DECOMPOSITION OF COMPLETE BIPARTITE GRAPHS INTOPATHS AND CYCLES USING 2−SIMPLEGRAPHOIDAL COVERS

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Abstract. Every nation's economy is centered on its transportation networks, which are also reshaping the global economy. Utilize graph decomposition techniques to optimize transportation networks to save travel times and fuel expenses. A 2−simple graphoidal cover(2-simple g.c) of *G* is a set ψ_G of (not necessarily open) paths in *G* such that every edge is in exactly one path in ψ_G and every vertex is an internal vertex of at most two paths in ψ_G and any two paths in ψ_G has at most one vertex in common. The minimum cardinality of the 2−simple graphoidal cover (2-simple g.c) of *G* is called the 2−simple graphoidal covering number of *G* and is denoted by η_{2s}. In this study, we discuss decomposition of complete bipartite graphs using 2-simple graphoidal covers.

Keywords: Simple Graphoidal Cover, 2-Simple graphoidal cover, bicyclic graphs
AMS Subject Classification: 05C70, 05C76

1. Introduction

A graph's decomposition is a collection of edge-disjoint subgraphs $G_i, i = 1, 2...n$ of the same graph *G*, where each edge of the original graph *G* is contained in exactly one G_i. A number of writers to discover several types of graph decomposition, apply different conditions and parameters. Acharya and Sampath Kumar [1, 2] developed the concept of graphoidal cover(g.c). Arumugam and Shahul Hamid developed the concept of a simple graphoidal cover (simple g.c) in their paper [4]. Das and Ratan Singh [5] proposed the idea of a 2-graphoidal cover. Motivation of 2-graphoidal cover, Venkat narayanan et al. [9] developed and discussed the idea of a 2-simple graphoidal cover on standard graphs. In this paper the authors discuss decompositions of complete bipartite graphs into paths and cycles. In chemical reactions and molecular interactions, complete bipartite graphs can be used to represent the association between two sets of molecules or functional groups. For example, in drug design, a complete bipartite graph can represent the interactions between a set of ligands and a set of target receptor sites. This representation helps in understanding the binding affinities and designing effective drug molecules

2. Preliminaries

A finite, simple, non-trivial, connected, and undirected graph is referred to as $G = (V,E)$. The symbols p and q, which stand for the number of elements in V , or order, and the number of elements in E, or size of G, respectively. For graph theoretic terminology we refer to Harary [6]

Definition 2.1. *[1] A graphoidal cover(g.c) of G is a set ψ_G of (not necessarily open) paths in G satisfying the following conditions.*

(i) *Every path in ψ_G has at least two vertices.*

(ii) *Every vertex of G is an internal vertex of at most one path in ψ_G*

(iii) *Every edge of G is in exactly one path in ψ_G.*

The minimum cardinality of a graphoidal cover of G is called the graphoidal covering number of G and is denoted by η.

Definition 2.2. *[4] A Simple graphoidal cover (simple g.c) of a graph G is a graphoidal cover ψ_G of G such that any two paths in ψ_G have at most one vertex in common. The minimum cardinality of a simple graphoidal cover of G is called simple graphoidal covering number of G and is denoted by η_s.*

Definition 2.3. *[9] A 2-simple graphoidal covering (2-simple g.c) of a graph G is a set ψ_G of paths in G such that every edge is in exactly one path in ψ_G, every vertex is an internal vertex of at most two paths and any two paths in ψ_G have at most one vertex in common. The minimum cardinality of 2-simple graphoidal cover ψ_G of G is known as 2-simple graphoidal covering number of G.*

Theorem 2.1. *[9] For any (p, q) graph, $\eta_{2s}(G) = q - p - t_2 + t$, where t_2 denotes the total number of internal vertices that appear exactly twice in paths of ψ_G, whereas t denotes the total number of external vertices in the paths of ψ_G.*

Corollary 2.1. *For any graph G, the following are equivalent*

(i) $\eta_{2s}(G) = q - p - t_2$.

(ii) *There exists a 2-simple g.c of G without exterior vertices*

Corollary 2.2. *There exists a 2−simple g.c ψ_G of G in which every vertex is an internal vertex in exactly 2 paths in ψ_G of if and only if $\eta_{2s}(G) = q - 2p$.*

3. Main Results

Theorem 3.1. *For the complete bipartite graph $K_{r,s}$, $r \geq 1, s \geq 1$*

$$(i) \quad \eta_{2s}(K_{1,s}) = \begin{cases} 1 & if \; s=1\,(or)\,s=2 \\ 2 & if \; s=3 \\ s-2 & if \; s \geq 4 \end{cases}$$

$$(ii) \quad \eta_{2s}(K_{2,s}) = \begin{cases} 1 & if \; s=2 \\ 3 & if \; s=3 \\ 4 & if \; s=4 \\ 5 & if \; s=5 \\ 2s-6 & if \; s \geq 6 \end{cases}$$

$$(ii) \quad \eta_{2s}(K_{3,s}) = \begin{cases} 5 & if \; s=3 \\ 6 & if \; s=4 \\ 7 & if \; s=5 \\ 8 & if \; s=6 \\ 3s-12 & if \; s \geq 7 \end{cases}$$

$$(iv) \quad \eta_{2s}(K_{4,s}) = \begin{cases} 8 & if \; s=4\,(or)\;5\;(or)\;6 \\ 10 & if \; s=7 \\ 13 & if \; s=8 \\ 4s-20 & if \; s \geq 9 \end{cases}$$

$$(v) \quad \eta_{2s}(K_{5,s}) = \begin{cases} 10 & if \; s=5 \\ 12 & if \; s=6,7 \\ q-2p & if \; s=8,9 \\ 21 & if \; s=10 \\ 5s-30 & if \; s \geq 11 \end{cases}$$

$$(vi) \quad \eta_{2s}(K_{6,s}) = \begin{cases} 14 & if \; s=6 \\ 17 & if \; s=7 \\ q-2p & if \; s=8,9,10,11,12,13 \\ 6s-39 & if \; s \geq 14 \end{cases}$$

$$(vii) \quad \eta_{2s}(K_{7,s}) = \begin{cases} q-2p & if \; 7 \leq s \leq 14 \\ 7s-42 & if \; s > 14 \end{cases}$$

Proof. It is observed that for any 2−simple g.c of $K_{r,s}$ any member of ψ_G is either a cycle of length 4 (Or) a path of length ≤ 2.

(i) Now let $X = \{r_1\}$ and $Y = \{y_1, y_2, y_3, ..., y_s\}$ be the bipartition of $K_{1,s}$ with $p = 1 + s$, $q = s$.

Case 1. Since $K_{1,1}$ and $K_{1,2}$ are paths. Therefore $\eta_{2s}(G) = 1$.

Case 2. When $s = 3$

Then $K_{1,3}$ is a tree with 3 pendant vertices and no vertex is of degree ≥ 4. Therefore $\eta_{2s}(K_{1,3}) = 3-1-0 = 2.$

Case 3. When $s \geq 4$

Then $K_{1,s}$ is a tree with s pendant vertices and one vertex is of degree ≥ 4. Therefore $\eta_{2s}\left(K_{1,s}\right) = s-1-1 = s-2.$

(ii) Now let $X = \{r_1, r_2\}$ and $Y = \{y_1, y_2, y_3, ..., y_s\}$ be the bipartition of $K_{2,s}$ with $p = 2 + s$, $q = 2s$.

Case 1. When s = 2

Then $\eta_{2s}\left(K_{2,2}\right) = \left(r_1, y_1, r_2, y_2, r_1\right) = 1.$

Case 2. When s = 3

Then $\psi_G = \{(r_1, y_1, r_2, y_2, r_1), (r_1, y_3), (r_2, y_3)\}$ is a 2−simple g.c of $K_{2,3}$ so that $\eta_{2s}\left(K_{2,3}\right) \leq 3$. Now, let ψ_G be any 2−simple g.c of $K_{2,3}$. Since no vertices is of degree ≥ 4, $t_2 = 0$ and if ψ_G contains one cycle, then $t_\psi = 2$ otherwise $t_\psi \geq 3$. Hence $t_2 = 0, t \geq 2$ so that $\eta_{2s}\left(K_{2,3}\right) = q - p - t_2 + t \geq 6 - 5 - 0 + 2 = 3$. Thus $\eta_{2s}\left(K_{2,3}\right) = 3.$

Case 3. When $s = 4$

Then $\psi_G = \{(r_1, y_1, r_2, y_2, r_1), (y_3, r_1, y_4)(r_2, y_3)(r_2, y_4)\}$ is a 2−simple g.c of $K_{2,4}$ so that $\eta_{2s}\left(K_{2,4}\right) \leq 4$. Now, let ψ_G be any 2−simple g.c of $K_{2,4}$ with $|\psi_G| \leq 3$. Case in which $|\psi_G| = 1$, is not possible since any paths in $K_{2,4}$ is either a cycle (or) path. If $|\psi_G| = 2$, then ψ_G contains exactly two cycles . If $|\psi_G| = 3$, then ψ_G contains exactly one cycle and two paths. In both cases, any two paths(cycles) in ψ_G contains more than one common vertex which is a contradiction. Therefore $|\psi_G| \geq 4$. Hence $\eta_{2s}\left(K_{2,4}\right) \geq 4$. Thus $\eta_{2s}\left(K_{2,4}\right) = 4$.

