FUZZY $\tilde{G}$ - HOMEOMORPHISMS

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Abstract

In this paper, we introduce f$\tilde{g}$ - homeomorphisms and prove that the collection of f$\tilde{g}^*$ - homeomorphisms forms a group under the operation composition of functions.

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1. Introduction and preliminaries

We recall the definitions of image and inverse image defined by Zadeh [13] and related properties proved by Chang [6]. Different types of generalizations of continuous function were introduced and studied by various authors in the recent development of fuzzy topology. For example ([6],[2],[9],[3],[8],[7],[1],[11],[10])

Definition 1.1. A fuzzy subset H of a fuzzy topological space (X,τ) is called a fuzzy $\tilde{g}$-closed set (shortly denotes $f\tilde{g}$-closed) [4] if $cl(H) \leq U$ whenever $H \leq U$ and U is fsg-open.

Proposition1.2. [5] If a fuzzy function $f : (X, F_\tau) \rightarrow (Y, F_\sigma)$ is fuzzy $\tilde{g}$ - irresolute then fuzzy $\tilde{g}$ - continuous but not conversely.

2. $f\tilde{g}$ - homeomorphisms

Definition 2.1. A fuzzy bijection $f : (X, F_\tau) \rightarrow (Y, F_\sigma)$ is called

(1) fuzzy $\tilde{g}$ - homeomorphism (briefly $f\tilde{g}$ - homeomorphism) if f is both $f\tilde{g}$ - continuous and $f\tilde{g}$ - open.

(2) Fuzzy $\tilde{g}^*$- homeomorphism (briefly $f\tilde{g}^*$ - homeomorphism) if both f and f^{-1} are $F\tilde{g}$ - irresolute.

We denote the concepts of all $f\tilde{g}$ - homeomorphisms (resp. $f\tilde{g}^*$- homeomorphisms and fuzzy homeomorphisms) of a fuzzy topological spaces (X, F_τ) onto itself by $f\tilde{g}$ - h (X, F_τ) (resp. $f\tilde{g}^*$-h (X, F_τ) and fh (X, F_τ)).

Proposition 2.2. Each fuzzy homeomorphism is a $f\tilde{g}$ - homeomorphism.

Proof. The proof is easy consequences of Definitions.

Remark 2.3. The following example show that the converse of Proposition 2.2 is need not be true.

Example 2.4. Consider X = Y = {m, n} and $F_\tau = \{0_X, A, B, 1_X\}$ where A is a fuzzy sets in X defined by A(m) = 0.5, A(n) = 0 and $F_\sigma = \{0_Y, B, 1_Y\}$ where B is a fuzzy set in Y defined by B(m) = 1, B(n) = 0.Then (X, F_τ) and (Y, F_σ) are fuzzy topological space. Let be $f : (X, F_\tau) \rightarrow (Y, F_\sigma)$ the identity fuzzy function. Then f is $f\tilde{g}$ - homeomorphism but not fuzzy homeomorphism.

Proposition 2.5. Each $f\tilde{g}$ - homeomorphism is a $f\omega$ – homeomorphism (resp. fg - homeomorphism).

Proof. Since any $f\tilde{g}$ - continuous function is $f\omega$ - continuous (resp. fg - continuous) and each $f\tilde{g}$ - open functions is $f\omega$ - open.

Remark 2.6. The following example show that the converse of Proposition 2.5 is need not be true.

Example 2.7. Consider X = Y = {m, n} and $F_\tau = \{0_X, A, 1_X\}$ where A is a fuzzy set in X defined by A(m) = 0.4, A(n) = 0.5 and $F_\sigma = \{0_Y, B, 1_Y\}$ where B is a fuzzy set in Y defined by B(m) = 0.3, B(n) = 0.5. Then (X, F_τ) and (Y, F_σ) are fuzzy topological space. Let $f : (X, F_\tau) \to (Y, F_\sigma)$ be the identity fuzzy function. Then f is $f\omega$ - homeomorphism but not $f\tilde{g}$ - homeomorphism.

Example2.8. The identity fuzzy function f in Example2.7, is fg - homeomorphism but not $f\tilde{g}$ - homeomorphism.

Theorem2.9. Let $f : (X, F_\tau) \to (Y, F_\sigma)$ be a bijective $f\tilde{g}$ - continuous function. Then the following statements are equivalent.

(1) F is a $f\tilde{g}$ - open function.

(2) F is a $f\tilde{g}$ - homeomorphism.

(3) F is a $f\tilde{g}$ - closed function.

Proof.

(1) ⇒ (2). Let M be an fuzzy open set of (X, F_τ). By assumption, $(f^{-1})^{-1}(M) = f(M)$ is $f\tilde{g}$ - open in (Y, F_σ) and so f is $f\tilde{g}$ - open.

(2) ⇒ (3). Let S be a fuzzy closed set of (X, F_τ). Then S^c is fuzzy open set in (X, F_τ). By assumption, $f(S^c)$ is $f\tilde{g}$ - open in (Y, F_σ). That is $f(S^c) = (f(S))^c$ is $f\tilde{g}$ - open in (Y, F_σ) and therefore $f(S)$ is $f\tilde{g}$ - closed in (Y, F_σ). Thus f is fuzzy $\tilde{g}$ - closed.

