_m is $6m$. The vertices and edges of R_m are traversed in the anticlockwise direction. First we label the vertices of the inner cycle then the vertices of the middle cycle followed by the vertices of the outer cycle. The edges are also labeled in the same sequence so that the edge sums are consecutive.

Input: The graph of convex polytope R_m , $m \geq 5$.

Algorithm:

Step 1:

For $1 \leq i \leq m$

$f(u_i) = 1$

$f(v_i) = m + 1$

$f(w_i) = 2m + 1$.

Step 2:

$f(u_i u_{i+1}) = i$, $1 \leq i \leq m - 1$

$f(u_m u_1) = m$.

The edge sums of the inner cycle are 3, 4, ..., $m + 2$.

Step 3:

$f(u_i v_i) = 2i - 1$, $1 \leq i \leq m$

$f(u_{i+1}\, v_i) = 2i,\ 1 \leq i \leq m - 1$
$f(u_1\, v_m) = 2m.$
The edge sums of the alternating band of triangles are from $m + 3$ to $3m + 2$
Step 4:
$f(v_i\, v_{i+1}) = m + i,\ 1 \leq i \leq m - 1$
$f(v_m\, v_1) = 2m$. The edge sums of the middle cycle are from $3m + 3$ to $4m + 2$.
Step 5: For $1 \leq i \leq m$
$f(v_i\, w_i) = m + i.$
Thus the edge sums of the edges joining the middle cycle and outer cycle are $4m + 3$ to $5m + 2$.
Step 6:
$f(w_i\, w_{i+1}) = m + i,\ 1 \leq i \leq m - 1$
$f(w_m\, w_1) = 2m.$
The edge sums of the outer cycle are $5m + 3$ to $6m + 2$.
Output: $tes(R_m) = \lceil (6m + 2)/3 \rceil = 2m + 1$.

Proof of Correctness:
We know by actual verification that the edge sums obtained are all distinct. Hence R_m is total edge k-irregular. Labeling of R_5 is shown in Figure 2.
Remark: The above results holds for R_3 and R_4.

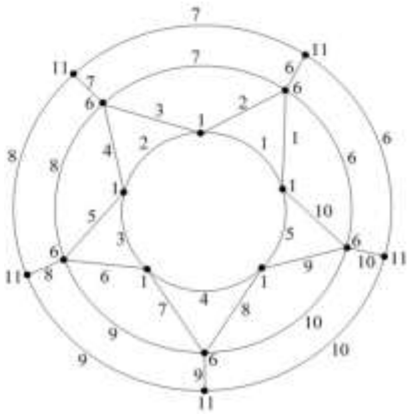

Figure 2: $tes(R_5) = 11$

2.2 Convex Polytope Q_m

Definition 2.2. The convex polytope Q_m can be obtained from S_m by deleting some lines (edges) that is $V(Q_m) = V(S_m)$ and $E(Q_m) = \{E(S_m) \setminus \{w_i\, w_{i+1} 1 \leq i \leq m - 1\}\} \cup \{w_m\, w_1\}$. See Figure 4. It consists of three-sided faces, four-sided faces, five-sided faces and m-sided face. The graph Q_m has two m-sided faces. One is the unbounded external face. We consider only the inner m-sided face. The vertex set and edge set of S_m are $V(S_m) = \{u_i, v_i, w_i, z_i, 1 \leq i \leq m\}$ and $E(S_m) = \{u_i\, u_{i+1}, 1 \leq i \leq m - 1\} \cup \{v_i\, v_{i+1}, 1 \leq i \leq m - 1\} \cup \{w_i\, w_{i+1}, 1 \leq i \leq m - 1\} \cup \{z_i\, z_{i+1}, 1 \leq i \leq m - 1\} \cup \{u_i\, v_i, 1 \leq i \leq m\} \cup \{v_i\, w_i, 1 \leq i \leq m\} \cup \{v_{i+1}\, w_i, 1 \leq i \leq m{-}1\} \cup \{w_i\, z_i, 1 \leq i \leq m\} \cup \{u_m\, u_1\} \cup \{v_m\, v_1\} \cup \{w_m\, w_1\} \cup \{z_m\, z_1\} \cup \{v_1 w_m\}$. See Figure 3.

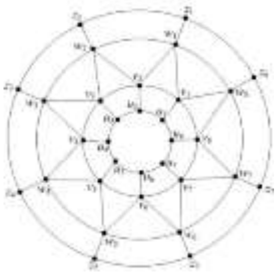

Figure 3: Convex Polytope S_8

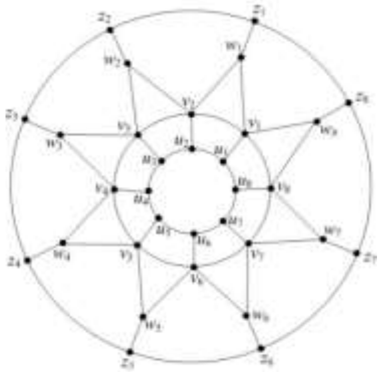

Figure 4: Convex Polytope Q_8

Theorem 2:

For every $m \geq 5$ the total edge irregularity strength of the convex polytope Q_m is $tes(Q_m) = \lceil (7m + 2)/3 \rceil = 2m + 3$.