Case 4. When s = 5

Then $\psi_G = \{(r_1, y_1, r_2, y_2, r_1), (y_3, r_1, y_4), (y_3, r_2, y_5), (r_1, y_5), (r_2, y_4)\}$ is a 2−simple g.c of $K_{2,5}$ so that $\eta_{2s}(K_{2,5}) \leq 5$. Now, let ψ_G be any 2−simple g.c of $K_{2,5}$ with $|\psi_G| \leq 4$. Case in which $|\psi_G| = 1(or)2$, is not possible since any paths in $K_{2,5}$ is either a cycle or path. If $|\psi_G| = 3$, then ψ_G contains two cycles and a path. If $|\psi_G| = 4$, then ψ_G contains two cycles and two edges. In both cases, any two paths(cycles) in ψ contains more than one common vertex which is a contradiction. Therefore $|\psi_G| \geq 5$. Hence $\eta_{2s}\left(K_{2,5}\right) \geq 5$. Thus $\eta_{2s}\left(K_{2,5}\right) = 5$.

Case 5. When $s \geq 6$

Then the collection paths are $P_1 = (r_1, y_1, r_2, y_2, r_1)$, $P_2 = (y_3, r_1, y_4), P_3 = (y_3, r_2, y_5)$ and $P_4 = (y_5, r_1, y_6)$. Then $\psi_G = \{P_i : i = 1, 2, 3, 4\} \cup \{Q\}$, where Q is the set of edges of $K_{2,s}$

not covered by $\{P_i : i = 1,2,3,4\}$ is a 2−simple g.c of $K_{2,s}$ so that $|\psi_G| = 4 + (2s-10) = 2s-6$. Hence $\eta_{2s}(K_{2,s}) \le 2s-6$. Now, let ψ_G be any 2−simple g.c of $K_{2,s}$. If ψ_G contains a cycle and three paths of length 2, the $t_2(\psi) = 2, t_\psi \ge (s-2)$ otherwise $t_2(\psi) = 2,\ t_\psi \ge (n-1)$. Hence $t_2 = 2, t \ge (s-2)$ so that $\eta_{2s}(K_{2,s}) \ge 2s - (2+s) - 2 + (s-2) = 2s-6$ Thus $\eta_{2s}(K_{2,s}) = 2s-6$.

(iii) Now let $X = \{r_1, r_2, r_3\}$ and $Y = \{y_1, y_2, y_3, ..., y_s\}$ be the bipartition of $K_{3,s}$ with $p = 3 + s$, $q = 3s$.

Case 1. When $s = 3$

Then $\psi = \{(r_1, y_1, r_2, y_2, r_1), (y_1, r_3, y_3), (r_1, y_3), (r_2, y_3), (r_3, y_2)\}$ is a 2−simple g.c of $K_{3,3}$ so that $\eta_{2s}(K_{3,3}) \le 5$. Now, let ψ_G be any 2−simple g.c of $K_{3,3}$. Since no vertices is of degree ≥ 4, $t_2 = 0$ and If ψ_G contains a cycle and a path, then $t_\psi = 2$ otherwise $t_\psi \ge 2$. Hence $t = 2, t_2 = 0$ so that $\eta_{2s}(K_{3,3}) \ge 9-6-0+2 = 5$. Thus $\eta_{2s}(K_{3,3}) = 5$.

Case 2. When $s = 4$

Then $\psi_G = \{(r_1, y_1, r_2, y_2, r_1), (r_1, y_3, r_3, y_4, r_1), (r_2, y_3), (r_2, y_4), (r_3, y_1), (r_3, y_2)\}$ is a 2−simple g.c of $K_{3,4}$ so that $\eta_{2s}(K_{3,4}) \le 6$. Now, let ψ_G be any 2−simple g.c of $K_{3,4}$ with $|\psi_G| \le 5$. Cases in which $|\psi_G| = 1(or)2$, is not possible since any paths in $K_{3,4}$ is either a cycle or path. If $|\psi_G| = 3$, then ψ_G contains exactly three cycles. If $|\psi_G| = 4$, then ψ_G contains two cycles and two paths of length 2. If $|\psi_G| = 5$, then ψ_G contains exactly one cycle and four paths of length 2. In all cases, any two paths(cycles) in ψ_G contains more than one common vertex which is a contradiction. Therefore $|\psi_G| \ge 6$. Hence $\eta_{2s}(K_{3,4}) \ge 6$. Thus $\eta_2(K_{3,4}) = 6$.

Case 3. When $s = 5$

Then $\psi_G = \{(r_1, y_1, r_2, y_2, r_1), (y_1, r_3, y_3), (y_2, r_3, y_4), (y_3, r_1, y_4), (y_3, r_2, y_5), (r_1, y_5, r_3), (r_2, y_4)\}$ is a 2−simple g.c of $K_{3,5}$ so that $\eta_{2s}(K_{3,5}) \le 7$. Now, let ψ_G be any 2−simple g.c of $K_{3,5}$ with $|\psi_G| \le 5$. Cases in which $|\psi_G| = 1(or)2(or)3(or)4$, is not possible since any paths in $K_{3,5}$ is either a cycle or path. If $|\psi_G| = 5$, then ψ_G contains exactly three cycle and one paths of length 2 and an edge. If $|\psi_G| = 6$, then ψ_G contains exactly three cycles and three edges of length 1. In all cases, any two cycles in ψ_G contains more than one common vertex which is a contradiction. Therefore $|\psi_G| \ge 7$. Hence $\eta_{2s}(K_{3,5}) \ge 7$. Thus $\eta_2(K_{3,5}) = 7$.

Case 4. When $s = 6$

Then $\psi = \{(r_1, y_1, r_2, y_2, r_1), (r_2, y_3, r_3, y_5, r_2), (y_5, r_1, y_6), (y_3, r_1, y_4), (y_4, r_2, y_6), (y_2, r_3, y_6), (r_3, y_1), (r_3, y_4)\}$ is a 2−simple g.c of $K_{3,6}$ so that $\eta_{2s}(K_{3,6}) \le 8$. Now, let ψ_G be any

2−simple g.c of $K_{3,6}$. If ψ_G contains two cycles and four paths of length 2, then $t_2(\psi)=3, t_\psi=2$ otherwise $t_2(\psi)\le 3, t_\psi \ge 4$. . Hence $t_2 \le 3, t \ge 2$ so that $\eta_{2s}(K_{3,6}) \ge 18-9-3+2=8$. Hence $\eta_{2s}(K_{3,6})=8$.

Case 5. When $s \ge 7$

Then the collection of paths are $P_1=(r_1,y_1,r_2,y_2,r_1), P_2=(r_2,y_3,r_3,y_4,r_2), P_3=(r_3,y_5,r_1,y_6,r_3), P_4=(y_3,r_1,y_7), P_5=(y_6,r_2,y_7)$ & $P_6=(y_2,r_3,y_7)$. Then $\psi=\{P_i, i=1,..,6\}\cup\{Q\}$ where Q is set of the edges of $K_{3,s}$ not covered by $\{P_i, i=1,2,...,6\}$ is a 2−simple g.c of $K_{3,s}$ so that $|\psi_G|=6+(3s-18)=3s-12$. Hence $\eta_{2s}(K_{3,s})\le 3s-12$. Now, let ψ_G be any 2−simple graphoidal path cover of $K_{3,s}$. If ψ_G contains three cycles and three paths, then $t_2(\psi)=3, t_\psi \ge (s-6)$ otherwise $t_2(\psi)=3, t_\psi \ge (s-3)$. Hence $t_2=3, t\ge(s-6)$ so that $\eta_{2s}(K_{3,s})=q-p-t_2+t\ge 3s-(3+s)-3+(s-6)=3s-12$. Hence $\eta_{2s}(K_{3,s})=3s-12$.

(iv) Now let $X=\{r_1,r_2,r_3,r_4\}$ and $Y=\{y_1,y_2,y_3,...,y_s\}$ be the bipartition of $K_{4,s}$ with $p=4+s$, $q=4s$.

