TWO DIMENSIONAL POLAR COORDINATE SYSTEMS IN BI HARMONIC EQUATIONS

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Abstract

Planar half spaces are fundamental in analysis structure. However many mathematical method used to solve the equation associated with those structure their applicability. Finding a solution usually requires a solution process where the form of a solution is assumed so that governing equation are satisfied. This process is usually accomplished with the help of an Airy stress function that is determined based on a few assumptions about the relationship between the stress function and the true stress. The process is tedious and often a single solution that is valid for a single.

In this work we represent a general solution to a bi harmonic equation using Fourier series that works for a variety of boundary condition in polar coordinates. This function is simplified with minor assumption about its form and solved analytically. A representative problem and its solution is also shown.

Keywords: *Introduction -Equations in polar coordinates- The stress – strain relations in polar coordinates are similar to those in the rectangular coordinate system- A circular hole in an infinite sheet subject under remote shear-Problems.*

1. Introduction

The main purpose of this address is to bring to the attention of the workers in Airy stress function and related branches of applied mathematics a simple general method of solution of several important classes of 2-dimensional boundary value problem. A 2-dimensional polar coordinate study in rings and disks, curved bars of narrow rectangular cross section with a circular axis etc. Using the polar coordinates is advantageous to solve in airy stress function.

If we represent the material particle under the state of stress by a square in the (r,θ) coordinate system, the components of the stress state are σ_{rr}, $\sigma_{\theta\theta}$, $\tau_{r\theta}$.We know that the two sets of the stress components are related as

$$\sigma_{rr} = \frac{\sigma_{xx}+\sigma_{yy}}{2} + \frac{\sigma_{xx}-\sigma_{yy}}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

$$\sigma_{\theta\theta} = \frac{\sigma_{xx}+\sigma_{yy}}{2} - \frac{\sigma_{xx}-\sigma_{yy}}{2}\cos 2\theta - \tau_{xy}\sin 2\theta$$

$$\tau_{r\theta} = \frac{\sigma_{xx}-\sigma_{yy}}{2}\sin 2\theta + \tau_{xy}\cos 2\theta$$

Equations in polar coordinates:

The Airy stress function is a function of the polar coordinates, $\varphi(x, y)$. The stresses are expressed in terms of the Airy stress function

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} = 0$$

The Airy stress function $\varphi(x, y)$ also needs to satisfy the stress compatibility

$$\nabla^2\left(\sigma_{xx} + \sigma_{yy}\right) = 0$$

$$\nabla^2\nabla^2\varphi = 0$$

The bi harmonic equation is

$$\nabla^4\varphi = 0$$

The Airy stress function is a function of the polar coordinates, $\varphi(r, \theta)$. The stresses are expressed in terms of the Airy stress function

$$\nabla^2\nabla^2\varphi(r, \theta) = 0$$

$$\sigma_{rr} = \frac{\partial^2\varphi}{r^2\partial\theta^2} + \frac{\partial\varphi}{r\partial r}$$

$$\sigma_{\theta\theta} = \frac{\partial^2\varphi}{\partial r^2}$$

$$\tau_{r\theta} = -\frac{\partial}{\partial r}\left(\frac{\partial\varphi}{r\partial\theta}\right)$$

The bi harmonic equation is

$$\left(\frac{\partial^2}{\partial r^2} + \frac{\partial}{r\partial r} + \frac{\partial^2}{r^2\partial\theta^2}\right)\left(\frac{\partial^2\varphi}{\partial r^2} + \frac{\partial\varphi}{r\partial r} + \frac{\partial^2\varphi}{r^2\partial\theta^2}\right) = 0$$

The stress – strain relations in polar coordinates are similar to those in the rectangular coordinate system:

$$\varepsilon_{rr} = \frac{\sigma_{rr}}{E} - v\frac{\sigma_{\theta\theta}}{E}$$

$$\varepsilon_{\theta\theta} = \frac{\sigma_{\theta\theta}}{E} - v\frac{\sigma_{rr}}{E}$$

$$\gamma_{r\theta} = \frac{2(1+v)}{E}\tau_{r\theta}$$

The strain –displacement relations are

$$\varepsilon_{rr} = \frac{\partial u_r}{\partial r}$$

$$\varepsilon_{\theta\theta} = \frac{u_r}{r} + \frac{\partial u_\theta}{r\,\partial\theta}$$

$$\gamma_{r\theta} = \frac{\partial u_r}{r\partial\theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}$$

A stress field symmetric about an axis.

Let the Airy stress function be $\varphi(r)$.

The bi harmonic equation is

$$\nabla^4\varphi(r) = 0$$

$$\nabla^2\nabla^2\varphi(r) = 0$$

$$\left(\frac{d^2}{dr^2} + \frac{d}{rdr}\right)\left(\frac{d^2\varphi(r)}{dr^2} + \frac{d\varphi(r)}{rdr}\right) = 0$$

This equation has the same dimension in the independent variable r such an ODE is known as an equi-dimensional equation. A solution to an equi-dimensional equation is of the form

$$\varphi = r^m$$

Inserting into the bi harmonic equation, we obtain that

$$m^2(m-2)^2 = 0$$

$$m = 0,0,2,2$$

The fourth order has a double root of 0 and a double root of 2.