Proof:

The number of vertices of Q_m is $4m$ and the number of edges of Q_m is $7m$. The vertices and edges of Q_m are traversed in the anticlockwise direction. First we label the vertices of the inner cycle, vertices of the middle cycle then the vertices joining the alternating band of triangles and outer cycle followed by the vertices of the outer cycle.The edges are also labeled in the same sequence so that the edge sums are consecutive.

Input: The graph of convex polytope Q_m for $m \geq 5$.

Algorithm:

Step 1:

For $1 \leq i \leq m$

$f(u_i) = 1$

$f(v_i) = m + 1$

$f(w_i) = 2m - 1$

$f(z_i) = 2m + 3$.

Step 2:

$f(u_i u_{i+1}) = i$, $1 \leq i \leq m - 1$

$f(u_m u_1) = m$.

The edge sums of the inner cycle are 3, 4, ..., $m + 2$.

Step 3:

The edges $u_i v_i$, $1 \leq i \leq m$ receive the labels from 1, 2, 3 to m so that the edge sums of the edges joining the inner cycle and middle cycle are from $m + 3$ to $2m + 2$.

Step 4:

$f(v_i v_{i+1}) = i$, $1 \leq i \leq m - 1$

$f(v_m v_1) = m$.

The edge sums of the middle cycle are $2m + 3$ to $3m + 2$.

Step 5:

$f(v_i w_i) = 2i + 1$, $1 \leq i \leq m$

$f(v_{i+1} w_i) = 2i + 2$, $1 \leq i \leq m - 1$

$f(v_1 w_m) = 2m + 2$.

The edge sums of the alternating band of triangles are from $3m + 3$ to $5m + 2$.

Step 6:

For $1 \le i \le m$
$f(w_i z_i) = m + i$.
Thus the edge sums of the edges joining the alternating band of triangles and outer cycle are $5m + 3$ to $6m + 2$.
Step 7:
$f(z_i z_{i+1}) = 2m + i - 4$, $1 \le i \le m - 1$
$f(z_m z_1) = 3m - 4$.
The edge sums of the outer cycle are $6m + 3$ to $7m + 2$.
Output: $tes(Q_m) = \lceil (7m + 2)/3 \rceil = 2m + 3$.
Proof of Correctness:
We know by actual verification that the edge sums obtained are all distinct. Hence Q_m is total edge k-irregular. Labeling of Q_5 is shown in Figure 5.
Remark: The above results holds for Q_3 and Q_4.

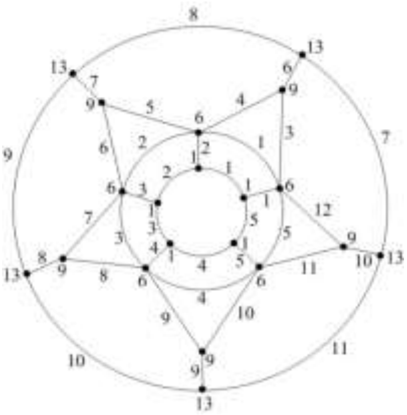

Figure 5: $tes(Q_5) = 13$

2.3 Convex Polytope B_m

Definition 2.3.1. The convex polytope B_m consists of $2m$ 4-sided faces, m 3- sided faces, m 5-sided faces and a pair of m-sided faces is obtained by the combination of the graph of convex polytope Q_m and graph of prism D_m.For our convenience, we call the cycle induced by $\{u_i, 1 \le i \le m\}$ the inner cycle, cycle induced by $\{v_i, 1 \le i \le m\}$ the interior cycle, the set of vertices $\{w_i, 1 \le i \le m\}$ the set of interior vertices, cycle induced by $\{y_i, 1 \le i \le m\}$ the exterior cycle, cycle induced by $\{z_i, 1 \le i \le m\}$ the outer cycle.The graph B_m has two m-sided faces. One is the unbounded external face. We consider only the inner m-sided face.

Notation:
The vertex set and edge set of B_m are defined as follows:
$V(B_m) = \{ u_i, v_i, w_i, y_i, z_i, 1 \le i \le m \}$ and $E(B_m) = \{ u_i u_{i+1}, v_i v_{i+1}, y_i y_{i+1}, z_i z_{i+1}, 1 \le i \le m\} \cup \{ u_i v_i, v_i w_i, v_{i+1} w_i, w_i y_i, y_i z_i, 1 \le i \le m\}$. See Figure 6.

