CERTAIN INTEGRALS INVOLVING HYPERGEOMETRIC FUNCTION OF N-VARIABLES

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Abstract:
In the present paper some integral involving hypergeometric function of n-variables are evaluated. Some particular cases are of interest.

Key words: Hypergeometric function, Lauricella's Function $F_D^{(n)}$, Saran's function F_G, F_N, F_S and E_α^m operator.

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INTRODUCTION and PRELIMINARY:

Let

$$E_\alpha f(\alpha) = f(\alpha+1), \quad E_\alpha^m f(\alpha) = f(\alpha+m) \qquad \dots (1)$$

Recently Agrawal [1] employed the operator E_α to evaluate the integrals.

$$\int_{-\infty}^{\infty} \frac{\sin[\pi x.(2p+1)]}{\sin(\pi x)\Gamma(\beta-x)\Gamma(\alpha+x)} {}_1F_1\left[\begin{matrix}\alpha'+\beta'; \\ \alpha+x;\end{matrix} v\right] {}_1F_1\left[\begin{matrix}\beta'; \\ \beta-x;\end{matrix} v\right] dx = \frac{2^{\alpha+\beta-2}}{\Gamma(\alpha+\beta-1)} {}_1F_1\left[\begin{matrix}\alpha'+\beta'; \\ \alpha+\beta-1;\end{matrix} 2v\right] \qquad \dots (2)$$

$$\int_{-\infty}^{\infty} \frac{1}{\Gamma(\alpha+x)\Gamma(\gamma+x)\Gamma(\beta-x)\Gamma(\delta-x)} {}_1F_1\left[\begin{matrix}\alpha+\beta-1; \\ \alpha+x;\end{matrix} v\right] {}_1F_1\left[\begin{matrix}\gamma+\delta-1; \\ \gamma-x;\end{matrix} v\right] dx =$$
$$\frac{\Gamma(\alpha+\beta+\gamma+\delta-3)}{\Gamma(\alpha+\beta-1)\Gamma(\beta+\gamma-1)\Gamma(\gamma+\delta-1)\Gamma(\delta+\alpha-1)} {}_2F_2\left[\begin{matrix}\frac{\alpha+\beta+\gamma+\delta-3}{2}, \alpha+\beta+\gamma+\delta-3; \\ \alpha+\beta-1,\ \delta+\alpha-1;\end{matrix} 4v\right]$$

$$\dots (3)$$

$$(\alpha+\beta+\gamma+\delta) > 3.$$

Re (□ + x) □1, p is an integer.

Re

Ragab and Simary [8] have also evaluated these integrals, by a different technique.

The classical Pochhammer symbol $(a)_n\ (a, n \in \mathbb{C})$ is define, in terms of the gamma function, by

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} \begin{cases} 1 & (n = 0; a \in \mathbb{C}\setminus\{0\}) \\ a(a+1)(a+2)\dots(a+n-1) & (n \in \mathbb{N};\ a \in \mathbb{C}) \end{cases}$$

And

$$\Gamma(a+n) = \Gamma(a)(a)_n \qquad \dots (4)$$

Exponential series in terms of hypergeometric function define by Gauss.

$$\exp(z) = {}_1F_1(a, a; z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

and

$$\exp(x_1 + x_2 + \dots + x_n) = \sum_{m_1, m_2, \dots, m_n = 0}^{\infty} \frac{x_1^{m_1}}{m_1!} \frac{x_2^{m_2}}{m_2!} \dots \frac{x_n^{m_n}}{m_n!} \qquad \dots (5)$$

In the present paper, following integral is evaluated which involve hypergeometric function

$F_D^{(n,k,r)}$ of n-variable

$$\int_{-\infty}^{\infty} \frac{\sin[(2p+1)\pi x]}{\sin(\pi x)\Gamma(\alpha+x)\Gamma(\beta-x)} F_D^{(n,k,r)}(\gamma,\gamma',\beta_1,\beta_2,...,\beta_n;\alpha+x,\beta-x;x_1,x_2,...,x_n)$$
$$= \frac{2^{\alpha+\beta-2}}{\Gamma(\alpha+\beta-1)} F_D^{(n,k,r)}(\gamma,\gamma',\beta_1,\beta_2,...,\beta_n;\alpha+\beta-1;2x_1,2x_2,...,2x_n) \quad \ldots (6)$$

Re $(\alpha+\beta)>1$

$F_D^{(n,k,r)}$ is hypergeometric function of n-variables defined by Singhal & Bhati [5].

