gΔ* and $g_s\Delta$*-closed sets in Ideal Spaces

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Abstract

We defined gΔ* and $g_s\Delta$*-closed in terms of ideal spaces in the paper, and some of their properties and characterizations are discussed. Further, we study its interrelationships between some existing generalizations and are investigated these closed sets. The paper's goal is to deepen the investigation of a few types of closed sets in ideal spaces which satisfies the Kuratowski closure operator. These sets exactly lies between δ* and I_g-closed sets.

Keywords and phrases: Ideal topological spaces, g-closed, gΔ*-closed, $g_s\Delta$*-closed, I_g- closed, δ-closed.

Preliminaries

The concept of generalized topology was introduced by A. Csaszar which is considered one of the indispensable developments of general topology. In the year 1970, generalized closed (g-closed) sets were introduced by Levine as generalization of closed sets, along with the $T1/2$- space class of topological spaces. Thereafter, closed sets were generalized based on these sets. S.P. Arya and T. Nour[8], Bhattacharya and Lahiri [12], Maki, Devi, and Balachandran [1], Velico [7], Abd El-Monsef [10], Dontchev and Ganster[2], Dontchev and Maki [3], Veerakumar [5] introduced and explored gs-closed set. Palaniappan and Rao [11] introduced and studied rg-closed in 1993. Kuratowski [4] also specified a local function of the ideal I with topology τ for the power set of X which is usually denoted as A*(I, τ) or simply A*.

Further, in 1945, Vaidyanathaswamy [4] expanded the research of the ideal space where we have τ is coarser than τ^*. Additionally, a study is done in the Kuratowski closure operator [9], I_g-closed
sets were researched and investigated by Dontchev,Ganster, and Noiri in 1999 and Navaneethakrishnan and Paulraj [6] carried out additional research on the topic. Let A be a subset of X ideal space which has δ*-local closure function of A which is denoted by A* (I, τ) [13].

We focus on the topology of dataset which is made up of points. Then, we discuss a particular way
of quantify the topology of continuous functions. The main advantages of the functions are its critical points. The topological data analysis is largely discussed with statistical and signal processing techniques. In the recent development of algebraic and computational topology of data science has explored into the new area which is called topological data analysis(TDA)[14].We can define the concept of image analysis using TDA. In particular , we analyze the optical results of liquid crystal sensors to air contaminants which is discussed in [15] and [16].

1.gΔ*-Closed Set

In the section, we introduce gΔ*-closed sets of ideal spaces. Additionally, we will discuss few of the results.

Definition 1 Let A be a subset of X in ideal space if $A_\delta^* \subseteq U$ provided $A \subseteq U$ with $U \in \tau$ is called gΔ*-closed.

Example 1 Let $X = \{a_1, b_1, c_1, d_1\}, \tau = \{X, \emptyset, \{a_1\}, \{b_1\}, \{c_1\}, \{a_1, b_1\}, \{a_1, c_1\}, \{b_1, c_1\}, \{a_1, b_1, c_1\}\}, I = \{\emptyset, \{a_1\}, \{b_1\}, \{a_1, b_1\}\}$. Then, $A = \{a_1, c_1, d_1\}$ is gΔ*-closed set.

Theorem 1 Any θg-closed of a subset is gΔ*-closed.

Proof. We have $C^* \subseteq cl_\theta(C) \Rightarrow C^* \subseteq U$ if any θg-closed with U is an ideal space provided

$C \subset U$ where C is of gΔ*-closed.

Remark 1 The Theorem 1 converse is not always true, as demonstrated by Example 2 which is given below.

Example 2 Let $X = \{a_1, b_1, c_1, d_1\}, \tau = \{X, \emptyset, \{b_1\}, \{c_1\}, \{d_1\}, \{b_1, c_1\}, \{b_1, d_1\}, \{c_1, d_1\}, \{b_1, c_1, d_1\}\}, I = \{\emptyset, \{a_1\}, \{c_1\}, \{a_1, c_1\}\}$. Then $C = \{c_1\}$ is gΔ*-closed but not θg-closed.