(3) ⇒ (1). Let N be a fuzzy closed set of (X, F_τ). By assumption, $f(N)$ is $f\tilde{g}$- closed in (Y, F_σ). But $f(N) = (f^{-1})^{-1}(N)$ and hence f^{-1} is $f\tilde{g}$- continuous.

Remark 2.10 .The following example show that composition of two $f\tilde{g}$- homeomorphism functions but not $f\tilde{g}$- homeomorphism function.

Example 2.11. Consider X = Y = Z = {m, n} with $F_\tau = \{0_X, A, B, 1_X\}$ where A,B are fuzzy sets in X defined by A(m) = 0.5, A(n) = 0 and B(m) = 1, B(n) = 0 and $F_\sigma = \{0_Y, C, 1_Y\}$ where C is a fuzzy set in Y defined by C(m) = 1, C(n) = 0 and $F_\eta = \{0_Z, D, E, 1_Z\}$ where D and E are fuzzy sets in Z defined by D(m) = 0.4, D(n) = 0 and E(m) = 1, E(n) = 0.Then (X, F_τ), (Y, F_σ) and (Z, F_η) are fuzzy topological space. Let $f : (X, F_\tau) \to (Y, F_\sigma)$ be the identity fuzzy function and $g : (Y, F_\sigma) \to (Z, F_\eta)$ be the identity fuzzy function. Then both f and g are $f\tilde{g}$ - homeomorphism but their composition $gof : (X, F_\tau) \to (Z, F_\eta)$ is not a $f\tilde{g}$- homeomorphism.

Proposition 2.12. Each $f\tilde{g}^*$ - homeomorphism is a $f\tilde{g}$ - homeomorphism but not conversely. i.e.,for any space (X, F_τ), $f\tilde{g}^*$-h $(X, F_\tau) \leq$ g-h (X, F_τ).

Proof. It is followed by the Proposition 1.2 and the fact that any fuzzy $\tilde{g}^*$ - open function is fuzzy $\tilde{g}$-open.

Remark 2.13. The following example show that the converse of Proposition 2.12 is need not be true.

Example 2.14. Consider X = Y = {m, n} and $F_\tau = \{0_X, A, B, 1_X\}$ where A and B are fuzzy sets in X defined by A(m) = 0.4, A(n) = 0, B(m) = 1, B(n) = 0 and $F_\sigma = \{0_Y, C, 1_Y\}$ where C is a fuzzy set in Y defined by C(m) = 1, C(n) = 0. Then (X, F_τ) and (Y, F_σ) are fts. Let $f : (X, F_\tau) \to (Y, F_\sigma)$ be the identity fuzzy function. Then f is a $f\tilde{g}$ - homeomorphism but not a $f\tilde{g}^*$- homeomorphism.

Proposition 2.15. Each $f\tilde{g}^*$ - homeomorphism is a fg - homeomorphism.

Proof. It follows from Propositions 2.5 and 2.12.

Remark 2.16. The following examples how that the converse of Proposition 2.15 is need not be true.

Example 2.17. The identity fuzzy function f in Example 2.7 is a fg - homeomorphism but not a $f\tilde{g}^*$ - homeomorphism.

Definition 2.18. A fuzzy bijective function $f : (X, F_\tau) \to (Y, F_\sigma)$ is said to be a fuzzy gc - homeomorphism if f and f^{-1} are fuzzy gc - irresolute functions.

Proposition 2.19. Each $f\tilde{g}^*$ - homeomorphism is a fuzzy gc - homeomorphism.

Proof. It follows from Definition 2.18 and Definition 2.1(1).

Remark 2.20. The following example show that the converse of Proposition 2.19 is need not be true.

Example 2.21. The identity fuzzy function f in Example 2.7 is fuzzy gc - homeomorphism but not $f\tilde{g}^*$ - homeomorphism.

Theorem 2.22. If $f : (X, F_\tau) \to (Y, F_\sigma)$ and $g : (Y, F_\sigma) \to (Z, F_\eta)$ are $f\tilde{g}^*$ - homeomorphisms, then their composition $gof : (X, F_\tau) \to (Z, F_\eta)$ is $f\tilde{g}^*$ - homeomorphism.

Proof. Let M be $f\tilde{g}$- open set in (Z, F_η). Now, $(gof)\ (M) = f^{-1}(g^{-1}(M)) = f^{-1}(N)$, where $N = g^{-1}(M)$. By hypothesis, N is $f\tilde{g}$ - open in (Y, F_σ) and so again by hypothesis, $f^{-1}(N)$ is $f\tilde{g}$ - open in (X, F_τ). Therefore, gof is $f\tilde{g}$ - irresolute.

Also for a $f\tilde{g}$ - open set U in (X, F_τ), we have $(gof)^{-1}(U) = g(f(U)) = g(V)$, where $V = f(U)$. By hypothesis $f(U)$ is $f\tilde{g}$ - open in (Y, F_σ) and so again by hypothesis, $g(f(U))$ is $f\tilde{g}$ - open in (Z, F_η) i.e., $(gof)\ (U)$ is $f\tilde{g}$ - open in (Z, F_η) and therefore $(gof)^{-1}$ is $f\tilde{g}$ - irresolute. Therefore gof is a $f\tilde{g}^*$ - homeomorphism.

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