$$F_D^{(n,k,r)}[\alpha,\alpha',\beta_1,\beta_2,...\beta_n;\gamma,\gamma';x_1,x_2,...x_n]$$
$$= \sum_{m_1,m_2,...,m_n=0}^{\infty} \frac{(\alpha)_{m_1+m_2+...+m_k}(\alpha')_{m_{k+1}+m_{k+2}+...+m_n}(\beta_1)_{m_1}(\beta_2)_{m_2}\cdots(\beta_n)_{m_n}}{(\gamma)_{m_1+m_2+...+m_r}(\gamma')_{m_{r+1}+m_{r+2}+...+m_n}} \frac{x_1^{m_1}}{m_1}\frac{x_2^{m_2}}{m_2}\cdots\frac{x_n^{m_n}}{m_n} \quad \ldots (7)$$

Where

Where **k** and **r** are distinct non-negative integer <n. By choosing k, r, and n appropriately, the function $F_D^{(n\,k\,r,\,,)}$ can be particularized to many hypergeometric functions of several variables.

(i) $F_D^{(n,0,0)}(-,\alpha',\beta_1,\beta_2,...,\beta_n;-,\gamma';x_1,x_2,...,x_n) = F_D^{(n)}(\alpha',\beta_1,\beta_2,...,\beta_n;\gamma';x_1,x_2,...,x_n)$

(ii) $F_D^{(n,n,n)}(\alpha,-,\beta_1,\beta_2,...,\beta_n;\gamma,-;x_1,x_2,...,x_n) = F_D^{(n)}(\alpha,\beta_1,\beta_2,...,\beta_n;\gamma;x_1,x_2,...,x_n)$

(iii) $F_D^{(3,3,1)}(\alpha,-,\beta_1,\beta_2,\beta_3;\gamma,\gamma';x_1,x_2,x_3) = F_G(\alpha,\alpha,\alpha,\beta_1,\beta_2,\beta_3;\gamma,\gamma';x_1,x_2,x_3)$

(iv) $F_D^{(3,2,1)}(\alpha,\alpha',\beta_1,\beta_2,\beta_3;\gamma,\gamma';x_1,x_2,x_3) = F_N(\beta_1,\beta_2,\beta_3,\alpha,\alpha',\alpha;\gamma,\gamma',\gamma';x_1,x_2,x_3)$

(v) $F_D^{(3,1,3)}(\alpha,\alpha',\beta_1,\beta_2,\beta_3;\gamma,-;x_1,x_2,x_3) = F_S(\alpha,\alpha',\alpha',\beta_1,\beta_2,\beta_3;\gamma,\gamma',\gamma';x_1,x_2,x_3)$

(vi) $F_D^{(2,2,1)}(\alpha,-,\beta_1,\beta_2;\gamma,\gamma';x_1,x_2) = F_2(\alpha,\beta_1,\beta_2;\gamma,\gamma';x_1,x_2)$

(vii) $F_D^{(2,1,2)}(\alpha,\alpha',\beta_1,\beta_2;\gamma,-;x_1,x_2) = F_3(\alpha,\alpha',\beta_1,\beta_2;\gamma;x_1,x_2)$

And one more relation is

$$F_D^{(n,k,r)}(\alpha,\alpha',\beta_1,\beta_2,...,\beta_n;\gamma,\gamma';x_1,x_2,...,x_n)$$
$$= F_D^{(n,n-k,n-r)}(\alpha',\alpha,\beta_n,\beta_{n-1},...,\beta_2,\beta_1;\gamma',\gamma;x_n,x_{n-1},...,x_2,x_1) \quad \ldots (8)$$

Where F_G, F_N *and* F_S are hypergeometric Saran's functions [11]; $F_D^{(n)}$ is Lauricelli's function, F_2 *and* F_3 are Appell's function. Relation (7) have been obtained under the assumption $k>r$, in which there is no loss of generality, since the case $k>r$ may be converted into $k'=(n-k)<(n-r)=r'$ by making use of (8).

In what follows we will apply the following results [cf [2] p224]

$$\int_{-\infty}^{\infty} \frac{\mathrm{Sin}[(2p+1)\pi x]}{\sin(\pi x)\Gamma(\alpha+x)\Gamma(\beta-x)}dx = \frac{2^{\alpha+\beta-2}}{\Gamma(\alpha+\beta-1)}, \qquad \mathrm{Re}(\alpha+\beta)>1 \quad \ldots (9)$$