Theorem 2 Each δg-closed is gΔ*-closed.

Proof. 2 Let $V \in \tau$ in ideal space and $B \subseteq V$, where B is of δg-closed since $B_\delta^* \subseteq cl_\delta(B)$ then
B is δ-closed. Then, $B_\delta^* \subseteq V$, we conclude that B is gΔ*-closed.

Remark 2 The counterpart of Theorem 2 does not hold always, as mentioned in Example 3

Example 3 Let $X = \{c, d, e, f\}$, $\tau = \{X, \emptyset, \{c\}, \{f\}, \{c, f\}\}$, $I = \{\emptyset, \{d\}, \{e\}, \{d, e\}\}$. In that case, $B = \{e\}$ is not have δg-closed but have gΔ*-closed.

Theorem 3 Each gΔ*-closed is I_g-closed.

Proof. 3 Assume E, a gΔ*-closed and $W \in \tau$ in ideal space such that $E \subseteq W$. Since we have
$E^* \subseteq E_\theta^*$ implies that $E^* \subseteq W$, where $W \in \tau$. Hence, E is I_g-closed set.

Theorem 4 Each δ-closed set has gΔ*-closed.

Proof. 4 Presume D is a δ-closed with V∈ τ such that D⊆ V .we have $D_\delta^* \subseteq cl_\delta(D) \Rightarrow D_\delta^* \subseteq V$ then D is gΔ*-closed. Hence, the result is true.

Remark 3 In general, the reverse of the implication of Theorem 4 does not satisfies always. Example 4 Take X={ a, b, e, f}, τ ={ X, ∅, {a},{e},{a, e}}, I={∅, {f}}. When D={f} is not have δ but has gΔ*-closed.

Characterizations

Using ideal space, we investigate a few gΔ*-closed sets and their characteristics and attributes.

Theorem 5 Let C be a non-empty ideal space of X. Consequently, they are equivalent:

1. C is gΔ*-closed.
2. $cl_\delta(C^*) \subseteq V$ each V ∈ τ such that C ⊆ V .
3. For each $y \in cl_\delta(C^*)$, cl({y}) ∩ C ≠ ∅.
4. $cl_\delta(C^*) - C$ contains no closed set which has non-empty.

Proof. 5 (i) ⇒ (ii). Allow C to be a gΔ*-closed set with C ⊆ V for each V ∈ τ. Then, $V_\delta^* \subseteq V$ implies that $cl(C_\delta^*) \subseteq V$ provided C ⊆ V with V is open.
(ii) ⇒ (iii). Suppose assume $y \in cl(C_\delta^*)$. If cl({y}) ∩ C = ∅, then C ⊆ X − cl({y}). By (ii),
$cl(C_\delta^*) \subseteq X - cl(\{y\})$. There is a contradiction. Hence, cl({y}) ∩ C≠ ∅.
(iii) ⇒ (iv). Suppose, $C \subseteq cl(C_\delta^*) - C$, where y is in C and C is closed. Since C is a set
of X − C, cl({y}) ⊂ cl(C) = C. Therefore, cl({y}) ⊆ X − C and thus, cl({y}) ∩ C = ∅. Since $y \in cl(C_\delta^*)$ by (iii), cl({y}) ∩ C=∅. There is a contradiction.
(iv) ⇒ (i). Let's assume, C the set does not have gΔ*-closed set. Then V can be of any open
set C ⊂ V, $C_\delta^* \not\subseteq V \Rightarrow C_\delta^* \cap (X - V) = \emptyset$. By which, we have $C_\delta^* \cap (X - V) \subseteq cl(C_\delta^*) \cap (X - V) \subseteq cl(C_\delta^*) \cap (X - C) = cl(C_\delta^*) - C$. Therefore, here is a contradiction with our assumption. Hence C is $g\Delta^*$-closed.

