Vertex k- Prime Labeling on graphs

Teresa Arockiamary S[1] and Vijayalakshmi G[2]

[1,2]Department of Mathematics, Stella Maris College (Autonomous),
Affiliated to University of Madras, Chennai.

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Abstract

A graph $G(V, E)$ with vertex set V is said to have a prime labeling if there exist a bijective function $f : V(G) \to 1, 2, ..., |V|$ such that for each edge $xy \in E(G)$, $gcd(f(x), f(y)) = 1$. In this paper, we introduce vertex k-prime labeling of a graph G and exhibit the existence of such a labeling by discussion through various cases.

Keywords: prime labeling, vertex k-prime labeling, planar graphs, complete graphs

1 Introduction

A labeling for a graph is a map that takes graph elements namely vertices, edges or both to numbers (positive integers) subject to certain conditions. Over the last three decades, there has been a vast literature dealing with various types of graph labelings and for a survey of various graph labeling findings we refer to Gallian [5].

Roger Entringer proposed the concept of prime labeling which was first introduced in a paper by Tout, Dabboucy and Howalla [11]. In 1980s, Entringer conjectured that all trees have a prime labeling. Path graph, star graph, caterpillar graph, complete binary trees, spider graph have prime labeling. Baskar Babujee and Vishnupriya [2] proved the following graphs have prime labelings: nP_2, $P_n \cup P_n \cup ... \cup P_n$, $B_{m,n}$. Baskar Babujee [3] further proved that the following graphs also satisfy the condition of prime labeling: $(P_m \cup nK_1) + \overline{K_2}$, $(C_m \cup nK_1) + \overline{K_2}$, $(P_m \cup C_n \cup \overline{K_r}) + \overline{K_2}$, $C_n \cup C_{n+1}$, $(2n-2)C_{2n}(n > 1)$, $C_n\ mP_k$ and the graph obtained by subdividing each edge of a star once. Seoud, Sonbaty and Mahran [8] provide necessary and sufficient conditions for a graph to be prime. Other graphs with prime labelings include all cycles and the disjoint union of C_{2k} and C_n [7]. The complete graph K_n does not have a prime labeling for $n \geq 4$ and W_n is prime if and only if n is even [6].

The concept of k-prime labeling was introduced by Vaidya and Prajapati [12]. They proved that every path graph P_m, $m \geq 1$ is k-prime for each positive integer k. k-prime labeling for cycle graphs C_n, tadpole graphs $T_{n,m}$, friendship graphs F_n, barycentric subdivision of cycle graphs $C_n(C_n)$, Y − tree P_n^3, X − tree P_n^4, one point union of path graph P_n^t are proved in [9, 10].

For our study we need the following definition of planar graph based on complete graphs. In [1], planar graphs are defined by J Basker Babujee as graphs obtained by deleting certain edges from the complete graph K_n. Pl_n denotes the class of planar graphs containing the maximum number of edges possible in a graph with *n* vertices.

Definition 1.1. *The graph $Pl_n = (V, E)$ where vertex set $V = \{1, 2, ..., n\}$ and edge set $E = \{E(K_n) \setminus \{(i, j) : 3 \le i \le n - 2$ and $i + 2 \le j \le n\}\}$ is a planar graph having the maximum number of edges with n vertices. Thus Pl_n is obtained by deleting $[(n - 4)(n - 3)]/2$ edges from K_n and it is a planar graph with $3n - 6$ edges.*

J. Baskar Babujee [4] proved the class of Planar graphs Pl_n for odd n admits primelabeling.

2 Main Results

In this section, we introduce vertex *k*-prime labeling of a graph G and prove the existence of such a labeling by discussion through various cases.
To begin with we first modify the definition of *k*-prime labeling given by Vaidya and Prajapati in [12] and redefine the labeling as vertex *k*- prime labeling.