The general solution to the ODE is

$$\varphi(r) = C_1 \log r + C_2 r^2 \log r + C_3 r^2 + C_4$$

Where C_1, C_2, C_3 and C_4 are constants. The components of the stress field are

$$\sigma_{rr} = \frac{\partial^2\varphi}{r^2\partial\theta^2} + \frac{\partial\varphi}{r\partial r} = \frac{C_1}{r^2} + C_2(1 + 2\log r) + 2C_3$$

$$\sigma_{\theta\theta} = \frac{\partial^2\varphi}{\partial r^2} = -\frac{C_1}{r^2} + C_2(3 + 2\log r) + 2C_3$$

$$\tau_{r\theta} = -\frac{\partial}{\partial r}\left(\frac{\partial\varphi}{r\partial\theta}\right) = 0$$

The stress field is linear in C_1, C_2 and C_3 .

The stress field in the sheet is

$$\sigma_{rr} = S(1 - (\frac{a}{r})^2)$$

$$\sigma_{\theta\theta} = S(1 + (\frac{a}{r})^2)$$

The stress concentration factor of this hole is 2. We may the spherical cavity in an infinite elastic solid under remote tension:

$$\sigma_{rr} = S(1 - \frac{1}{2}(\frac{a}{r})^3)$$

$$\sigma_{\theta\theta} = S(1 + \frac{1}{2}(\frac{a}{r})^3)$$

The stresses can be obtained from the Airy stress function

$$= \frac{\sigma_0}{4}\mathrm{r}^2(1 - \cos 2\theta)$$

Problem

We will write the Airy stress function and the stresses in polar coordinates for a plate pulled in the X-direction by a stress σ_0.

$$\sigma_x = \sigma_0, \qquad \sigma_y = \tau_{xy} = 0$$

The Airy stress function that would give this stress is

$$\varphi = \frac{\sigma_0}{2}y^2$$

$$\mathrm{y} = \mathrm{r}\sin\theta$$

$$\varphi = \frac{\sigma_0}{2}(\mathrm{r}\sin\theta)^2$$

$$\varphi = \frac{\sigma_0}{2} \mathrm{r}^2 (\sin\theta)^2$$
$$\sigma_r = \frac{\sigma_0}{2}(1 + \cos 2\theta)$$
$$\tau_{r\theta} = \frac{\sigma_0}{2}\sin\theta$$
$$\sigma_\theta = \frac{\sigma_0}{2}(1 - \cos 2\theta)$$

A circular hole in an infinite sheet subject under remote shear:

The sheet is a state of pure shear

$$\tau_{xy} = S, \sigma_{xx} = \sigma_{yy} = 0$$

We know that

$$\sigma_{rr} = \frac{\partial^2\varphi}{r^2\partial\theta^2} + \frac{\partial\varphi}{r\partial r}$$
$$\sigma_{\theta\theta} = \frac{\partial^2\varphi}{\partial r^2}$$
$$\tau_{r\theta} = -\frac{\partial}{\partial r}\left(\frac{\partial\varphi}{r\partial\theta}\right)$$

The stress function must be in the form

$$\varphi(r,\theta) = f(r)\sin 2\theta$$

The bi harmonic equation is

$$\left(\frac{\partial^2}{\partial r^2} + \frac{\partial}{r\partial r} + \frac{\partial^2}{r^2\partial\theta^2}\right)\left(\frac{\partial^2\varphi}{\partial r^2} + \frac{\partial\varphi}{r\partial r} + \frac{\partial^2\varphi}{r^2\partial\theta^2}\right) = 0$$
$$\left(\frac{\partial^2}{\partial r^2} + \frac{\partial}{r\partial r} - \frac{4}{r^2}\right)\left(\frac{\partial^2 f}{\partial r^2} + \frac{\partial f}{r\partial r} - \frac{4f}{r^2}\right) = 0$$

A solution to this equi-dimentional

$$f(r) = r^m$$

Inserting into the bi harmonic equation, we obtain that

$$((m-2)^2 - 4)^2(m^2 - 4) = 0$$

$(m-2)^2 - 4)^2 = 0$ or $(m^2 - 4) = 0$

$$m = 2, -2, 0, 4$$

The stress function is

$$\varphi(r,\theta) = \left(C_1 r^2 + C_2 r^4 + \frac{C_3}{r^2} + C_4\right)\sin 2\theta$$