Case 1. When $s=4$

Then $\psi_G=\{(r_1,y_1,r_2,y_2,r_1),(r_3,y_1,r_4,y_3,r_3),(r_1,y_4,r_3),(r_2,y_4,r_4),(r_1,y_3),(r_2,y_3),(r_3,y_2),(r_4,y_2)\}$ is a 2−simple g.c of $K_{4,4}$ so that $\eta_{2s}(K_{4,4})\le 8$. Now, let ψ_G be any 2−simple g.c of $K_{4,4}$ with $|\psi_G|\le 7$. Cases in which $|\psi_G|=1(or)2(or)3$ is not possible, since any paths in $K_{4,4}$ is either a cycle (or) path. If $|\psi_G|=4$, then ψ_G contains exactly four cycles. If $|\psi_G|=5$, then ψ_G contains exactly three cycles and two paths. If $|\psi_G|=6$, then ψ_G contains exactly two cycles and four paths. If $|\psi_G|=7$, then ψ_G contains exactly a cycle and six paths. In all cases, any two cycles in ψ_G contains more than one common vertex which is a contradiction. Therefore $|\psi_G|\ge 8$. Hence $\eta_{2s}(K_{4,4})\ge 8$. Thus $\eta_{2s}(K_{4,4}) = 8$.

Case 2. When $s=5$

Then $\psi_G=\{(r_1,y_1,r_2,y_2,r_1),(r_2,y_3,r_3,y_4,r_2),(r_1,y_4,r_4,y_5,r_1),(y_1,r_4,y_3),(y_1,r_3,y_5),(r_3,y_2,r_4),(r_1,y_3),(r_2,y_5)\}$ is a 2−simple g.c of $K_{4,5}$ so that $\eta_{2s}(K_{4,5})\le 8$. Now, let ψ_G be any 2−simple g.c of $K_{4,5}$. If ψ_G contains three cycles and three paths, then $t_2(\psi)=4, t_\psi=1$ otherwise $t_2(\psi)=4, t_\psi\ge 4$. Hence $t_2=4, t\ge 1$ so that $\eta_{2s}(K_{4,5})\ge 20-9-4+1=8$. Thus $\eta_{2s}(K_{4,5})=8$.

Case 3. When $s=6$

Then $\psi = \{(r_1, y_1, r_2, y_2, r_1), (r_2, y_3, r_3, y_4, r_2), (r_3, y_1, r_4, y_6, r_3), (r_4, y_3, r_1, y_5, r_4), (y_4, r_1, y_6)$ $(y_5, r_2, y_6), (y_2, r_3, y_5), (y_2, r_4, y_4)\}$ is a 2−simple g.c of $K_{4,6}$ so that $\eta_{2s}(K_{4,6}) \leq 8$.Now, let ψ_G be any 2−simple g.c of $K_{4,8}$. If ψ_G contains four cycles and four paths, then $t_2(\psi) = 6$, $t_\psi = 0$, otherwise $t_2(\psi) \leq 4, t_\psi \geq 3$. Hence $t_2 \leq 6, t \geq 0$ so that $\eta_{2s}(K_{4,6}) \geq 8$. Thus $\eta_{2s}(K_{4,6}) = 8$.

Case 4. When $s = 7$

Then $\psi = \{(r_1, y_1, r_2, y_2, r_1), (r_2, y_3, r_3, y_4, r_2), (r_3, y_1, r_4, y_6, r_3), (r_4, y_3, r_1, y_5, r_4), (r_2, y_7, r_4)$ $,(r_1, y_7, r_3), (y_4, r_1, y_6), (y_5, r_2, y_6), (y_2, r_3, y_5), (y_2, r_4, y_4)\}$ is a 2−simple g.c of $K_{4,7}$ so that $\eta_{2s}(K_{4,7}) \leq 10$. Now, let ψ_G be any 2−simple g.c of $K_{4,7}$. If ψ_G contains four cycles and four paths , then $t_2(\psi) = 7, t_\psi = 0$, otherwise $t_2(\psi) \leq 5, t_\psi \geq 3$. Hence $t_2 \leq 7, t \geq 0$ so that $\eta_{2s}(K_{4,7}) \geq 28 - 11 - 7 + 0 = 10$. Thus $\eta_{2s}(K_{4,7})$=10.

Case 5. When $s = 8$

Then the collection of paths are $\{(r_1, y_1, r_2, y_2, r_1), (r_2, y_3, r_3, y_4, r_2), (r_2, y_5, r_4, y_6, r_2),$ $(r_4, y_3, r_1, y_7, r_4), (r_1, y_5, r_3, y_8, r_1), (y_4, r_1, y_6), (y_7, r_2, y_8), (r_3, y_1, r_4), (y_2, r_4, y_8), (r_3, y_2), (r_3,$ $y_6), (r_3, y_7), (r_4, y_4)\}$ is a 2−simple g.c of $K_{4,8}$ so that $\eta_{2s}(K_{4,8}) \leq 13$. Now, let ψ_G be any 2−simple g.c of $K_{4,8}$. If ψ_G contains five cycles and four paths, then $t_2(\psi) = 7, t_\psi = 0$ otherwise $t_2(\psi) \leq 5, t_\psi \geq 3$. Hence $t_2 \leq 7, t \geq 0$ so that $\eta_{2s}(K_{4,8}) \geq 32 - 12 - 7 + 0 = 13$. Thus $\eta_{2s}(K_{4,8}) = 13$.

Case 6. When $s \geq 9$

Then the collection of paths are $P_1 = (r_1, y_1, r_2, y_2, r_1), P_2 = (r_2, y_3, r_3, y_4, r_2), P_3 = (r_2, y_5,$ $r_4, y_6, r_2), P_4 = (r_4, y_3, r_1, y_7, r_4), P_5 = (r_1, y_5, r_3, y_8, r_1), P_6 = (r_3, y_2, r_4, y_9, r_3), P_7 = (y_4, r_1, y_6)$ and $P_8 = (y_7, r_2, y_8)$. Then $\psi = \{P_i : i = 1, 2, ..., 8\} \cup Q$ where Q is a set of edges of $K_{4,s}$ not covered by $\{P_i : i = 1,..,8\}$, is a 2−simple g.c of $K_{4,s}$ so that $|\psi_G| = 8 + (4s - 28) = 4s - 20$ Hence $\eta_{2s}(K_{4,s}) \leq 4s - 20$. Now, let ψ_G be any 2−simple graphoidal path cover of $K_{4,s}$. If ψ_G contains six cycles and two paths , then $t_2(\psi) = 7, t_\psi = s - 9$, otherwise $t_2(\psi) \leq 5$, $t_\psi \geq s - 5$. Hence $t_2 \leq 7, t \geq s - 9$ so that $\eta_{2s}(K_{4,s}) \geq 4s - (4 + s) - 7 + (s - 9) = 4s - 20$. –20. Thus $\eta_{2s}(K_{4,s}) = 4s - 20$.

(v) Now let $X = \{r_1, r_2, r_3, r_4, r_5\}$ and $Y = \{y_1, y_2, y_3, ..., y_s\}$ be the bipartition of $K_{5,s}$ with p =5 + s, q = 5s.

Case 1. When $s = 5$

Then $\psi = \{(r_1, y_1, r_2, y_2, r_1), (r_2, y_3, r_4, y_4, r_2), (r_4, y_1, r_5, y_5, r_4), (r_3, y_3, r_1, y_5, r_3), (y_1, r_3,$ $y_4)(y_2, r_5, y_4), (r_3, y_2, r_4), (r_1, y_4), (r_2, y_5), (r_5, y_3)\}$ is a 2–simple g.c of $K_{5,5}$ so that $\eta_{2s}(K_{5,5}) \leq 10$. Now, let ψ_G be any 2−simple g.c of $K_{5,5}$. If ψ_G contains four cycles and three paths, then $t_2(\psi)=5, t_\psi=0$, otherwise $t_2(\psi)\le 2, t_\psi \ge 0$. Hence $t_2 \le 5, t \ge 0$ so that $\eta_{2s}(K_{5,5}) \ge 25-10-5+0=10$. Thus $\eta_{2s}(K_{5,5})=10$.

Case 2. When s = 6

Then $\psi=\{(r_1,y_1,r_2,y_2,r_1),(r_2,y_3,r_4,y_4,r_2),(r_3,y_1,r_5,y_5,r_3),(r_4,y_5,r_1,y_6,r_4),(r_2,y_6,r_5),$ $(r_1,y_4,r_3),(r_1,y_3,r_5),(y_3,r_3,y_6),(r_3,y_2,r_4),(y_2,r_5,y_4),(r_2,y_5),(r_4,y_1)\}$ is a 2−simple g.c of $K_{5,6}$ so that $\eta_{2s}(K_{5,6})\le 12$. Now, let ψ_G be any 2−simple g.c of $K_{5,6}$. If ψ_G contains four cycles and six paths , then $t_2(\psi)=7, t_\psi=0$, otherwise $t_2(\psi)\le 6, t_\psi \ge 3$. Hence $t_2 \le 7$, $t \ge 0$, so that $\eta_{2s}(K_{5,6})=q-p-t_2+t \ge 30-11-7+0=12$. Thus $\eta_{2s}(K_{5,6})=12$.