Definition 2.1. A vertex *k*-prime labeling of a graph G is a bijective function $f: V \rightarrow \{k, k+1, k+2, ..., k+|V|-1\}$ for some positive integer *k* such that $gcd\left(f(u), f(v)\right) = 1$ $\forall\ e = uv \in E(G)$. A graph G that admits vertex *k*-prime labeling is called a vertex *k*-prime graph.

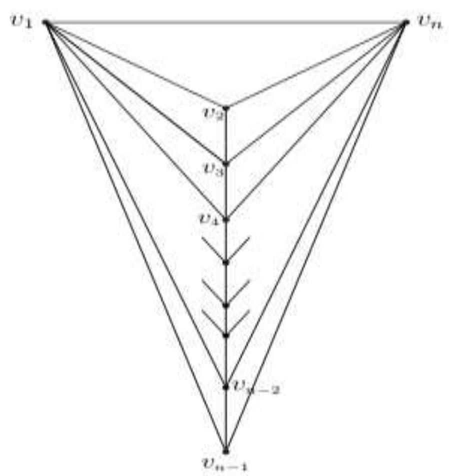

Figure 1. Planar graphs Pl_n

Theorem 2.1. *The class Pl_n is vertex k-prime for $k \geq n$, odd n and k, $k \geq 3$ except for k and $k+n-1$ not prime.*

Proof. Consider the planar graph $Pl_n(V, E)$ with n vertices $v_1, v_2, ..., v_n$ and $3(n-2)$ edges for odd $n \geq 5$. We use the following embedding for the Pl_n graph: Place the vertices $v_2, v_3, ..., v_{n-1}$ in that sequence along a vertical line, with v_{n-1} at the bottom with degree 3 and v_2 at the top. The degree of the vertices on the path $v_2, v_3, ..., v_{n-2}$ is 4. Now place the vertices v_1 and v_n with deg v_1 and deg v_n to be $n-1$ as the end points of a horizontal line segment with v_1 to the left of v_n so that the vertices v_1, v_2 and v_n form a triangular face. The edges of the graph Pl_n can be drawn without any crossings. All the faces of this graph are of length 3. The vertex set and edge set of G is denoted as $V(G) = \{v_1, v_2, ..., v_n\}$ and $E(G) = E_1 \cup E_2 \cup E_3$ where $E_1 = \{v_1v_i, v_nv_i : 2 \leq i \leq n-1\}$, $E_2 = \{v_iv_{i+1} : 2 \leq i \leq n-2\}$ and $E_3 = \{v_1v_n\}$. See Figure 1. A bijective function f from $V(Pl_n)$ to $\{k, k+1, ..., k+n-1\}$ is defined as follows. We consider three cases:

Case 1: k and $k+n-1$ are prime numbers

Define $f : V \rightarrow \{k, k+1, ..., k+n-1\}$ by

$f(v_1) = k$

$f(v_n) = k+n-1$

$f(v_i) = k+i-1, \quad 2 \leq i \leq n-1$

For any edge $v_1v_i \in E_1$, $gcd(f(v_1), f(v_i)) = gcd(k, k+i-1) = 1$ since k is a prime number. For any edge $v_nv_i \in E_1$, $gcd(f(v_n), f(v_i)) = gcd(k+n-1, k+i-1) = 1$ since $k+n-1$ is a prime number. For any edge $v_iv_{i+1} \in E_2$, $gcd(f(v_i), f(v_{i+1})) = gcd(k+i-1, k+i) = 1$ since $k+i-1$ and $k+i$ are labeled with consecutive positive integers. For the edge $v_1v_n \in E_3$, $gcd(f(v_1), f(v_n)) = gcd(k, k+n-1) = 1$ since k and $k+n-1$ are prime numbers.