2. ANALYSIS OF THE RESULTS: -

To prove (6) multiply both side of (9) by

$$\Gamma(\gamma)\Gamma(\gamma')\Gamma(\beta_1)\Gamma(\beta_2)\Gamma(\beta_3)...\Gamma(\beta_n)$$

and applying the operator exp

$$\int_{-\infty}^{\infty}\frac{\operatorname{Sin}[\pi x(2p+1)]\Gamma(\gamma)\Gamma(\gamma')\Gamma(\beta_1)\Gamma(\beta_2)...\Gamma(\beta_n)}{\sin(\pi x)\Gamma(\alpha+x)\Gamma(\beta-x)}dx = \frac{2^{\alpha+\beta-2}\Gamma(\gamma)\Gamma(\gamma')\Gamma(\beta_1)\Gamma(\beta_2)...\Gamma(\beta_n)}{\Gamma(\alpha+\beta-1)} \quad ...(10)$$

Were

$$\left[\begin{array}{c}\{(x_1E_{\beta_1}+x_2E_{\beta_2}+...+x_rE_{\beta_r})E_\alpha+(x_{r+1}E_{\beta_{r+1}}+...+x_kE_{\beta_k})E_\beta\}E_\gamma\\ +\left(x_{m_{k+1}}E_{\beta_{k+1}}+...+x_nE_{\beta_n}\right)E_\beta E_{\gamma'}\end{array}\right]$$

$$\exp\left[\begin{array}{c}\{(x_1E_{\beta_1}+x_2E_{\beta_2}+...+x_rE_{\beta_r})E_\alpha+(x_{r+1}E_{\beta_{r+1}}+...+x_kE_{\beta_k})E_\beta\}E_\gamma\\ +\left(x_{k+1}E_{\beta_{k+1}}+...+x_nE_{\beta_n}\right)E_\beta E_{\gamma'}\end{array}\right] \quad ...(11)$$

$$=\sum_{m_1,m_2,...,m_n=0}^{\infty}E_\alpha^{m_1+m_2+...+m_r}E_\beta^{m_{r+1}+...+m_n}E_\gamma^{m_1+m_2+...+m_k}E_{\gamma'}^{m_{k+1}+...+m_n}E_{\beta_1}^{m_1}E_{\beta_2}^{m_2}...E_{\beta_n}^{m_n}\frac{x_1^{m_1}x_2^{m_2}...x_n^{\beta_n}}{m_1!m_2!...m_n!}$$

We obtain

$$\int_{-\infty}^{\infty}\frac{\operatorname{Sin}[(2p+1)\pi x]}{\operatorname{Sin}(\pi x)}\sum_{m_1,m_2,...,m_n=0}^{\infty}\frac{\Gamma(\gamma+m_1+m_2+...+m_k)\Gamma(\gamma'+m_{k+1}+...+m_n)}{\Gamma(\alpha+x+m_1+m_2+...+m_r)\Gamma(\beta-x+m_{r+1}+...+m_n)}\frac{\Gamma(\beta_1+m_1)...\Gamma(\beta_n+m_n)x_1^{m_1}x_2^{m_2}...x_n^{m_n}}{m_1!m_2!...m_n!}dx$$

$$=\sum_{m_1,m_2,...,m_n=0}^{\infty}\frac{2^{\alpha+m_1+m_2+...+m_r}2^{\beta+m_{r+1}+...+m_n-2}\Gamma(\gamma+m_1+m_2+...+m_k)\Gamma(\gamma'+m_{k+1}+...+m_n)}{\Gamma(\alpha+\beta-1+m_1+m_2+...+m_n)}\frac{\Gamma(\beta_1+m_1)...\Gamma(\beta_n+m_n)x_1^{m_1}x_2^{m_2}...x_n^{m_n}}{m_1!m_2!...m_n!} \quad ...(12)$$

$$\int_{-\infty}^{\infty}\frac{\operatorname{Sin}[(2p+1)\pi x]\Gamma(\gamma)\Gamma(\gamma')\Gamma(\beta_1)\Gamma(\beta_2)\Gamma(\beta_3)...\Gamma(\beta_n)}{\operatorname{Sin}(\pi x)\Gamma(\alpha+x)\Gamma(\beta-x)}\sum_{m_1,m_2,...m_n=0}^{\infty}\frac{(\gamma)_{m_1+m_2+...+m_k}(\gamma')_{m_{k+1}+...+m_n}(\beta_1)_{m_1}(\beta_2)_{m_2}\cdots(\beta_n)_{m_n}}{(\alpha+x)_{m_1+m_2+...+m_r}(\beta-x)_{m_{r+1}+...+m_n}}\frac{x_1^{m_1}x_2^{m_2}...x_n^{m_n}}{m_1!m_2!...m_n!}$$