Case 3. When s = 7

Then $\psi=\{(r_1,y_1,r_2,y_2,r_1),(r_2,y_3,r_4,y_4,r_2),(r_3,y_1,r_5,y_5,r_3),(r_4,y_5,r_1,y_6,r_4),(r_1,y_3,r_3,y_7$ $,r_1),(r_5,y_6,r_2,y_7,r_5),(r_1,y_4,r_5),(y_4,r_3,y_6),(r_3,y_2,r_4),(y_1,r_4,y_7),(y_2,r_5,y_3),(r_2,y_5)\}$ is a 2−simple g.c of $K_{5,7}$ so that $\eta_{2s}(K_{5,7})\le 12$. Now, let ψ_G be any 2−simple g.c of $K_{5,7}$. If ψ_G contains six cycles and five paths , then $t_2(\psi)=11, t_\psi=0$, otherwise $t_2(\psi)\le 7, t_\psi \ge 3$. Hence $t_2 \le 11, t \ge 0$ so that $\eta_{2s}(K_{5,7}) \ge 35-12-11+0=12$. Thus $\eta_{2s}(K_{5,7})=12$.

Case 4. When s = 8

The collection of paths are $P_1=(r_1,y_1,r_2,y_2,r_1), P_2=(r_2,y_3,r_4,y_4,r_2), P_3=(r_3,y_1,r_5,y_5$ $,r_3), P_4=(r_4,y_5,r_1,y_6,r_4), P_5=(r_1,y_3,r_3,y_7,r_1), P_6=(r_3,y_2,r_4,y_8,r_3), P_7=(r_5,y_6,r_2,y_7,r_5),$ $P_8=(r_5,y_4,r_1,y_8,r_5), P_9=(y_4,r_3,y_6)$ & $P_{10}=(y_2,r_5,y_3)$. Then $\psi=\{P_i : i=1,2,...,10\}$ together with remaining edges form a minimum 2−simple g.c of $K_{5,8}$ in which all the vertices are made internal exactly twice in a path. By corollary 2.2, $\eta_{2s}(K_{5,8})=q-2p$.

Case 5. When s = 9

The collection of paths are $P_1=(r_1,y_1,r_2,y_2,r_1), P_2=(r_2,y_3,r_4,y_4,r_2), P_3=(r_3,y_1,r_5, y_5,$ $r_3), P_4=(r_4,y_5,r_1,y_6,r_4), P_5=(r_1,y_3,r_3,y_7,r_1), P_6=(r_3,y_2,r_4,y_8,r_3), P_7=(r_5,y_6,r_2,y_7,r_5),$ $P_8=(r_5,y_4,r_1,y_8,r_5), P_9=(y_4,r_3,y_6), P_{10}=(y_2,r_5,y_3), P_{11}=(r_2,y_9,r_3)$ & $P_{12}=(r_4,y_9,r_5)$. Then $\psi=\{P_i : i= 1,2,...,12\}$ together with remaining edges form a minimum 2−simple g.c of $K_{5,8}$ in which all the vertices are made internal exactly twice in a path. By corollary 2.2, $\eta_{2s}(K_{5,8})=q-2p$.

Case 6. When s = 10

Then the collection of paths are $P_1=(r_1,y_1,r_2,y_2,r_1), P_2=(r_2,y_3,r_4,y_4,r_2), P_3=(r_3,y_1,$ $r_5,y_5,r_3), P_4=(r_4,y_5,r_1,y_6,r_4),$ $P_5=(r_1,y_3,r_3,y_7,r_1),$ $P_6=(r_3,y_2,r_4,y_8,r_3),$ $P_7=(r_5,y_6,r_2,$ $y_7,r_5), P_8=(r_5,y_4,r_1,y_8,r_5),$ $P_9=(r_4,y_9,r_5,y_{10},r_4),$ $P_{10}=(r_2,y_9,r_3)$ & $P_{11}=(y_4,r_3,y_6)$. Then $\psi=\{P_i : i= 1,2,...,11\}\cup\{Q\}$ where Q is a set of the edges of $K_{5,10}$ not covered by

$\{P_i : i = 1,2,...,11\}$ is a 2−simple g.c of $K_{5,10}$ so that $\eta_{2s}(K_{5,10}) \le 21$. Now, let ψ_G be any 2−simple g.c of $K_{5,10}$. If ψ_G contains nine cycles and two paths , then $t_2(\psi) = 14, t_\psi = 0$ otherwise $t_2(\psi) \le 6, t_\psi \ge 1$. Hence $t_2 \le 14, t \ge 0$ so that $\eta_{2s}(K_{5,10}) \ge 50 - 15 - 14 = 21$.

Thus $\eta_{2s}(K_{5,10}) = 21$..

Case 7. When $s \ge 11$

The collection of paths are $P_1 = (r_1, y_1, r_2, y_2, r_1), P_2 = (r_2, y_3, r_4, y_4, r_2), P_3 = (r_3, y_1, r_5, y_5, r_3), P_4 = (r_4, y_5, r_1, y_6, r_4), P_5 = (r_1, y_3, r_3, y_7, r_1), P_6 = (r_3, y_2, r_4, y_8, r_3), P_7 = (r_5, y_6, r_2, y_7, r_5), P_8 = (r_5, y_4, r_1, y_8, r_5), P_9 = (r_4, y_9, r_5, y_{10}, r_4)$ & $P_{10} = (r_2, y_9, r_3, y_{11}, r_2)$. Then $\psi = \{P_i, i = 1,2,..,10\} \cup \{Q\}$ where Q is set of edges of $K_{5,s}$ not covered by $\{P_i : i = 1,..,10\}$ is a 2−simple g.c of $K_{5,s}$ so that $|\psi_G| = 10 + (5s - 40) = 5s - 30$. Hence $\eta_{2s}(K_{5,s}) \le 5s - 30$. Now, let ψ_G be any 2−simple g.c of $K_{5,s}$. If ψ_G contains ten cycles, then $t_2(\psi) = 14, t_\psi = s - 11$ otherwise $t_2(\psi) \le 10, t_\psi \ge s - 5$. Hence $t_2 \le 14, t \ge s - 11$ so that $\eta_{2s}(K_{5,s}) \ge 5s - (5 + s) - 14 + (s - 11)$

$= 5s - 30$. Thus $\eta_{2s}(K_{5,s}) = 5s - 30$.

(vi) Now let $X = \{r_1, r_2, r_3, r_4, r_5, r_6\}$ and $Y = \{y_1, y_2, y_3, ..., y_s\}$ be the bipartition of $K_{6,s}$ with $p = 6 + s$, $q = 6s$.

Case 1. When $s = 6$

Then $\psi = \{(r_1, y_1, r_2, y_2, r_1), (r_2, y_3, r_3, y_4, r_2), (r_3, y_1, r_4, y_5, r_3), (r_4, y_3, r_1, y_6, r_4), (r_5, y_2, r_6, y_4, r_5), (r_6, y_5, r_2, y_6, r_6), (y_1, r_6, y_3), (y_2, r_3, y_6), (y_1, r_5, y_6), (y_4, r_1, y_5), (r_4, y_2), (r_4, y_4), (r_5, y_5), (r_5, y_3)\}$ form a 2−simple g.c of $K_{6,6}$ so that $\eta_{2s}(K_{6,6}) \le 14$. Now, let ψ_G be any 2−simple g.c of $K_{6,6}$. If ψ_G contains six cycles and four paths, then $t_2(\psi) = 10, t_\psi = 0$ otherwise $t_2(\psi) \le 6$, $t_\psi \ge 0$. Hence $t_2 \le 10, t \ge 0$, so that $\eta_{2s}(K_{6,6}) \ge 36 - 12 - 10 = 14$. Thus $\eta_{2s}(K_{6,6}) = 14$.

Case 2. When $s = 7$

Then $\psi = \{(r_1, y_1, r_2, y_2, r_1), (r_2, y_3, r_3, y_7, r_2), (r_3, y_4, r_4, y_6, r_3), (r_4, y_5, r_5, y_7, r_4), (r_5, y_2, r_6, y_4, r_5), (r_6, y_3, r_1, y_6, r_6), (r_3, y_5, r_6, y_1, r_3), (y_4, r_1, y_7), (y_4, r_2, y_5), (y_2, r_4, y_3), (y_1, r_5, y_3), (r_1, y_5)$

$(r_2, y_6), (r_3, y_2), (r_4, y_1), (r_5, y_6), (r_6, y_7)\}$ is a 2−simple g.c of $K_{6,7}$ so that $\eta_{2s}(K_{6,7}) \le 17$. Now, let ψ_G be any 2−simple g.c of $K_{6,7}$. If ψ_G contains seven cycles and three paths, then $t_2(\psi) = 12, t_\psi = 0$, otherwise $t_2(\psi) \le 7, t_\psi \ge 0$. Hence $t_2 \le 12, t \ge 0$, so that $\eta_{2s}(K_{6,7}) \ge 42 - 13 - 12 = 17$. Thus $\eta_{2s}(K_{6,7}) = 17$.

