Non split Monophonic sets of the Join and Corona product of graphs

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Abstract

A monophonic set M of a connected graph G is said to be a *non split monophonic set* if $M = V(G)$ or the subgraph $\langle V - M\rangle$ is connected. The minimum cardinality of the *non split monophonic set* is the non split monophonic number of G and is denoted by $m_{ns}(G)$. In this paper we have characterised the non split monophonic sets of the join and corona of two graphs. Also, we have calculated the non split monophonic number of the join and corona product of two graphs.

Keywords: Monophonic set, Non split monophonic set, Join, Corona product.

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Introduction

The graph G considered in this paper is finite, simple, undirected and connected with vertex set $V(G)$ and edge set $E(G)$ respectively. The order and size of G is denoted by n and m respectively. [3, 4] is referred for basic graph theoretic definitions. The *distance* $d(u, v)$ between two vertices u and v is the length of a shortest $u - v$ path in G. The *neighborhood* of a vertex v denoted by $N(v)$ is the set of all vertices adjacent to v. For any subset $S \subseteq V(G)$, the *induced subgraph* $\langle S\rangle$ is the maximal subgraph of G with the vertex set S. A vertex v is said to be an *extreme vertex* if the subgraph $\langle N(v)\rangle$ is complete. The join of graphs G and H, denoted by $G + H$ is the graph with $V(G + H) = V(G) \cup V(H)$ and $E(G + H) = E(G) \cup E(H) \cup \{uv : u \in V(G) and v \in V(H)$. Two vertices (x_1, y_1) and (x_2, y_2) in the cartesian product $G\square H$ is adjacent if and only if either $x_1 = x_2$ and y_1 is adjacent to y_2 or $y_1 = y_2$ and x_1 is adjacent to x_2. $F \subseteq V(G)$ is said to be a dominating set if every vertex in $V(G) - F$ is adjacent to at least one vertex in F. The least order of the dominating sets of G is said to be the domination number of G and is denoted by

$\gamma(G)$. The monophonic number of a graph was studied by Pelayo et.al in [1, 6]. Any chordless path connecting the vertices u and v is called a $u - v$ m-path. The monophonic closure of a subset S of $V(G)$ is given by $J_G[S] = \cup_{u,v \in S} J_G[u,v]$, where $J_G[u,v]$ is the set containing the vertices u, v and all vertices lying on some $u - v$ m-path. If $J_G[S] = V(G)$, then S is known to be a *monophonic set* in G. A monophonic set in G of minimum order is called a minimum monophonic set of G. The order of a minimum monophonic set in G is called the *monophonic number* of G and is denoted by $m(G)$. The monophonic number of the join and composition of graphs by Paluga et al.[2] motivates us to study the non split monophonic number of join and corona of graphs. The non split monophonic number of a graph was introduced and studied by P. Arul Paul Sudhahar and Merlin Sugirtha in [5]. A monophonic set M of a connected graph G is said to be a *non split monophonic set* if $M = V(G)$ or the subgraph $\langle V - M \rangle$ is connected. The minimum cardinality of the *non split monophonic set* is the non split monophonic number of G and is denoted by $m_{ns}(G)$.
In the second section non split monophonic sets and the non split monophonic number of the join of two graphs is studied. In the third section non split monophonic sets and the non split monophonic number of the corona of two graphs is studied.

1 Non split monophonic set of the Join of Graphs

Theorem 2.1. Let G be a non complete graph and $S \subseteq V(G + K_p)$. If $S \cap V(G)$ is a non split monophonic set of G, then S is a non split monophonic set of $G + K_p$.

Proof. Let $T = S \cap V(G)$ be a non split monophonic set of G. Then there exists vertices $x, y \in T$ such that $a \in J_G[x,y]$, for every $a \in V(G)\backslash T$. Also, for any vertex $u \in V(K_p)$ the monophonic path x, u, y contains all the vertices of $V(K_p)$. Hence $J_{G+K_p}[T] = V(G)$. Hence $S \cap V(G)$ is a monophonic set of $G + K_p$. Since $T \subseteq S$, S is a monophonic set of $G + K_p$. Since T is a non split monophonic set $\langle V(G) - T \rangle$ is connected.

Theorem 2.2. Let G be a non complete graph. $S \subseteq V(G + K_p)$ is a non split monophonic set of $G + K_p$ if and only if $S \cap V(G)$ is a monophonic set of G.