Theorem 6 If C ⊆ X in ideal space, then these statements are equivalent:

1. C is gΔ*-closed.
2. $Cl_\delta^*(C) \subseteq W$ provided C ⊆ W and W ∈ τ.

Proof. 6 (i) ⇒ (ii). Let C ⊆ W and W ∈ τ be open. Because of C is gΔ*-closed, $C_\delta^* \subseteq W$ by which $cl_\delta(C) = C \cup C_\delta^* \subset W$.

(ii) ⇒ (i). Since $C_\delta^* \subseteq cl_\delta^*(C)$, the proof holds.

If $I = \emptyset$, then the following Corollary 1 holds.

Corollary 1 The following statements are equivalent when X is an ideal space with $G \subset X$.

1. G is δg-closed
2. $cl_{\delta}(G) \subseteq V$ wherein $G \subseteq V$ with $V \in \tau$.

Corollary 2 The ideal space with $G \subseteq X$. We can prove, $cl^*(G) \subseteq X$ provided $G \subseteq V$ and V is an open set if G is $g\Delta^*$-closed set.

Proof. 7 Let's assume G has $g\Delta^*$-closed with $G \subseteq V$, $V \in \tau$ which implies, $G_{\delta}^* \subset V$. Then, whenever $G \subseteq V$ and V is open where $G^* \subset V$ implies $cl^*(G) = G \cup G^* \subset V$

Remark 4 In general the Corollary 2 opposite inference does not hold.

Example 5 Let $X = \{a_1, b_1, d_1, f_1\}$, $\tau = \{X, \emptyset, \{a_1\}\}$, $I = \{\emptyset, \{a_1\}\}$.Then, $B = \{b_1, d_1, f_1\}$ is not $g\Delta^*$-closed.

Theorem 7 In ideal space whenever joining of any two $g\Delta^*$-closed are always $g\Delta^*$-closed.

Proof. 8 Let $V \in \tau$ in (X, τ, I) with C and F are any two sets of V with $C \cup F \subseteq V \Rightarrow C \subseteq V$ and $F \subseteq V$ .we have C and F are $g\Delta^*$-closed sets then $C_{\delta}^* \subseteq V$ and $F_{\delta}^* \subseteq V \Rightarrow (C \cup F)_{\delta}^* = C_{\delta}^* \cup F_{\delta}^*$. Then, $(C \cup F)_{\delta}^* \subseteq V$ which implies $C \cup F$ is $g\Delta^*$-closed.

Theorem 8 When $G \subseteq X$ is an ideal space, the results are identical as shown below.

1. G is $g\Delta^*$-closed.
2. G has only the empty sets in X, $g\Delta^* - G$.

Proof. 9 (i) $\Rightarrow$ (ii). Suppose G may have closed set thereby $G \subseteq G_{\delta}^* - G$ that is $G \subseteq X - G$ then $G_{\delta}^* \subseteq X - G$ hence $G \subseteq X - G_{\delta}^*$.Therefore, $G \subseteq (G_{\delta}^* - G) \cap (X - G_{\delta}^*) = (G_{\delta}^* \cap (X - G)) \cap (X - G_{\delta}^*) = \emptyset$
(ii) $\Rightarrow$ (i). By the assumption that G is not $g\Delta^*$-closed. Then V is open, which means that $C \subseteq V$, $G_{\delta}^* \not\subset V$ so $G_{\delta}^* \cap (X - V) \neq \emptyset$. We can have as, $G_{\delta}^* \cap (X - V) \subseteq G_{\delta}^* \cap (X - G)$. Whence, $G_{\delta}^* \cap (X - G)$ is a closed set. Hence our assumption is wrong. So, G is $g\Delta^*$-closed.

Corollary 3 The $F_{\delta}^* - F$ does not include any set of regular closed sets in ideal space whenever the F will have $g\Delta^*$-closed

Theorem 9 Let $C \subseteq X$ be an ideal space. Assume, D is a $g\Delta^*$-closed when C is of $g\Delta^*$-closed set of X with $C \subseteq D \subseteq C_{\delta}^*$

Proof. 10 Since $D_{\delta}^* - D \subseteq C_{\delta}^* - C$ and C is $g\Delta^*$-closed, by Theorem 8, $C_{\delta}^* - C$ have any non-empty closed subset and hence $D_{\delta}^* - D$ there is no non-empty closed subset of D

⇒ D is gΔ*-closed set again by Theorem 8.