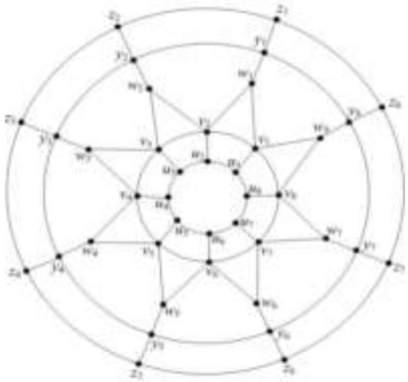

Figure 6: Convex Polytope B_8

Theorem 3:
For every $m \geq 5$ the total edge irregularity strength of the convex polytope B_m is $tes(B_m) = \lceil (9m + 2)/3 \rceil = 3m + 1$.

P roof: The number of vertices of B_m are $5m$ and the number of edges of B_m is $9m$. The vertices and edges of B_m are traversed in the anticlockwise direction. First we label the vertices of the inner cycle, the interior cycle, the set of interior vertices then the vertices of the exterior cycle followed by the vertices of the outer cycle. The edges are also labeled in the same sequence so that the edge sums are consecutive.

Input: The graph of convex polytope B_m for $m \geq 3$.

Algorithm:

Step 1: For $1 \leq i \leq m$

$f(u_i) = 1$

$f(v_i) = m + 1$

$f(w_i) = m + 3$

$f(y_i) = (3m + 1) - 1$.

$f(z_i) = 3m + 1$.

Step 2:

$f(u_iu_{i+1}) = i$, $1 \leq i \leq m - 1$

$f(u_mu_1) = m$.

Thus the edge sums of the inner cycle are 3, 4, ..., $m + 2$.

Step 3: The edges u_iv_i , $1 \leq i \leq m$ receive the labels from 1, 2, 3 to m so that the edge sums are $m + 3$ to $2m + 2$.

Step 4:

$f(v_iv_{i+1}) = i$, $1 \leq i \leq m - 1$

$f(v_mv_1) = m$.

Thus the edge sums of the interior cycle are from $2m + 3$ to $3m + 2$.

Step 5:

$f(v_1w_1) = m - 1$

$f(v_iw_i) = (m - 3) + 2i$, $2 \leq i \leq m$

$f(v_2w_1) = m$

$f(v_{i+1}w_i) = (m - 2) + 2i$, $2 \leq i \leq m - 1$

$f(v_1w_m) = 3m - 2$.

Thus the edge sums of the alternating band of triangles are from $3m + 3$ to $5m + 2$.

Step 6: For $1 \leq i \leq m$

$f(w_iy_i) = (m - 1) + i$.

Thus the edge sums of the edges joining the set of interior vertices and exterior cycle are $5m + 3$ to $6m + 2$.

Step 7:
$f(y_iy_{i+1}) = i + 2,\ 1 \le i \le m - 1$
$f(y_my_1) = m + 2$.
Thus the edge sums of the exterior cycle are $6m + 3$ to $7m + 2$.
Step 8: For $1 \le i \le m$
$f(y_iz_i) = m + i + 1$.
Thus the edge sums of the edges joining the vertices of the exterior cycle and outer cycle are $7m + 3$ to $8m + 2$.
Step 9:
$f(z_iz_{i+1}) = 2m + i,\ 1 \le i \le m - 1$
$f(z_mz_1) = 3m$. Thus the edge sums of the outer cycle are $8m + 3$ to $9m + 2$.
Output: $tes(B_m) = \lceil (9m + 2)/3 \rceil = 3m + 1$.

Proof of Correctness: We know by actual verification from the above stepwise procedure that the edge sums obtained are all distinct. Hence B_m is total edge k-irregular. Labeling of B_5 is shown in Figure 7.

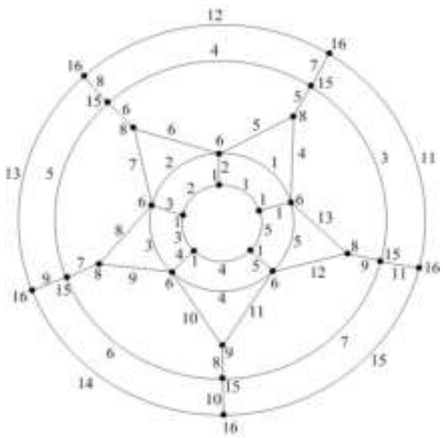

Figure 7: $tes(B_5) = 16$

3. Conclusion

In this paper, we have proved that the convex polytopes admits edge irregular total k-labeling. Our future study is extended to other structures of graphs.

Acknowlegement
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