Case 3. When s = 8

Then $\psi=\{(r_1,y_1,r_2,y_2,r_1),(r_2,y_4,r_3,y_6,r_2),(r_3,y_3,r_1,y_5,r_3),(r_4,y_2,r_5,y_7,r_4),(r_5,y_3,r_6,y_8,r_5),(r_6,y_1,r_4,y_6,r_6),(r_1,y_4,r_6,y_7,r_1),(r_4,y_5,r_2,y_8,r_4),(y_6,r_1,y_8),(y_2,r_3,y_8),(y_3,r_4,y_4),(y_1,r_5,y_4)\}$ together with remaining edges form a 2−simple g.c of $K_{6,8}$ in which all vertices are twice made internal. By the corollary 2.2, $\eta_{2s}\left(K_{6,8}\right)=q-2p$.

Case 4. When s = 9

Then $\psi=\{(r_1,y_1,r_2,y_2,r_1),(r_1,y_3,r_4,y_4,r_1),(r_5,y_2,r_3,y_3,r_5),(r_4,y_1,r_5,y_6,r_4),(r_3,y_5,r_1,y_6,r_3),(r_5,y_4,r_6,y_5,r_5),(r_6,y_7,r_1,y_8,r_6),(r_2,y_7,r_5,y_9,r_2),(r_3,y_8,r_4,y_9,r_3),(y_3,r_2,y_5),(y_1,r_3,y_4),(y_2,r_6,y_6)\}$ together with remaining edges form a 2−simple g.c of $K_{6,9}$ in which all vertices are twice made internal. By the corollary 2.2, $\eta_{2s}\left(K_{6,9}\right)=q-2p$.

Case 5. When s = 10

Then $\psi=\{(r_1,y_1,r_2,y_2,r_1),(r_1,y_3,r_4,y_4,r_1),(r_5,y_2,r_3,y_3,r_5),(r_4,y_1,r_5,y_6,r_4),(r_3,y_5,r_1,y_6,r_3),(r_5,y_4,r_6,y_5,r_5),(r_6,y_7,r_1,y_8,r_6),(r_2,y_7,r_5,y_9,r_2),(r_3,y_8,r_4,y_9,r_3),(y_3,r_2,y_5),(y_1,r_3,y_4),(y_2,r_6,y_6),(r_1,y_{10},r_5),(r_2,y_{10},r_6)\}$ together with remaining edges form a 2−simple g.c of $K_{6,10}$ in which all vertices are twice made internal. By the corollary 2.2, $\eta_{2s}\left(K_{6,10}\right)=q-2p$.

Case 6. When s = 11

Then $\psi=\{(r_1,y_1,r_2,y_2,r_1),(r_1,y_3,r_4,y_4,r_1),(r_5,y_2,r_3,y_3,r_5),(r_4,y_1,r_5,y_6,r_4),(r_3,y_5,r_1,y_6,r_3),(r_5,y_4,r_6,y_5,r_5),(r_6,y_7,r_1,y_8,r_6),(r_2,y_7,r_5,y_9,r_2),(r_3,y_8,r_4,y_9,r_3),(y_3,r_2,y_5),(y_1,r_3,y_4),(y_2,r_6,y_6),(r_1,y_{10},r_5),(r_2,y_{10},r_6),(r_2,y_{11},r_4),(r_3,y_{11},r_6)\}$ together with remaining edges form a 2−simple g.c of $K_{6,11}$ in which all vertices are twice made internal. By the corollary 2.2, $\eta_{2s}\left(K_{6,11}\right)=q-2p$.

Case 7. When s = 12

Then $\psi=\{(r_1,y_2,r_6,y_4,r_1),(r_3,y_7,r_4,y_9,r_3),(r_5,y_8,r_2,y_4,r_5),(r_2,y_6,r_1,y_{11},r_2),(r_5,y_2,r_3,y_{10},r_5),(r_6,y_1,r_4,y_3,r_6),(r_1,y_1,r_5,y_9,r_1),(r_1,y_3,r_3,y_8,r_1),(r_5,y_7,r_6,y_{11},r_5),(r_6,y_5,r_2,y_{12},r_6),(r_4,y_5,r_5,y_6,r_4),(r_4,y_{10},r_1,y_{12},r_4)$ together with remaining edges form a 2−simple g.c of $K_{6,12}$ in which all vertices are twice made internal. By the corollary 2.2, $\eta_{2s}\left(K_{6,12}\right)=q-2p$.

Case 8. When s = 12

Then $\psi=\{(r_1,y_2,r_6,y_4,r_1),(r_3,y_7,r_4,y_9,r_3),(r_5,y_8,r_2,y_4,r_5),(r_2,y_6,r_1,y_{11},r_2),(r_5,y_2,r_3,y_{10},r_5),(r_6,y_1,r_4,y_3,r_6),(r_1,y_1,r_5,y_9,r_1),(r_1,y_3,r_3,y_8,r_1),(r_5,y_7,r_6,y_{11},r_5),(r_6,y_5,r_2,y_{12},r_6),(r_4,y_5,r_5,y_6,r_4),(r_4,y_{10},r_1,y_{12},r_4),(r_2,y_{13},r_4),(r_3,y_{13},r_6)\}$ together with remaining edges form a 2−simple g.c of $K_{6,13}$ in which all vertices are twice made internal. By the corollary 2.2, $\eta_{2s}\left(K_{6,13}\right)=q-2p$.

Case 9. When $s \geq 14$

Then the collection of paths are $P_1 = (r_1, y_2, r_6, y_4, r_1), P_2 = (r_3, y_7, r_4, y_9, r_3), P_3 = (r_5, y_8, r_2, y_4, r_5), P_4 = (r_2, y_6, r_1, y_{11}, r_2), P_5 = (r_5, y_2, r_3, y_{10}, r_5), P_6 = (r_6, y_1, r_4, y_3, r_6), P_7 = (r_1, y_1, r_5, y_9, r_1), P_8 = (r_1, y_3, r_3, y_8, r_1), P_9 = (r_5, y_7, r_6, y_{11}, r_5), P_{10} = (r_6, y_5, r_2, y_{12}, r_6), P_{11} = (r_4, y_5, r_5, y_6, r_4), P_{12} = (r_4, y_{10}, r_1, y_{12}, r_4), P_{13} = (r_2, y_{13}, r_4), P_{14} = (r_3, y_{13}, r_6)$ & $P_{15} = (r_2, y_{14}, r_3)$. Then $\psi = \{P_i, i = 1,..,15\} \cup \{Q\}$ where Q is set of edges of $K_{6,s}$ not covered by $\{P_i, i = 1,..,15\}$ is a 2−simple g.c of $K_{6,s}$ so that $|\psi_G| = 27 + (6s - 66) = 6s - 39$. Hence $\eta_{2s}(K_{6,s}) \leq 6s - 39$. Now, let ψ_G be any 2−simple g.c of $K_{6,s}$. If ψ_G contains twelve cycles and three paths then $t_2(\psi) = 19, t_\psi = (s - 14)$ otherwise $t_2(\psi) \leq 11, t_\psi \geq s - 9$. Hence $t_2 \leq 19$, $t \geq (s - 14)$ so that $\eta_{2s} \geq 6s - (6 + s) - 19 + (s - 14) = 6s - 39$. Hence $\eta_{2s}(K_{6,s}) = 6s - 39$.

(vii) Now let $X = \{r_1, r_2, r_3, r_4, r_5, r_6, r_7\}$ and $Y = \{y_1, y_2, y_3, ..., y_s\}$ be the bipartition of $K_{7,s}$ with $p = 7 + s, q = 7s$.