Theorem 10 Let C and E be subsets in (X, τ, I) with $C \subseteq E \subseteq C^*$ When C is gΔ*-closed we prove that E is I_g-closed

Proof. 11 Given that C is of gΔ*-closed, it follows that C includes no closed sets that are not empty. Here, we have $E^* - E \subseteq C^* - C$. Therefore, since E takes a closed set containing empty then, E is I_g-closed according to theorem 4.

Theorem 11 A set E is gΔ*-closed if Ker(E) is gΔ*-closed when $E \subseteq X$ in ideal space.
Proof. 12 Given, Ker(E) is of gΔ*-closed. If $E \subseteq V$ and $V \in \tau$, we have $Ker(E) \subseteq V$. Since Ker(E) is gΔ*-closed ⇒$(Ker(E)_\delta^* \subseteq V$ and $E_\delta^* \subseteq (Ker(E))_\delta^*$. Hence, E is gΔ*-closed.

Theorem 12 A set G is in ideal space such that gΔ*-closed iff $G_\delta^* \subseteq Ker(G)$.

Proof. 13 Necessary: It is essential to assume the G is gΔ*-closed set and also that $y \notin Kerf(G)$. We have an open set V in a way $G \subseteq V$ with $y \notin V$.Because G is gΔ*-closed, $G_\delta^* \subseteq V$ which leads to a contradiction. Hence,$G^* \subseteq Ker(G)$.
Sufficiency: Given, then $G_\delta^* \subseteq Ker(G)$ and $V \in \tau$ provided $G \subseteq V$. Further, $Ker(G) \subseteq V$ by which $G_\delta^* \subseteq V$. Whence, G is gΔ*-closed of ideal space.

Theorem 13 Assume that $y \in X$, and either {y} is closed or $\{y\}^C$ is a gΔ* in ideal space then, $X = X_c \cup X_{g\Delta^*}$.

Proof. 14 Assume {y} does not have a closed set in ideal space. Since $\{y\}^C$ be not having an open set which is containing an open that does so $\{y\}^C$ is X. Consequently, $(\{y\}^C)_\delta^* \subseteq X$ hence $\{y\}^C$ is of gΔ*-closed.

2. $g_s\Delta^*$-Closed Set

In next succeeding section, we begin with $g_s\Delta^*$-closed sets and examine how they relate to other classes of sets that are well-known.

Definition 2 In an ideal space G be a set defined as $g_s\Delta^*$-closed if $G_\delta^* \subseteq U$ when $G \subseteq U$ provided U is semi-open in ideal space. In general, the opposite of $g_s\Delta^*$-closed in ideal space is $g_s\Delta^*$-open.

Theorem 14 Every δ-closed is $g_s\Delta^*$-closed.

Proof. 15 Suppose G is any δ-closed set, $V \in \tau$ in ideal space with $G \subseteq V$. Then, we have G as δ-closed $G_\delta^* \subseteq cl_\delta(G) \Rightarrow G_\delta^* \subseteq V$. Hence, G is $g_s\Delta^*$-closed.

Remark 5 From the following example 7, we can conclude that Theorem 14 will not true always.

Example 7 Let $X = \{a, b, d, e\}$, $\tau = \{\emptyset, X, \{a\}, \{d\}, \{e\}, \{a, d\}, \{a, e\}, \{d, e\}, \{a, d, e\}\}$, $I = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$. Consequently, when $G = \{b\}$ is $g_s\Delta^*$-closed which is not δ-closed.

Theorem 15 Each θ-closed is $g_s\Delta^*$-closed.

Proof. 16 Take V any semi-open set of an ideal space with S being any set of closed so that $S \subseteq V$. We had when S is θ-closed then $S_\delta{}^* \subseteq cl_\theta(S)$ which implies that $S_\delta^* \subseteq V$. Thus, S is $g_s\Delta^*$-closed.