Where C_1, C_2, C_3 and C_4 are constants. The components of the stress field are

$$\sigma_{rr} = \frac{\partial^2\varphi}{r^2\partial\theta^2} + \frac{\partial\varphi}{r\partial r} = -\left(2C_1 + \frac{6C_3}{r^4} + \frac{4C_4}{r^2}\right)\sin 2\theta$$
$$\sigma_{\theta\theta} = \frac{\partial^2\varphi}{\partial r^2} = \left(2C_1 + 12C_2 r^2 + \frac{6C_3}{r^4}\right)\sin 2\theta$$
$$\tau_{r\theta} = -\frac{\partial}{\partial r}\left(\frac{\partial\varphi}{r\partial\theta}\right) = \left(-2C_1 - 6C_2 r^2 + \frac{6C_3}{r^4} + \frac{2C_4}{r^2}\right)\cos 2\theta$$

To determine the constants C_1, C_2, C_3 and C_4, We invoke the boundary conditions:

1. Remote from the hole, namely, $r \to \infty, \sigma_{rr} = S\sin 2\theta$ $, \tau_{r\theta} = S\cos 2\theta$, giving $C_1 = -\frac{S}{2}, C_2 = 0$.
2. O n the surface of the hole, namely, $r = a$, $\sigma_{rr} = 0$ $, \tau_{r\theta} = 0$, giving $C_3 = -\frac{Sa^4}{2}$, $C_4 = Sa^2$.

The stress field inside the sheet is

$$\sigma_{rr} = S[1 + 3(\frac{a}{r})^4 - 4(\frac{a}{r})^2]\sin 2\theta$$

$$\sigma_{\theta\theta} = S[1 + 3(\frac{a}{r})^4]\sin 2\theta$$

$$\tau_{r\theta} = S[1 - 3(\frac{a}{r})^4 + 2(\frac{a}{r})^2]\cos 2\theta$$

Problem:

Use the Airy stress function to guess at a solution for a hole in a plate problem.

Using the Airy stress function for a plate with no hole, We guess the solution for the plate with a hole to be of the form $\varphi(r,\theta) = f_1(r) + f_2(r)\cos 2\theta$

The Airy stress function in to the bi harmonic equation in polar coordinate .We obtain

$$\left(\frac{d^2}{dr^2} + \frac{d}{rdr}\right)\left(\frac{d^2 f_1}{dr^2} + \frac{df_1}{rdr}\right) = 0$$

$$\left(\frac{d^2}{dr^2} + \frac{d}{rdr} - \frac{4}{r^2}\right)\left(\frac{d^2 f_2}{dr^2} + \frac{df_2}{rdr} - \frac{4f_2}{r^2}\right) = 0$$

$$\frac{d}{rdr}\{r\frac{d}{dr}[\frac{d}{rdr}\left(r\frac{df_1}{rdr}\right)]\} = 0$$

$$\frac{d}{rdr}(\frac{d}{r^3dr}\{r^3\frac{d}{dr}[\frac{d}{r^3dr}(rf_2)]\} = 0$$

The solution are

$$f_1 = C_1 r^2\log r + C_2 r^2 + C_3\log r + C_4$$

$$f_2 = C_5 r^2 + C_6 r^4 + \frac{C_7}{r^2} + C_8$$

The stresses are

$$\sigma_r = C_1(1 + 2\log r) + 2C_2 + \frac{C_3}{r^2} - (2C_5 + \frac{6C_7}{r^4} + \frac{4C_8}{r^2})\cos 2\theta$$

$$\sigma_\theta = C_1(3 + 2\log r) + 2C_2 - \frac{C_3}{r^2} + (2C_5 + 12C_6 r^2 + \frac{6C_7}{r^4})\cos 2\theta$$

$$\tau_{r\theta} = (2C_5 + 6C_6 r^2 - \frac{6C_7}{r^4} - \frac{2C_8}{r^2})\sin 2\theta$$

The constants we use the boundary conditions:

C_1 and C_6 become zeros because the stresses must be finite as $r \to \infty$,they must assume the forms for the plate with no hole giving us the expression for C_2 and C_5

$$\sigma_0 = -4C_5 = 4C_2$$

At $r = a, \sigma_r = \tau_{r\theta} = 0$ giving as the relations

$$2C_2 + \frac{C_8}{a^2} = 0$$

$$2C_5 + \frac{6C_7}{a^4} + \frac{4C_8}{a^2} = 0$$

$$2C_5 - \frac{6C_7}{a^4} - \frac{2C_8}{a^2} = 0$$

Finally, stresses are

$$\sigma_r = \frac{1}{2}\sigma_0[(1 - \left(\frac{a}{r})^2\right) + (1 + 3\frac{a^4}{r^4} - 4\frac{a^2}{r^2})\cos 2\theta$$

$$\sigma_\theta = \frac{1}{2}\sigma_0[(1 - \left(\frac{a}{r})^2\right) + (1 + 3\frac{a^4}{r^4})\cos 2\theta$$

$$\tau_{r\theta} = \frac{-1}{2}\sigma_0(1 - 3\frac{a^4}{r^4} + 2\frac{a^2}{r^2})\sin 2\theta$$

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