Case 2: k is prime and $k+n-1$ is not prime

Let l_1 be the largest prime number such that $k+1 \leq l_1 \leq k+n-1$. Define $f : V \rightarrow \{k, k+1, ..., k+n-1\}$ by

$f(v_1) = k$

$f(v_n) = l_1$

$$f(v_i) = \begin{cases} k+i-1 & if\ 2 \leq i \leq l_1 - k \\ l_1 + (n-i) & if\ l_1 - k + 1 \leq i \leq n-1 \end{cases}$$

For any edge $v_1v_i \in E_1$, $gcd(f(v_1), f(v_i)) = 1$ since $f(v_1)$ is a prime number. For any edge $v_nv_i \in E_1$, $gcd(f(v_n), f(v_i)) = gcd(l_1, f(v_i)) = 1$ since l_1 is a prime number. For any edge $v_iv_{i+1} \in E_2$, $gcd(f(v_i), f(v_{i+1})) = 1$ since $f(v_i)$ and $f(v_{i+1})$ are labeled with consecutive positive integers. For the edge $v_1v_n \in E_3$, $gcd(f(v_1), f(v_n)) = gcd(k, l_1) = 1$ since k and l_1 are prime numbers.

Case 3: k is not prime and $k+n-1$ is prime

Let l_1 be the largest prime number which is $k+n-1$ and l_2 be the second largest prime number such that $k+1 \leq l_2 \leq k+n-2$. Define $f : V \rightarrow \{k, k+1, \ldots, k+n-1\}$ by

$f(v_1) = l_2$

$f(v_n) = l_1$

$$f(v_i) = \begin{cases} l_2 - (i-1) & if\ 2 \le i \le l_2 - k + 1 \\ l_1 - (n-i) & if\ l_2 - k + 2 \le i \le n - 1 \end{cases}$$

For any edge $v_1v_i \in E_1$, $gcd(f(v_1), f(v_i)) = gcd(l_2, f(v_i)) = 1$ since l_2 is a prime number.

For any edge $v_nv_i \in E_1$, $gcd(f(v_n), f(v_i)) = gcd(l_1, f(v_i)) = 1$ since l_1 is a prime number. For any edge $v_iv_{i+1} \in E_2$, $gcd(f(v_i), f(v_{i+1})) = 1$ since $f(v_i)$ and $f(v_{i+1})$ are labeled with consecutive positive integers. For the edge $v_1v_n \in E_3$, $gcd(f(v_1), f(v_n)) = gcd(l_2, l_1) = 1$ since l_2 and l_1 are prime numbers.

Thus Pl_n is vertex k-prime if $k \ge n$ and at least one of k and $k + n - 1$ is not prime. A simple illustration for case 2 is shown in Figure 2.

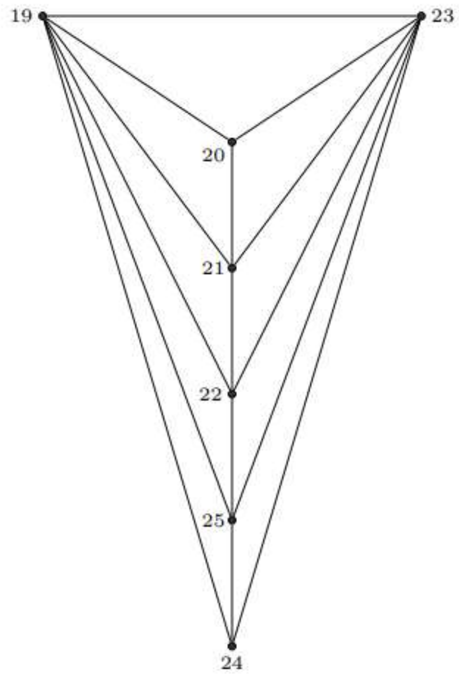

Figure 2. Vertex k-prime labeling of Pl_7 for $k = 19$

Theorem 2.2. *The class $Pl_n : n \ge 5$, odd $k \ge n$ and $k \ge 3$ is not vertex k-prime labeling if both k and $k + n - 1$ are not prime.*