Proof. Let $S \subseteq V(G + K_p)$ be a non split monophonic set of $G + K_p$. If $S \cap V(G) = V(G)$, then it is clear that $S \cap V(G)$ is a monophonic set of G. Suppose $S \cap V(G) \neq V(G)$. Let $a \in V(G)\backslash[S \cap V(G)]$. Then there exists vertices $x, y \in S$ such that $a \in J_{G+K_p}[x,y]$. Suppose $x \in V(G)$ and $y \in V(K_p)$. Then $d_{G+K_p}[x,y] = 1$, which implies that $a \notin J_{G+K_p}[x,y]$, a contradiction. Also, if $\{x,y\} \in V(K_p)$, we get a contradiction. Hence the only possibility is $\{x, y\} \in V(G)$. Since a is arbitrary, it can be shown that $a \in J_{G+K_p}[x, y]$, where $x, y \in S \cap V(G)$. Thus $S \cap V(G)$ is a monophonic set of G. Conversely, assume that $S \cap V(G)$ is a monophonic set of G. Since G is non complete, there exists vertices $a_1, a_2 \in S \cap V(G)$ and $a_1a_2 \notin E(G)$. a_1, u, a_2 will be the monophonic path connecting the vertex u of K_p. Since u is arbitrary, it is clear that $J_{G+K_p}[S \cap V(G)] \supseteq V(K_p)$. Hence $S \cap V(G)$ is a monophonic set of G. Since every vertex of G is adjacent to each vertex of K_p, $\langle V(G) - [S \cap V(G)]\rangle$ is connected. Hence $S \cap V(G)$ is a non split monophonic set of $G + K_p$. Since $[S \cap V(G)] \subseteq S$, it follows that S is a non split monophonic set of G.

Corollary 2.3. Let G be a non complete graph. $S \subseteq V(G + K_p)$ is a minimum non split monophonic set of $G + K_p$ if and only if $S \cap V(G)$ is a minimum monophonic set of G.

Proof. By Theorem 2.2, it is clear that $S \subseteq V(G + K_p)$ is a minimum non split monophonic set of $G + K_p$ if and only if $S \cap V(G)$ is a minimum monophonic set of G.

Corollary 2.4. For the non complete graph G, $m_{ns}(G + K_p) = m(G)$.

Proof. It follows from Corollary 2.3.

Theorem 2.5. For the connected graph G of order n and the complete graph K_p,

$$m_{ns}(G + K_p) = \begin{cases} m(G) & if \ \ G \neq K_n \\ n + p & if \ \ G = K_n \end{cases}.$$

Proof. Let us assume that $G = K_n$. Then, $G + K_p \cong K_{n+p}$. Since the monophonic set contains all the vertices of K_{n+p}, it follows that $m_{ns}(K_n + K_p) = n + p$. Next, let us consider the case when $G \neq K_n$. It holds from Corollary 2.4.

Theorem 2.6. Let G and H be non complete graphs and $S \subseteq V(G + H)$. If $S \cap V(G)$ is a non split monophonic set of G, then S is a non split monophonic set of $G + H$.

Proof. Let $S_1 = S \cap V(G)$ be a non split monophonic set of G. Then there exists vertices $x, y \in S_1$ such that $a \in J_{G+H}[x, y]$ for all $a \in V(G)\setminus S_1$. Also, for any vertex $u \in V(H)$ the monophonic path x, u, y will cover all the vertices of H. Hence $J_{G+H}[S_1] = V(G + H)$. Since $S_1 \subseteq S$, it follows that S is a monophonic set of $G + H$. Since S_1 is a non split monophonic set $\langle V(G) - S_1\rangle$ is connected. Since every vertex of G is adjacent to each vertex of H it is clear that $\langle V(G) - S\rangle$ is connected. Hence S is a non split monophonic set of $G + H$.

Theorem 2.7. Let G and H be non complete graphs and $S \subseteq V(G+H)$. If $S \cap V(H)$ is a non split monophonic set of H, then S is a non split monophonic set of $G + H$.

Proof. Proof is similar to that of Theorem 2.6.

Theorem 2.8. $S \subseteq V(G + H)$ is a non split monophonic set of $G + H$ if and only of at least one of the following conditions holds

1. $S \cap V(G)$ is a monophonic set of G.
2. $S \cap V(H)$ is a monophonic set of H.
3. There exist vertices $u, v \in S \cap V(G)$ and $x, y \in S \cap V(H)$ such that $xy \notin E(H)$ and $uv \notin E(G)$.

Proof. Let $S \subseteq V(G + H)$ be a non split monophonic set of $G + H$. Let $x \in V(G) \backslash S$. Since S is a non split monophonic set of $G + H$, there exist non adjacent vertices $a, b \in S \subseteq V(G + H)$ such that $x \in J_{G+H}[a, b]$.
Case 1: $a \in V(G)$ and $b \in V(H)$. Then by the definition of join we have, $d_{G+H}[a, b] = 1$. Thus $x \notin J_{G+H}[a, b]$. Hence S is not a non split monophonic set of $G + H$, which is a contradiction.