Remark 6 From the next example 8 the counter-statement of the above theorem 15 does not hold always.

Example 8 Assume $X = \{a, c, d, e\}$, $\tau = \{\emptyset, X, \{a\}, \{e\}, \{a, e\}\}$, $I = \{\emptyset, \{a\}, \{c\}, \{a, c\}\}$. Then $S = \{c, d\}$ is of $g_s\Delta^*$-closed but which does not of having θ-closed.

Theorem 16 Each $g_s\Delta^*$-closed set of I_g-closed.

Proof. 17 Specify T as any $g_s\Delta^*$-closed set and $W \in \tau$ such that $T \subseteq W$. As a result $T_\delta^* \subseteq T^*$ and each open set is in semi open $T_\delta^* \subseteq T \Rightarrow T^* \subseteq W$. Therefore, T is I_g-closed.

Remark 7 Theorem 16 does not satisfy always which is explained by example 9.

Example 9 Let $X = \{a, e, f, g\}$, $\tau = \{\emptyset, X, \{e\}, \{f, g\}, \{e, f, g\}\}$, $I = \{\emptyset, \{a\}, \{e\}, \{a, e\}\}$. Where, $T = \{a, b\}$ is I_g-closed but, not having the $g_s\Delta^*$-closed.

Theorem 17 Each $g_s\Delta^*$-closed is $g\Delta^*$-closed.

Proof. 18 Consider F to be a, $g_s\Delta^*$-closed with $V \in \tau \Rightarrow F \subseteq V$. Since all open sets are semi-open then $F^* \subseteq V$, where F is $g\Delta^*$-closed.

Remark 8 The case reveals that it is generally false for the inverse of Theorem 17.

Example 10 Let $X = \{a, b, e, f\}$, $\tau = \{\emptyset, X, \{a\}, \{b\}, \{e\}, \{a, b\}, \{a, e\}, \{b, e\}, \{a, b, e\}\}$, $I = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$. Here,the set F is of $g\Delta^*$-closed all but the $g_s\Delta^*$-closed if $F = \{b, f\}$

Theorem 18 Each τ_δ*closed is $g_s\Delta^*$-closed.

Proof. 19 Assume H is a τ_δ^*-closed with W is of semi-open with $H \subseteq W$. We have, $H_\delta^* \subseteq cl_\delta^*(H)$, with H is τ_δ^*-closed $\Rightarrow$ $H_\delta^* \subseteq W$. Therefore, H is $g_s\Delta^*$-closed.

Remark 9 The example demonstrates that the inverse of the Theorem 18 does not frequently hold.

Example 11 Let X ={a, c, f, g}, τ = {∅, X, {a}, {c}, {f}, {a, c}, {a, f}, {c, f}, {a, c, f}},
I ={∅, {a}, {c}, {a, c}}. Then, B is $g_s\Delta^*$-closed all but τ_δ^* -closed when B = {c, g}.

Characterizations

In this part, we explore few of the essential characteristics of $g_s\Delta^*$-closed an ideal space.

Theorem 19 If C is a $g_s\Delta^*$-closed in ideal space, then $C_\delta^* - C$ contains none of the non-empty closed an ideal space.

Proof. 20 Let C represents any of the closed sets in ideal space, then $C \subseteq C_\delta^* - C$ thereby
$C \subseteq X - C$. In (X, τ, I), $X - C$ is open then semi-open. Though C is of $g_s\Delta^*$-closed, $C^* \subseteq X - C$.
Thus, $C \subseteq X - C_\delta^*$ As a result $C \subseteq (C_\delta^* - C) \cap (X - C_\delta^*) = \emptyset$.

Remark 10 According to Example 12, Theorem 19 does not apply generally.

Example 12 Let X = {a, c, f, g}, τ = {∅, X, {a}}, I = {∅, {a}}, the $g_s\Delta^*$-closed sets, {∅, X, {a}}.Then, $C_\delta^* - C$ is empty but C is not $g_s\Delta^*$-closed when C = {c, f, g}
Theorem 20 A set H of an ideal then $cl(H) - H$ is of $g_s\Delta^*$-closed iff $H \cup (X - cl(H))$ is
of $g_s\Delta^*$ -open.