Proof. Let $G = Pl_n$ be a complete planar graph where k and $k + n - 1$ are not prime. Let l_1 be the largest prime number from $k \le l_1 \le k + n - 1$ and let l_2 be the second largest prime number from $k \le l_2 \le l_1 - 1$. Define a bijective function $f : V(Pl_n) \to \{k, k + 1, . . . , k + n - 1\}$ by $f(v_2) = k$; $f(v_1) = l_2$ and $f(v_n) = l_1$. The vertices labeled$l_2 - 1$ and $l_2 + 1$ are adjacent and will be labeled with even integers since l_2 is prime. For any

edge $v_iv_{i+1} \in E(G)$, $gcd(f(v_i), f(v_{i+1})) = gcd(l_2 - 1, l_2 + 1) > 1$ since $l_2 - 1$ and $l_2 + 1$ are both even intergers.

Similarly, define a bijective function $f : V(Pl_n) \to \{k, k + 1, \dots, k + n - 1\}$ by $f(v_2) =$

k; $f(v_1) = l_2$ and $f(v_n) = l_1$. The vertex labeled $l_1 - 1$ is adjacent to the vertex labeled $l_1 + 1$ will also be labeled with even integers since l_1 is prime. For any edge $v_i v_{i+1} \in E(G)$, $gcd(f(v_i), f(v_{i+1})) = gcd(l_1 - 1, l_1 + 1) > 1$ since $l_1 - 1$ and $l_1 + 1$ are both even integers. Therefore, the graph Pl_n is not vertex k-prime when k and $k+n-1$ are not prime.

Theorem 2.3. *The class Pl_n is not vertex k-prime for even n.*

Proof. Let $G = Pl_n$ be a complete planar graph for even n. As a contrary, let us assume G is vertex k-prime for even n. Let l_1 be the largest prime number from $k \le l_1 \le k+n-1$ and let l_2 be the second largest prime number from $k \le l_2 \le l_1 - 1$.

Case 1: $k \equiv 1 (mod\ 2)$

Define a bijective function $f : V(Pl_n) \to \{k, k+1, ..., k+n-1\}$ by $f(v_2) = k$; $f(v_1) = l_2$ and $f(v_n) = l_1$. For odd k, $k + n - 1$ will be an even integer for even n. The adjacent vertices labeled $l_1 - 1$ and $l_1 + 1$ will be even integers since l_1 is largest prime. For any edge $v_i v_{i+1} \in E(G)$, $gcd(f(v_i), f(v_{i+1})) = gcd(l_1 - 1, l_1 + 1) > 1$ since $l_1 - 1$ and $l_1 + 1$ areboth even integers. This is a contradiction to our assumption.

Case 2: $k \equiv 0 (mod\ 2)$

Define a bijective function $f : V(Pl_n) \to \{k, k+1, ..., k+n-1\}$ by $f(v_2) = k; f(v_1) = l_2$ and $f(v_n) = l_1$. For k even, $k + n - 1$ will be an odd integer for even n.

Subcase 1. Suppose $k+n-1$ is prime, the vertices labeled $l_2 - 1$ and $l_2 + 1$ are adjacent and even since l_2 is a prime number. Hence for any edge $v_i v_{i+1} \in E(G)$, $gcd(f(v_i), f(v_{i+1})) = gcd(l_2 - 1, l_2 + 1) > 1$ which is a contradiction.

Subcase 2. Suppose $k + n - 1$ is not prime, the adjacent vertices labeled with $l_2 - 1$ and $l_2 + 1$ are even integers. Similarly, the adjacent vertices labeled with $l_1 - 1$ and $l_1 + 1$ are even integers since both l_1 and l_2 are prime number. Hence for any edge $v_i v_{i+1} \in E(G)$, $gcd(f(v_i), f(v_{i+1})) = gcd(l_2 - 1, l_2 + 1) > 1$ which contradicts our assumption. Similarly for any
edge $v_i v_{i+1}$ $E(G)$, $gcd(f(v_i), f(v_{i+1})) = gcd(l_1 - 1, l_1 + 1) > 1$. This is a contradiction to our assumption.