Case 2: $a \in V(G)$ and $b \in V(G)$. Since S is a non split monophonic set of $G + H$, $x \in J_G[a, b]$. Thus, $S \cap V(G)$ is a monophonic set of G.
Case 3: $a, b \in V(H)$. Suppose $T_1 = \{a, b\}$ and it is a non split monophonic set of H then it comes under (2). Now, let us consider the case when $T_1 = \{a, b\}$ is not a non split monophonic set of H. Since $ab \notin E(H)$, $J_{G+H}[a, b] = V(G) \cup J_H[a, b]$. Hence T_1 forms a monophonic set of G. If $S = S_1$, then we arrive at a contradiction. Thus $S \neq S_1$. By taking two non adjacent vertices from G we can form a monophonic set of $G + H$. Let $T_2 = T_1 \cup \{u, v\}$, where $\{u, v\} \in V(G)$ and $uv \notin E(G)$. It follows that $J_{G+H}[T_2] = V(G + H)$ and $\langle V(G + H) - T_2 \rangle$ is connected. Therefore, $S = T_2$ forms a non split monophonic set $G + H$ with $\{a, b\} \in S \cap V(H)$ and $\{u, v\} \in S \cap V(G)$. Hence (3) holds. By considering a vertex $x \in V(H) \backslash S$ and working on as above we can prove (2).
Conversely, assume that $S \cap V(G)$ is a monophonic set of G. It follows that $J_G[S \cap V(G)] = V(G)$. Also, $J_{G+H}[S \cap V(G)] = V(G + H)$. Hence $S \cap V(G)$ is a monophonic set of G. Since every vertex of G is adjacent to each vertex of H , $\langle V(G + H) - [S \cap V(G)] \rangle$ is connected. Hence S is a non split monophonic set of $G + H$.
Next let us assume that $S \cap V(H)$ is a monophonic set of H. It follows that $J_H[S \cap V(H)] = V(H)$. Also, $J_{G+H}[S \cap V(H)] = V(G + H)$. Hence $S \cap V(H)$ is a monophonic set of H. Since every vertex of G is adjacent to each vertex of H, $\langle V(G + H) - [S \cap V(H)] \rangle$ is connected. Hence S is a non split monophonic set of $G + H$.

Now let us assume that there exists vertices $u, v \in S \cap V(G)$ and $x, y \in S \cap V(H)$ such that $xy \notin E(H)$ and $uv \notin E(G)$. Let $K = \{u, v, x, y\}$ and $J_{G+H}[K] = V(G + H)$ and $\langle V(G + H) - K \rangle$ is connected. Hence K will be a non split monophonic set of $G + H$. Since $K \subseteq S$, it follows that S is a non split monophonic set of $G + H$.

Theorem 2.9. For the non complete graphs G and H , $m_{ns}(G + H) = min\{4, m(G), m(H)\}$.

Proof. It follows from Theorem 2.8.

2 Non split monophonic set of the Corona product of graphs

The *corona product* $G \circ H$ of two graphs G (with p_1 vertices) and H (with p_2 vertices) is defined as the graph obtained by taking one copy of G and p_1 copies of H and then joining the i^{th} vertex of G to every point in the i^{th} copy of H. For each vertex $v \in V(G)$, let H_v denotes the copy of H whose vertices are joined to the vertex v. Also, $v + H_v$ denotes the subgraph of the corona $G \circ H$ corresponding to the join $\langle \{v\} + H_v \rangle$. Let $|S|$ denotes the cardinality of S.

Theorem 3.1. Let G be a connected graph of order n and H be a connected graph of order m. If S is a non split monophonic set of $G \circ H$, then $S \cap V(H_v) \neq \phi$ for every $v \in V(G)$.

Proof. Suppose $S \cap V(H_v) = \phi$ for some $v \in V(G)$. The vertices of H_v is adjacent only to the vertex $v \in V(G)$ apart from the vertices in the copy H_v. Hence $V(H_v) \notin J_{G \circ H}[S]$. Since $\langle V(G) - S \rangle$ is not connected. It follows that S is not a non split monophonic set of $G \circ H$, which is a contradiction. Hence $S \cap V(H_v) \neq \phi$ for every $v \in V(G)$.

Theorem 3.2. Let G and H be connected graphs of order n *and* m respectively. If S is a minimum non split monophonic set of $G \circ H$, then $S \cap V(G) = \phi$.