Proof. 21 Necessary: Here, H is $g_s\Delta^*$-closed with $E = cl(H) - H$ as a result of $X - H$ is $g_s\Delta^*$-open. Now, $X - H = X \cap (X - H) = X \cap (X - (cl(H) - H)) = H \cup (X - cl(H))$. Hence, $H \cup (X - cl(H))$ is $g_s\Delta^*$-open.
Sufficiency: Assume $V = H \cup (X - cl(H))$ is $g_s\Delta^*$-open. Now, $X - V = X - (H \cup (X - cl(H))) = (X - H) \cap cl(H) = cl(H) - H$. So, $cl(H) - H$ is $g_s\Delta^*$-closed.
Theorem 21 Every subset X is $g_s\Delta^*$-closed an ideal space, and all open subsets of X are I_g-closed.

Proof. 22 Assume that X contains only $g_s\Delta^*$-closed. Assuming V is of any open of X,
V is $g_s\Delta^*$-closed as a result $V_\delta^* \subseteq V$, indicating that every open set is semi-open. Accordingly, $V^* \subseteq V$.
Corollary 4 If all of X's subsets are $g_s\Delta^*$-closed then all of X's subsets that of δ-open subsets of X are I_g-closed.

Corollary 5 If all of X's subsets are $g_s\Delta^*$-closed then all of X's subsets that of θ-open subsets of X are I_g-closed.

Remark 11 The Examples below mentioned are reverse of the above Corollaries not generally Valid.

Example 13 Let $X = \{c, d, e, f\}$, $\tau = \{\emptyset, X, \{c\}, \{d, f\}, \{c, d, f\}\}$, $I = \{\emptyset, \{c\}\}$.We have I_g- closed but not $g_s\Delta^*$-closed set for each θ-open of X when $E = \{c, e\}$

Example 14 Let $X = \{b, d, e, f\}$, $\tau = \{\emptyset, X, \{b\}\}$, $I = \{\emptyset, \{b\}\}$.We have I_g-closed but not $g_s\Delta^*$-closed of each δ-open of X when $F = \{d, e, f\}$.

Theorem 22 For each $y \in X$, if X is of an ideal space where either $\{y\}$ is semi-closed, or $X - \{y\}$is $g_s\Delta^*$-closed.

Proof. 23 Assume that $\{y\}$ is only semi-open that contains $X - \{y\}$ if $\{y\}$ and X are not semi-closed or semi-open respectively. Therefore,$(X - \{y\})_\delta^* \subseteq X \Rightarrow X - \{y\}$ is $g_s\Delta^*$-closed.

Theorem 23 Consider the ideal space where F is a set of X. So, whenever F is of $g_s\Delta^*$-closed and semi-open then $F_\delta^* \subseteq F$.

Proof. 24 Suppose F is a $g_s\Delta^*$-closed and semi-open an ideal space. Though, F is semi-open, By hypothesis $F_\delta^* \subseteq F$.

Theorem 24 We have, $E \cup F$ is a $g_s\Delta^*$-closed in (X, τ, I) iff E and F are $g_s\Delta^*$-closed sets.

Proof. 25 Assume $E \cup F \subseteq V$, wherein V is semi-open in (X, τ, I). Then $E \subseteq V$, $F \subseteq V$. When $E^*_\delta \subseteq V \Rightarrow F_\delta^* \subseteq V$ since E and F are $g_s\Delta^*$-closed in (X, τ, I). In this case,$(E \cup F)_\delta^* = E_\delta^* \cup F_\delta^*$. Hence, $(E \cup F)_\delta^* \subseteq V$, whenever V is semi-open. So, $E \cup F$ is $g_s\Delta^*$-closed in (X, τ, I).

Theorem 25 A set G is an ideal space where G is gsΔ*-closed if then $G\delta^* \subseteq \cap\{W/G \subseteq W,$ W is semi-open$\}$.