$$=\frac{2^{\alpha+\beta-2+m_1+m_2+...+m_n}\Gamma(\gamma)\Gamma(\gamma')\Gamma(\beta_1)\Gamma(\beta_2)\Gamma(\beta_3)...\Gamma(\beta_n)}{\Gamma(\alpha+\beta-1+m_1+m_2+...+m_n)}\frac{(\gamma)_{m_1+m_2+...+m_k}(\gamma')_{m_{k+1}+...+m_n}(\beta_1)_{m_1}(\beta_2)_{m_2}\cdots(\beta_n)_{m_n}x_1^{m_1}x_2^{m_2}...x_n^{m_n}}{m_1!m_2!...m_n!} \quad ...(13)$$

$$\int_{-\infty}^{\infty}\frac{\operatorname{Sin}[(2p+1)\pi x]}{\operatorname{Sin}(\pi x)\Gamma(\alpha+x)\Gamma(\beta-x)}F_D^{(n,k,r)}(\gamma,\gamma',\beta_1,\beta_2,...,\beta_n;\alpha+x,\beta-x;x_1,x_2,...,x_n)$$

$$=\frac{2^{\alpha+\beta-2}}{\Gamma(\alpha+\beta-1)}F_D^{(n,k,r)}(\gamma,\gamma',\beta_1,\beta_2,...,\beta_n;\alpha+\beta-1;2x_1,2x_2,...,2x_n)$$

Which complete the proof of (6)

3. PARTICULAR CASES: -

(i) By putting n=3, k=3 and r=1 in (6) we get

$$\int_{-\infty}^{\infty}\frac{\operatorname{Sin}[(2p+1)\pi x]}{\operatorname{Sin}(\pi x)\Gamma(\alpha+x)\Gamma(\beta-x)}F_G(\gamma,-,\beta_1,\beta_2,\beta_3;\alpha+x,\beta-x;x_1,x_2,x_3)dx$$

$$=\frac{2^{\alpha+\beta-2}}{\Gamma(\alpha+\beta-1)}F_D(\gamma,\beta_1,\beta_2,\beta_3;\alpha+\beta-1;2x_1,2x_2,2x_3)$$

(ii) By putting n=3, k=2 and r=1 in (6) we get

$$\int_{-\infty}^{\infty} \frac{\text{Sin}[(2p+1)\pi x]}{\text{Sin}(\pi x)\Gamma(\alpha+x)\Gamma(\beta-x)} F_N(\gamma,\gamma',\beta_1,\beta_2,\beta_3;\alpha+x,\beta-x;x_1,x_2,x_3)\,dx$$

$$= \frac{2^{\alpha+\beta-2}}{\Gamma(\alpha+\beta-1)} F_S(\gamma,\gamma',\beta_1,\beta_2,\beta_3;\alpha+\beta-1;2x_1,2x_2,2x_3) \qquad \text{Re}(\alpha+\beta)>1$$

(iii) By putting n=3, k=1 and r=3 in (6) we get

$$\int_{-\infty}^{\infty} \frac{\text{Sin}[(2p+1)\pi x]}{\text{Sin}(\pi x)\Gamma(\alpha+x)\Gamma(\beta-x)} F_S(\gamma,\gamma',\beta_1,\beta_2,\beta_3;\alpha+x,\alpha+x;x_1,x_2,x_3)\,dx$$

$$= \frac{2^{\alpha+\beta-2}}{\Gamma(\alpha+\beta-1)} F_S(\gamma,\gamma',\beta_1,\beta_2,\beta_3;\alpha+\beta-1;2x_1,2x_2,2x_3)$$

$\text{Re}(\alpha+\beta)>1$

(iv) By putting n=2, k=2 and r=1 2 in (6) we get

$$\int_{-\infty}^{\infty} \frac{\text{Sin}[(2p+1)\pi x]}{\text{Sin}(\pi x)\Gamma(\alpha+x)\Gamma(\beta-x)} F_2(\gamma,\beta_1,\beta_2;\alpha+x,\beta-x;x_1,x_2)\,dx$$

$$= \frac{2^{\alpha+\beta-2}}{\Gamma(\alpha+\beta-1)} F_S(\gamma,\beta_1,\beta_2;\alpha+\beta-1;2x_1,2x_2)$$

$\text{Re}(\alpha+\beta)>1$

(v) By putting n=2, k=1 and r=2 in (6) we get

$$\int_{-\infty}^{\infty} \frac{\text{Sin}[(2p+1)\pi x]}{\text{Sin}(\pi x)\Gamma(\alpha+x)\Gamma(\beta-x)} F_3(\gamma,\gamma',\beta_1,\beta_2;\alpha+x;x_1,x_2)\,dx$$

$$= \frac{2^{\alpha+\beta-2}}{\Gamma(\alpha+\beta-1)} F_3(\gamma,\gamma',\beta_1,\beta_2;\alpha+\beta-1;2x_1,2x_2)$$

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