Case 1. When $7 \leq s \leq 14$

Then the collection of paths are $P_1 = (r_1, y_1, r_2, y_2, r_1), P_2 = (r_2, y_3, r_5, y_7, r_2), P_3 = (r_3, y_3, r_4, y_4, r_3), P_4 = (r_4, y_5, r_7, y_2, r_4), P_5 = (r_5, y_5, r_6, y_6, r_5), P_6 = (r_6, y_1, r_3, y_7, r_6), P_7 = (r_7, y_4, r_1, y_6, r_7), P_8 = (y_5, r_1, y_7), P_9 = (y_4, r_2, y_5), P_{10} = (y_2, r_3, y_6), P_{11} = (y_1, r_4, y_6), P_{12} = (y_2, r_5, y_4), P_{13} = (y_2, r_6, y_3), P_{14} = (y_1, r_7, y_3), Q_i = (r_k, y_{7+i}, r_l), R_i = (r_p, y_{7+i}, r_q) : k \neq l \neq p \neq q, i = 1, 2, ..., 7$ and $7 + i \leq s$. Then $\psi = \{P_i : i = 1, 2, .., 14\} \cup \{Q_i : i = 1, .., 7\} \cup \{R_i : i = 1, ..., 7\}$ together with remaining edges form a minimum 2−simple g.c in which all the vertices are made internal twice. By the corollary 2.2, $\eta_{2s}(K_{7,s}) = q - 2p$.

Case 2. When $s \geq 15$

Then the collection of paths are $P_1 = (r_1, y_1, r_2, y_2, r_1), P_2 = (r_2, y_3, r_5, y_7, r_2), P_3 = (r_3, y_3, r_4, y_4, r_3), P_4 = (r_4, y_5, r_7, y_2, r_4), P_5 = (r_5, y_5, r_6, y_6, r_5), P_6 = (r_6, y_1, r_3, y_7, r_6), P_7 = (r_7, y_4, r_1, y_6, r_7), P_8 = (y_5, r_1, y_7), P_9 = (y_4, r_2, y_5), P_{10} = (y_2, r_3, y_6), P_{11} = (y_1, r_4, y_6), P_{12} = (y_2, r_5, y_4), P_{13} = (y_2, r_6, y_3), P_{14} = (y_1, r_7, y_3), Q_i = (r_k, y_{7+i}, r_l)$ and $R_i = (r_p, y_{7+i}, r_q) : k \neq l \neq p \neq q, i = 1, 2, .., 7, 7 + i \leq s$. Then $\psi = \{P_i : i = 1, .., 14\} \cup \{Q_i : i = 1, .., 7\} \cup \{R_i : i = 1, ..., 7\}$ together with remaining edges form a minimum 2−simple g.c in which $\{y_i : i = 15, 16, ...\}$ cannot be made internal. Thus $|\psi_G| = 28 + (7s - 70) = 7s - 42$. Hence $\eta_{2s}(K_{7,n}) \leq 7s - 42$. Now, let ψ_G be any 2−simple g.c of $K_{7,n}$. If ψ_G contains seven cycles and twenty one paths, then $t_2(\psi) = 21, t_\psi = s - 14$ otherwise $t_2(\psi) \leq 15, t_\psi \geq (s - 13)$. Hence $t_2 \leq 21, t \geq s - 14$ so that $\eta_{2s}(7, s) \geq 7s - (7 + s) - 21 + (s - 14) = 7s - 42$. Hence $\eta_{2s}(K_{7,s}) = 7s - 42$.

Theorem 3.2. *For a complete bipartite graph* $K_{r,s},(r \geq 8)$ *and r is even, then*

$$\eta_{2s}(K_{r,s}) = \begin{cases} rs - 2r - 2s & \text{if } r \leq s \leq \left\lfloor \dfrac{r^2+r}{4} \right\rfloor \\ \left\lfloor \dfrac{2rs - r^2 - 5r}{2} \right\rfloor & \text{if } s > \left\lfloor \dfrac{r^2+r}{4} \right\rfloor \end{cases}$$

Proof. Now let $X = \{r_1, r_2, r_3, ..., r_r\}$ and $Y = \{y_1, y_2, y_3, ..., y_s\}$ be the bipartition of $K_{r,s}$ with p = r + s, $q = rs$. Now there are two cases.

Case 1. When $r \leq s \leq \lfloor (r^2+r)/4 \rfloor$

Then the collection of paths are

$P_i = (r_i, y_i, r_{(i+1)}, y_{(i+1)}, r_i) : i = 1, 3, ..., (r-1)$

$Q = (r_2, y_3, r_5, y_8, r_2)$

$R_i = (r_{2i}, y_{(2i+1)}, r_{(2i+3)}, y_{(2i+6)}, r_{2i}) : i = 2, 3, ..., ((r-6)/2), (r > 8)$

$R_{r-4} = (r_{(r-4)}, y_{(r-3)}, r_{(r-1)}, y_2, r_{(r-4)})$; $R_{r-2} = (r_{(r-2)}, y_{(r-1)}, r_1, y_4, r_{(r-2)})$

$R_r = (r_r, y_1, r_3, y_6, r_r)$; $S_{i-2} = (y_i, r_{(i-2)}, y_{(i+2)}), i = 3, 4, ..., (r-2)$

$T_1 = (y_2, r_{(r-3)}, y_4)$; $T_2 = (y_1, r_{(r-2)}, y_3)$; $T_3 = (y_1, r_{(r-1)}, y_4)$

$T_4 = (y_4, r_s, y_5)$; $U_i = (r_k, y_{(r+i)}, r_l)$

$V_i = (r_r, y_{(r+i)}, r_s), k \neq l \neq s : i = 1, 2, ..., \lfloor ((r^2 - 3m)/4) \rfloor$ *and* $r + i \leq s.$

Then $\psi = \{P_i : i = 1, 3, ...(r-1)\} \cup \{Q\} \cup \{R_i : i = 2, 3, ..., \left(\dfrac{r-6}{2}\right)\} \cup \{R_{r-4}\} \cup \{R_{r-2}\} \cup \{R_r\}$

$\cup \{S_{i-2} : i = 3, 4, ..., (r-2)\} \cup \{T_i : i = 1, 2, 3, 4\} \cup \{U_i\} \cup \{V_i : i = 1, 2, ..., \left\lfloor \dfrac{r^2 - 3m}{4} \right\rfloor$

together with remaining edges form a minimum 2−simple g.c in which all the vertices are internal twice. By the corollary 2.2, $\eta_{2s}(K_{r,s}) = q - 2p = rs - 2(r+s) = rs - 2r - 2s.$

Case 2. When $s > \lfloor (r^2+r)/4 \rfloor$, then there are two sub cases

Subcase 2.1. When $r \equiv 0 \ (mod 4)$

Then the collection of paths are

$P_i = (r_i, y_i, r_{(i+1)}, y_{(i+1)}, r_i) : i = 1, 3, ..., (r-1)$

$Q = (r_2, y_3, r_5, y_8, r_2)$

$R_i = (r_{2i}, y_{(2i+1)}, r_{(2i+3)}, y_{(2i+6)}, r_{2i}) : i = 2, 3, ..., ((r-6)/2), (r > 8)$

$R_{r-4} = (r_{(r-4)}, y_{(r-3)}, r_{(r-1)}, y_2, r_{(r-4)})$; $R_{r-2} = (r_{(r-2)}, y_{(r-1)}, r_1, y_4, r_{(r-2)})$

$R_r = (r_r, y_1, r_3, y_6, r_r)$; $S_{i-2} = (y_i, r_{(i-2)}, y_{(i+2)}), i = 3, 4, ..., (r-2)$

$T_1 = (y_2, r_{(r-3)}, y_4)$; $T_2 = (y_1, r_{(r-2)}, y_3)$; $T_3 = (y_1, r_{(r-1)}, y_4)$

$T_4 = (y_4, r_s, y_5)$; $U_i = (r_k, y_{(r+i)}, r_l)$

$V_i = (r_r, y_{(r+i)}, r_s), k \neq l \neq s : i = 1,2,...,\lfloor((r^2 - 3m)/4)\rfloor$ *and* $r + i \leq s.$

Then $\psi_G = \{P_i : i = 1,3,....(r-1)\} \cup \{Q\} \cup \{R_i : i = 2,3,...,\left(\frac{r-6}{2}\right)\} \cup \{R_{r-4}\} \cup \{R_{r-2}\} \cup \{R_r\}$

$\cup \{S_{i-2} : i = 3,4,...,(r-2)\} \cup \{T_i : i = 1,2,3,4\} \cup \{U_i\} \cup \{V_i : i = 1,2,...,\left\lfloor \frac{(r^2 - 3m)}{4} \right\rfloor$ is a $2-$ simple g.c in which $\{y_i : i = \left\lfloor \frac{(r^2+r+4)}{4} \right\rfloor,....\}$ cannot be made internal. Thus $|\psi_G| =$ $\left\lfloor \frac{r^2+r}{2} \right\rfloor + \lfloor rs - r^2 - 3r \rfloor = \left\lfloor \frac{2rs - r^2 - 5r}{2} \right\rfloor$. Hence $\eta_{2s} \leq \left\lfloor \frac{2rs - r^2 - 5r}{2} \right\rfloor$. Now, let ψ_G be any $2-$simple g.c of $K_{r,s}$ If ψ_G contains r cycles and $\left\lfloor \frac{r^2 - r}{2} \right\rfloor$ paths, then $t_2(\psi) \leq$ $\left\lfloor \frac{r^2+5r}{4} \right\rfloor$, $t_\psi \geq \left\lfloor \frac{4s - r^2 - r}{4} \right\rfloor$,otherwise $t_2(\psi) \leq \left\lfloor \frac{r^2 - r}{4} \right\rfloor$, $t_\psi \geq \left\lfloor \frac{4s - r^2 + r}{4} \right\rfloor$. Hence $t_2 \leq \left\lfloor \frac{r^2+5r}{4} \right\rfloor, t \geq \left\lfloor \frac{4s - r^2 - r}{4} \right\rfloor$ so that $\eta_{2s} \geq \left\lfloor \frac{2rs - r^2 - 5r}{2} \right\rfloor$.Thus $\eta_{2s} = \left\lfloor \frac{2rs - r^2 - 5r}{2} \right\rfloor$.