Proof. Suppose $S \cap V(G) \neq \phi$. Let $v \in S \cap V(G)$. By (1), $S \cap V(H_v) \neq \phi$ for every $v \in V(G)$. For any two vertices $x_v, y_v \in S \cap V(H_v)$, we have $v \in J_{G \circ H}[x_v, y_v]$. Hence $S - \{v\}$ is a monophonic set of $G \circ H$. Also, $\langle V(G \circ H) - [S - \{v\}] \rangle$ is connected. Thus $S - \{v\}$ is a non split monophonic set of $G \circ H$, which is a contradiction. Therefore, $S \cap V(G) = \phi$.

Theorem 3.3. Let G and H be connected graphs. $S \subseteq V(G \circ H)$ is a non split monophonic set of $G \circ H$, then $S_v = S \cap V(v + H_v)$ is a non split monophonic set of H_v for every $v \in V(G)$.

Proof. Let $S \subseteq V(G \circ H)$ be a non split monophonic set of $G \circ H$. We have to prove that $S_v = S \cap V(v + H_v)$ is a non split monophonic set of H_v for every

$v \in V(G)$. Let $x \in V(v + H_v)\backslash S_v$. If $x = v$, then $x \in J_{v+H_v}[a_v, b_v]$ for some $a_v, b_v \in S_v$ and $a_v b_v \notin E(H_v)$. Hence S_v is a monophonic set of H_v. Since x is adjacent to every vertex of H_v, $\langle V(v + H_v) - S_v \rangle$ is connected. It follows that S_v is a non split monophonic set of H_v for every $v \in V(G)$. Now let us consider the case when $x \neq v$. It implies that $x \in V(H_v)$ for some $v \in V(G)$. Since S is a non split monophonic set of $G \circ H$, there exist vertices a_v, b_v in S_v such that $x \in J_{G \circ H}[a_v, b_v]$. Also, $\langle V(v + H_v) - S_v \rangle$ is connected. Hence S_v is a non split monophonic set of H_v for every $v \in V(G)$.

Theorem 3.4. Let G and H be connected graphs. $S \subseteq V(G \circ H)$ is a non split monophonic set of $G \circ H$, then $S_v = S \cap V(v + H_v)$ is a monophonic set of H_v for every $v \in V(G)$.

Proof. By Theorem 3.3, it is clear that S_v is a non split monophonic set of H_v for every $v \in V(G)$. Hence the proof follows.

Corollary 3.5. Any minimal non split monophonic set of $G \circ H$ contains exactly $m(H)$ vertices from each copy H_v of $G \circ H$.

Proof. Let S be a minimum non split monophonic set of $G \circ H$. By Theorem 3.4, we have $S_v = S \cap V(v + H_v)$ is a monophonic set of H_v for every $v \in V(G)$. Thus, the non split monophonic set of $G \circ H$ has at least $m(G)$ vertices from each copy H_v of $G \circ H$. It follows that the minimum non split monophonic set of $G \circ H$ contains exactly $m(H)$ vertices from each copy H_v of $G \circ H$.

Theorem 3.6. Let G and H be two graphs of order p and q respectively. Then, $m_{ns}(G \circ H) = p.m(H)$.

Proof. Let S be a minimum non split monophonic set of $G \circ H$. By Corollary 3.5 it follows that S contain exactly $m(H)$ vertices from each copy H_v of $G \circ H$. Since there exist p copies of H, it follows that $m_{ns}(G \circ H) = p.m(H)$.

4. Conclusion

The non split monophonic sets and non split monophonic number of the join and corona product of graphs is determined in this paper. This method can also be extended to calculate the non split monophonic sets of some other binary operations of graphs.

References

[1] C. Hernando, T. Jiang, M. Mora, I. M. Pelayo, and C. Seara, On the steiner, geodetic and hull numbers of graphs, Discrete Mathematics, vol. 293, no. 1-3, 139–154, 2005.

[2] E. M. Paluga and S. R. Canoy Jr, Monophonic numbers of the join and composition of connected graphs, Discrete Mathematics, vol. 307, no. 9-10, 1146–1154, 2007.

[3] F. Buckley and F. Harary, Distance in graphs, Addison Wesley, Redwood City, 1990.

[4] F. Harary, Graph theory, Addison Wesley, 1969.

[5] F. Merlin Sugirtha, Monophonic distance related parameters in graphs, PhD thesis, Manonmaniam Sundaranar University, Tirunelveli, 2020.

[6] I. M. Pelayo, Comment on "the steiner number of a graph" by g. chartrand and p. zhang: [discrete mathematics 242 (2002) 41–54], Discrete mathematics, vol. 280, no. 1-3, 259–263, 2004.