Proof. 26 It is necessary that G can have a gsΔ*-closed with $y \in G\delta^*$. Assume, $y \notin \cap\{W/G \subseteq W$, W is semi-open$\}$, which is a semi-open set, and G follows contradiction. Therefore, $G\delta^* \subseteq \cap\{W/G \subseteq W$, W is semi-open$\}$.

$y \notin W$. Which leads to a Sufficiency: Suppose that, $G_\delta^* \subseteq \cap\{W/G \subseteq W$, G is semi-open$\}$. Let $G \subseteq W$, W is semi-open. Then, $\cap\{W/G \subseteq W$, W is semi-open$\} \subseteq W$ and so $G_\delta^* \subseteq W$. So, G is $g_s\Delta^*$-closed.

Theorem 26 Suppose X is in ideal space, $X_1 \cap A_\delta^* \subseteq \cap\{U/A \subseteq U$, U is semi-open} for all set A of X, where $X_1 = \{z \in X : \{z\}$ is pre-open}.

Proof. 27 Suppose that $z \in X_1 \cap A_\delta^*$ and $z \notin \cap\{U/A \subseteq U, U$ is semi-open}. Because $z \in X_1$, $\{z\} \subseteq Int(cl(\{z\}))$ and so $scl(\{z\}) = Int(cl\{z\})$. Since $y \in A_\delta^*$, $A \cap Int(cl(U)) \notin I$ and consequently $A \cap Int(cl(U)) \neq \emptyset$, for every open set U that contains z. Choose U = $Int(cl(\{z\}))$,
then $A \cap Int(cl(\{z\})) \neq \emptyset$. Fix $z \in A \cap Int(cl(\{z\}))$. Given, though $z \notin \cap\{U/A \subset U, U$ is semi- open},we can find a semi-open G in which $A \subset G$ and $z \notin G$. If $V = X - G$, G is of a semi-closed provided $z \in V \subset X - A$. So, $scl(\{z\}) = Int(cl(\{z\})) \subseteq U$, $z \in A \cap G$, a contradiction. Therefore, $z \in \cap\{U/A \subseteq U, U$ is semi-open }.

Definition 3 If either X or A is contained in the non-empty $g_s\Delta^*$-closed then A is defined to be maximal $g_s\Delta^*$-closed where A is an ideal space. By Example 15 we can verify the existence of maximal $g_s\Delta^*$-closed in ideal spaces.

Example 15 Let $X = \{c, f, g, h\}$, $\tau = \{\emptyset, X, \{c\}, \{g\}, \{c, g\}\}$, $I = \{\emptyset, \{f\}, \{g\}, \{f, g\}\}$ then $g_s\Delta^*$-closed sets are $\{\emptyset, X, \{f\}, \{g\}, \{c, f\}, \{f, g\}, \{f, h\}, \{c, f, h\}, \{f, g, h\}\}$. Hence, If $C = \{c, f, h\}$ is the maximal $g_s\Delta^*$-closed set.

Theorem 27 Assuming that X is in (X, τ, I) the conditions hold.

1. Assume D and E are of $g_s\Delta^*$-closed and maximal $g_s\Delta^*$-closed respectively. In that case either $E \cup D = X$ or $D \subseteq E$.
2. When H is of $g_s\Delta^*$-closed with I contains maximal $g_s\Delta^*$-closed then either $H \cup I = X$ or $H = I$.

Proof. 28 1. Let D be of $g_s\Delta^*$-closed and F be of maximal $g_s\Delta^*$-closed set. The result is obvious when $E \cup D = X$. Consider, $E \cup D \neq X$ which brought $D \subseteq D \cup E$. According to Theorem 24, $E \cup D$ is $g_s\Delta^*$-closed. Given that E is maximally $g_s\Delta^*$-closed, then either $E \cup D = X$ or $E \cup D = E$. Follows, $E \cup D = E$ then, $D \subseteq E$.

2. Consider H, I are maximal $g_s\Delta^*$-closed then if $H \cup I = X$, then the result is obvious. The result is not true, we can have, $I \subseteq H$ and $H \subseteq I$, following $H = I$.

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