Subcase 2.2. When $r \equiv 2 \ (mod\, 4)$

Then the collection of paths are

$P_i = (r_i, y_i, r_{(i+1)}, y_{(i+1)}, r_i) : i = 1,3,...,(r-1)$

$Q = (r_2, y_3, r_5, y_8, r_2)$

$R_i = (r_{2i}, y_{(2i+1)}, r_{(2i+3)}, y_{(2i+6)}, r_{2i}) : i = 2,3,...,((r-6)/2),(r > 8)$

$R_{r-4} = (r_{(r-4)}, y_{(r-3)}, r_{(r-1)}, y_2, r_{(r-4)})$; $R_{r-2} = (r_{(r-2)}, y_{(r-1)}, r_1, y_4, r_{(r-2)})$

$R_r = (r_r, y_1, r_3, y_6, r_r)$; $S_{i-2} = (y_i, r_{(i-2)}, y_{(i+2)}), i = 3,4,...,(r-2)$

$T_1 = (y_2, r_{(r-3)}, y_4)$; $T_2 = (y_1, r_{(r-2)}, y_3)$; $T_3 = (y_1, r_{(r-1)}, y_4)$

$T_4 = (y_4, r_s, y_5)$; $U_i = (r_k, y_{(r+i)}, r_l)$

$V_i = (r_r, y_{(r+i)}, r_s), k \neq l \neq s : i = 1,2,...,\lfloor((r^2 - 3m)/4)\rfloor$ *and* $r + i \leq s.$

$W = \left(r_r, y_{\lfloor(r^2+r+4)/4\rfloor}, r_q \right). \ r \neq q \neq k \neq l \neq s$

Then $\psi_G = \{P_i : i = 1,3,...(r-1)\} \cup \{Q\} \cup \{R_i : i = 2,3,..,\left(\frac{r-6}{2}\right)\} \cup \{R_{r-4}\} \cup \{R_{r-2}\} \cup \{R_r\} \cup$

$\{S_{i-2} : i = 3,4,...,(r-2)\} \cup \{T_i : i = 1,..,4\} \cup \{U_i\} \cup \{V_i : i = 1,2,..,\left\lfloor \frac{(r^2 - 3m)}{4} \right\rfloor \cup \{W\} \cup \{X\}$

Where X is remaining edges not covered $K_{r,s}$ form a 2–simple g.c in which $\{y_i : i = \left\lfloor \frac{(r^2+r+8)}{4} \right\rfloor, ...\}$ cannot be made internal. Thus $|\psi_G| = \left\lfloor \frac{r^2+r}{2} \right\rfloor +$ $\lfloor rs - r^2 - 3r \rfloor = \left\lfloor \frac{2rs - r^2 - 5r}{2} \right\rfloor$. Hence $\eta_{2s} \le \left\lfloor \frac{2rs - r^2 - 5r}{2} \right\rfloor$. Now, let ψ_G be any 2–simple g.c of $K_{r,s}$ If ψ_G contains r cycles and $\left\lfloor \frac{r^2+22r-140}{4} \right\rfloor$ paths, then $t_2(\psi) \le$ $\left\lfloor \frac{r^2+5r}{4} \right\rfloor$, $t_\psi \ge \left\lfloor \frac{4s - r^2 - r}{4} \right\rfloor$ otherwise $t_2(\psi) \le \left\lfloor \frac{r^2 - r}{4} \right\rfloor$, $t_\psi \ge \left\lfloor \frac{4s - r^2 + r - 2}{4} \right\rfloor$. Hence $t_2 \le \left\lfloor \frac{r^2+5r}{4} \right\rfloor$, $t \ge \left\lfloor \frac{4s - r^2 - r}{4} \right\rfloor$ so that $\eta_{2s} \ge \left\lfloor \frac{2rs - r^2 - 5r}{2} \right\rfloor$. Thus $\eta_{2s} = \left\lfloor \frac{2rs - r^2 - 5r}{2} \right\rfloor$. **Theorem 3.3.** *For a complete bipartite graph* $K_{r,s}, (r \ge 9)$ *and r is odd, then*

$$\eta_{2s}(K_{r,s}) = \begin{cases} rs - 2r - 2s & \text{if } r \le s \le \left\lfloor \frac{r^2+r-2}{4} \right\rfloor \\ \left\lfloor \frac{2rs - r^2 - 5r + 2}{2} \right\rfloor & \text{if } s > \left\lfloor \frac{r^2+r-2}{4} \right\rfloor \end{cases}$$

Proof. Now let $X = \{r_1, r_2, r_3, ..., r_r\}$ and $Y = \{y_1, y_2, y_3, ..., y_s\}$ be the bipartition of $K_{r,s}$ $(r \ge 9 \text{ and odd})$ with p = r + s, $q = rs$. Now there are two cases.

Case 1. When $r \le s \le \left\lfloor \frac{(r^2+r-2)}{4} \right\rfloor$

Then the collection of paths are

$P_i = (r_i, y_i, r_{(i+1)}, y_{(i+1)}, r_i) : i = 1, 3, ..., (r-2)$

$Q = (r_2, y_3, r_5, y_8, r_2)$

$R_i = (r_{2i}, y_{(2i+1)}, r_{(2i+3)}, y_{(2i+6)}, r_{2i}) : i = 2, 3, ..., \left(\frac{r-7}{2}\right), (r > 9)$

$R_{r-5} = (r_{(r-5)}, y_{(r-4)}, r_{(r-2)}, y_2, r_{(r-5)})$; $R_{r-3} = (r_{(r-3)}, y_{(r-2)}, r_1, y_4, r_{(r-3)})$

$R_{r-1} = (r_{(r-1)}, y_1, r_3, y_6, r_{(r-1)})$; $S_{i-2} = (y_i, r_{(i-2)}, y_{(i+2)})$: $i = 3, 4, ..., (r-2)$

$T_1 = (y_1, r_{(r-3)}, y_3)$; $T_2 = (y_1, r_{(r-2)}, y_4)$; $T_3 = (y_2, r_{(r-1)}, y_4)$; $T_4 = (r_1, y_r, r_3)$

$T_5 = (r_2, y_r, r_4)$; $T_6 = (y_5, r_s, y_8)$; $T_7 = (y_6, r_s, y_9)$; $U_i = (r_k, y_{(r+i)}, r_l)$

$V_i = (r_r, y_{(r+i)}, r_s), k \ne l \ne r \ne s, i = 1, 2, ..., \left\lfloor \frac{r^2+r-2}{4} \right\rfloor, r+i \le s$

Then $\psi=\{P_i: i=1,3,..(r-2)\}\cup\{Q\}\cup\{R_i: i=2,3,.,\left(\frac{r-7}{2}\right)\}\cup\{R_{r-5}\}\cup\{R_{r-3}\}\cup\{R_{r-1}\}\cup\{S_{i-2}$ $: i=3,4,..,(r-2)\}\cup\{T_i: i=1,..,7\}\cup\{U_i: i=1,2,..,\left\lfloor\frac{r^2+r-2}{4}\right\rfloor\}\cup\{V_i: i=1,2,..,\left\lfloor\frac{r^2+r-2}{4}\right\rfloor$

together with remaining edges form a minimum 2−simple g.c in which all the vertices are internal twice. By the corollary 2.2, $\eta_{2s}\left(K_{r,s}\right)=q-2p=rs-2(r+s)=rs-2r-2s.$

Case 2. When $s>\left\lfloor\frac{(r^2+r-2)}{4}\right\rfloor$**,** then there are two sub cases

Subcase 2.1. When $r\equiv 1\ (mod\ 4)$

Then the collection of paths are

$P_i=\left(r_i, y_i, r_{(i+1)}, y_{(i+1)}, r_i\right): i=1,3,\ldots,(r-2)$

$Q=\left(r_2, y_3, r_5, y_8, r_2\right)$

$R_i=\left(r_{2i}, y_{(2i+1)}, r_{(2i+3)}, y_{(2i+6)}, r_{2i}\right): i=2,3,\ldots,\left(\frac{r-7}{2}\right),(r>9)$