Therefore, the graph Pl_n is not vertex k-prime for even n.

Theorem 2.4. *Complete graph $K_n : n \ge 4$ is not vertex k-prime for every k.*

Proof. Let $G = K_n$ be complete graph for $n \ge 4$. By contradiction, assume that K_n is vertex k-prime for $n \ge 4$. Let $V(G) = \{v_1, v_2, ..., v_n\}$be n vertices of K_n and $E(G) = \{v_i v_{i+1} \ / \ \forall\ i\}$ be $\frac{n(n-1)}{2}$ edges of K_n. Define a bijective function $f : V(G)$ $\{k, k+1, ..., k+n-1\}$ by

$f(v_i) = k + i - 1,\ 1 \le i \le n$

For any edge $v_i v_{i+2} \in E(G)$, $gcd(f(v_i), f(v_{i+2})) = gcd(k + i - 1,\ k + i + 1) > 1$ for $f(v_i)$ to be even which contradicts our assumption.

For any edge $v_{i+1} v_{i+3} \in E(G)$, $gcd(f(v_{i+1}), f(v_{i+3})) = gcd(k + i, k + i + 2) > 1$ for $f(v_{i+1})$to be even which contradicts our assumption.

Hence K_n is not vertex k-prime for $n \ge 4$.

A simple illustration is shown in Figure 3.

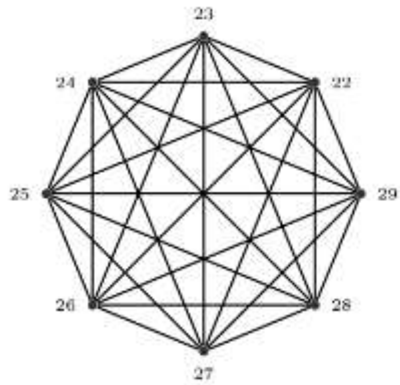

Figure 3. Complete graph K_8 for $k = 22$

Theorem 2.5. *If $G_1(p_1, q_1)$ and $G_2(p_2, q_2)$ has vertex k-prime labeling, then $G_1 \cup G_2$ admits vertex k-prime labeling.*

Proof. Let $f_1 : V(G_1) \longrightarrow \{k, k+1, ..., k+p_1-1\}$ and $f_2 : V(G_2) \longrightarrow \{k, k+1, ..., k+p_2-1\}$ be vertex k-prime labeling of G_1 and G_2. Let $\{u_i, 1 \leq i \leq p_1\}$ be the vertex set of G_1 let $\{v_j, 1 \leq j \leq p_2\}$ be the vertex set of G_2 respectively. Define $f : V(G_1) \cup V(G_2) \longrightarrow \{k, k+1, ..., k+p_1+p_2-1\}$by

$f(u_i) = f_1(u_i), \quad 1 \leq i \leq p_1$

$f(v_j) = f_2(v_j), \quad 1 \leq j \leq p_2$

For any edge $u_iu_{i+1} \in E(G_1) \cup E(G_2)$, $gcd(f(u_i), f(u_{i+1})) = gcd(f_1(u_i), f_1(u_{i+1})) = 1$since f_1 is a vertex k-prime labeling.

For any edge $v_jv_{j+1} \in E(G_1) \cup E(G_2)$, $gcd(f(v_j), f(v_{j+1})) = gcd(f_2(v_j), f_2(v_{j+1})) = 1$since f_2 is a vertex k-prime labeling.

Thus $G_1 \cup G_2$ satisfies the condition of vertex k-prime labeling.

3 Conclusion

In this paper we have proved that the class of planar graphs Pl_n for odd n and $G \cup K_{1,n}$ are vertex k-prime and the class of planar graphs Pl_n for even n and complete graph K_n are not vertex k-prime. The study of the existence of vertex *k*prime labeling for other families of graphs is an area for further investigation.

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