$R_{r-5}=\left(r_{(r-5)}, y_{(r-4)}, r_{(r-2)}, y_2, r_{(r-5)}\right)$; $R_{r-3}=\left(r_{(r-3)}, y_{(r-2)}, r_1, y_4, r_{(r-3)}\right)$

$R_{r-1}=\left(r_{(r-1)}, y_1, r_3, y_6, r_{(r-1)}\right)$; $S_{i-2}=\left(y_i, r_{(i-2)}, y_{(i+2)}\right): i=3,4,\ldots,(r-2)$

$T_1=\left(y_1, r_{(r-3)}, y_3\right)$; $T_2=\left(y_1, r_{(r-2)}, y_4\right)$; $T_3=\left(y_2, r_{(r-1)}, y_4\right)$; $T_4=\left(r_1, y_r, r_3\right)$

$T_5=\left(r_2, y_r, r_4\right)$; $T_6=\left(y_5, r_s, y_8\right)$; $T_7=\left(y_6, r_s, y_9\right)$; $U_i=\left(r_k, y_{(r+i)}, r_l\right)$

$V_i=\left(r_r, y_{(r+i)}, r_s\right), k\neq l\neq r\neq s, i=1,2,\ldots,\left\lfloor\frac{r^2+r-2}{4}\right\rfloor, r+i\leq s$

Then $\psi_G=\{P_i: i=1,3,..(r-2)\}\cup\{Q\}\cup\{R_i: i=2,3,.,\left(\frac{r-7}{2}\right)\}\cup\{R_{r-5}\}\cup\{R_{r-3}\}\cup\{R_{r-1}\}\cup\{S_{i-2}:$ $i=3,4,.,(r-2)\}\cup\{T_i: i=1,.,7\}\cup\{U_i: i=1,2,..,\left\lfloor\frac{r^2+r-2}{4}\right\rfloor\}\cup\{V_i: i=1,2,..,\left\lfloor\frac{r^2+r-2}{4}\right\rfloor\cup\{W\}$

Where W is remaining edges not covered $K_{r,s}$ form a 2–simple g.c in which $\{y_i: i=\left\lfloor\frac{r^2+r+2}{4}\right\rfloor,\ldots\}$ cannot be made internal. Thus $|\psi_G|=\left\lfloor\frac{r^2+r+2}{4}\right\rfloor+\left\lfloor rs-r^2-3r\right\rfloor=\left\lfloor\frac{2rs-r^2-5r}{2}\right\rfloor$. Hence $\eta_{2s}\leq\left\lfloor\frac{2rs-r^2-5r+2}{2}\right\rfloor$. Now, let ψ_G be any 2− simple g.c of $K_{r,s}$ If ψ_G contains (r–1) cycles and $\left\lfloor\frac{r^2-r+4}{2}\right\rfloor$ paths, then $t_2(\psi)\leq\left\lfloor\frac{r^2+5r-2}{4}\right\rfloor$, $t_\psi\geq\left\lfloor\frac{4s-r^2-r+2}{4}\right\rfloor$ otherwise $t_2(\psi)\leq\left\lfloor\frac{r^2-r}{4}\right\rfloor, t_\psi\geq\left\lfloor\frac{4s-r^2+r}{4}\right\rfloor$. Hence

$t_2 \le \left\lfloor \frac{r^2+5r-2}{4} \right\rfloor$, $t \ge \left\lfloor \frac{4s-r^2-r+2}{4} \right\rfloor$ so that $\eta_{2s} \ge \left\lfloor \frac{2rs-r^2-5r+2}{2} \right\rfloor$. Thus $\eta_{2s} = \left\lfloor \frac{2rs-r^2-5r+2}{2} \right\rfloor$.

Subcase 2.2. When $r \equiv 3 \ (mod\ 4)$

Then the collection of paths are

$P_i = \left(r_i, y_i, r_{(i+1)}, y_{(i+1)}, r_i\right) : i = 1, 3, \ldots, (r-2)$

$Q = \left(r_2, y_3, r_5, y_8, r_2\right)$

$R_i = \left(r_{2i}, y_{(2i+1)}, r_{(2i+3)}, y_{(2i+6)}, r_{2i}\right) : i = 2, 3, \ldots, \left(\frac{r-7}{2}\right), (r > 9)$

$R_{r-5} = \left(r_{(r-5)}, y_{(r-4)}, r_{(r-2)}, y_2, r_{(r-5)}\right)$; $R_{r-3} = \left(r_{(r-3)}, y_{(r-2)}, r_1, y_4, r_{(r-3)}\right)$

$R_{r-1} = \left(r_{(r-1)}, y_1, r_3, y_6, r_{(r-1)}\right)$; $S_{i-2} = \left(y_i, r_{(i-2)}, y_{(i+2)}\right) : i = 3, 4, \ldots, (r-2)$

$T_1 = \left(y_1, r_{(r-3)}, y_3\right)$; $T_2 = \left(y_1, r_{(r-2)}, y_4\right)$; $T_3 = \left(y_2, r_{(r-1)}, y_4\right)$; $T_4 = \left(r_1, y_r, r_3\right)$

$T_5 = \left(r_2, y_r, r_4\right)$; $T_6 = \left(y_5, r_s, y_8\right)$; $T_7 = \left(y_6, r_s, y_9\right)$; $U_i = \left(r_k, y_{(r+i)}, r_l\right)$

$V_i = \left(r_r, y_{(r+i)}, r_s\right), k \ne l \ne r \ne s, i = 1, 2, \ldots, \left\lfloor \frac{r^2+r-2}{4} \right\rfloor, r+i \le s$

$W = \left(r_p, y_{\left\lfloor (r^2+r-2)/4 \right\rfloor}, r_q\right) \ p \ne q \ne k \ne l \ne r \ne s$

Then $\psi_G = \{P_i : i = 1, 3, ..(r-2)\} \cup \{Q\} \cup \{R_i : i = 2, ., \left(\frac{r-7}{2}\right)\} \cup \{R_{r-5}\} \cup \{R_{r-3}\} \cup \{R_{r-1}\} \cup \{S_{i-2}$ $: i = 3, 4, ., (r-2)\} \cup \{T_i : i = 1, ., 7\} \cup \{U_i : i = 1, .., \left\lfloor \frac{r^2+r-2}{4} \right\rfloor\} \cup \{V_i : i = 1, .., \left\lfloor \frac{r^2+r-2}{4} \right\rfloor \cup \{W\}$ is a 2–simple g.c in which $\{y_i : i = \left\lfloor \frac{(r^2+5r+6)}{4} \right\rfloor, ..\}$ cannot be made internal. Thus $|\psi_G| = \left\lfloor \frac{(r^2+r+2)}{4} \right\rfloor + \left\lfloor rs - r^2 - 3r \right\rfloor = \left\lfloor \frac{2rs-r^2-5r+2}{2} \right\rfloor$. Hence $\eta_{2s} \le \left\lfloor \frac{2rs-r^2-5r+2}{2} \right\rfloor$. Now, let ψ_G be any 2− simple g.c of $K_{r,s}$ If ψ_G contains (r–1) cycles and $\left\lfloor \frac{r^2-r+4}{2} \right\rfloor$ paths, then $t_2(\psi) \le \left\lfloor \frac{r^2+5r-2}{4} \right\rfloor$, $t_\psi \ge \left\lfloor \frac{4s-r^2-r+2}{4} \right\rfloor$ otherwise $t_2(\psi) \le \left\lfloor \frac{r^2-r-2}{4} \right\rfloor$, $t_\psi \ge \left\lfloor \frac{4s-r^2+r-2}{4} \right\rfloor$. Hence $t_2 \le \left\lfloor \frac{r^2+5r-2}{4} \right\rfloor$, $t \ge \left\lfloor \frac{4s-r^2-r+2}{4} \right\rfloor$ so that $\eta_{2s} \ge \left\lfloor \frac{2rs-r^2-5r+2}{2} \right\rfloor$. Thus $\eta_{2s} = \left\lfloor \frac{2rs-r^2-5r+2}{2} \right\rfloor$.

4. Conclusions

Complete bipartite graphs find applications in materials science, particularly in the study of surface science and adsorption phenomena. The bipartite graph can represent the interaction between adsorbate molecules and surface sites on a solid material. This decomposition of complete bicyclic graphs helps in understanding the adsorption behavior, surface reactions, and the design of new materials with desired properties.

Acknowledgment. The author expresses gratitude to the editor and anonymous referees for reviewing this manuscript and providing helpful suggestions and comments.

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