Coupled fixed points in bicomplex partial metric space and an application

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Abstract: In this study, we develop coupled fixed points for a self map on a bi complex partial metric space that satisfy certain generalized contraction conditions. We support our findings using examples. We solve the existence and uniqueness solution of a Fredholm type integral type equation as an application.

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1. Introduction and Preliminaries

Segre [8] made a first attempt at developing special algebra. Complex numbers, bicomplex numbers, tricomplex numbers, and so on were envisioned as elements of an infinite set of algebras. Researchers contributed to this field in the 1930s [2, 9,10]. Recent research on this topic [1, 11] discovered several important applications in mathematics as well as other sectors of science and industry. Several researchers have published a substantial quantity of study, to which we refer [1- 6].

Bicomplex Numbers[6]**.** The set of bicomplex numbers indicated by $\mathbb{C}_2$ is the first of an infinite sequence of multicomplex sets that are generalisations of the set of complex numbers. In this case, we recollect the set of bicomplex numbers$\mathbb{C}_2$, for example, [6,7] as:

$$\mathbb{C}_2 = \{w = \mathfrak{d}_0 + i_1\mathfrak{d}_1 + i_2\mathfrak{d}_2 + i_1 i_2\mathfrak{d}_3 : \mathfrak{d}_p \in \mathbb{R} \ (p = 0,1,2,3)\}$$

$\mathbb{C}_2$ can also be expressed as

$$\mathbb{C}_2 = \{\mathfrak{y}_1 + i_2\mathfrak{y}_2 : \mathfrak{y}_1, \mathfrak{y}_2 \in \mathbb{C}_1\}$$

$i.e., \mathbb{C}_2 = \{\mathbb{Z} : \mathfrak{y}_1 + i_2\mathfrak{y}_2 : \mathfrak{y}_1, \mathfrak{y}_2 \in \mathbb{C}_1\}$

where $\mathfrak{y}_1 = \mathfrak{d}_0 + i_1\mathfrak{d}_1, \mathfrak{y}_2 = \mathfrak{d}_2 + i_1\mathfrak{d}_3$, i_1 and i_2 are imaginary independent units such that

$i_1^2 = -1 = i_2^2$. The product of $i_1 i_2 = j$such that $j^2 = 1$ product of units is defined as

$$i_1 j = -i_2, i_2 j = -i_1.$$

The norm of $w = \mathfrak{y}_1 + i_2\mathfrak{y}_2$, is denoted by $\|w\|$and is defined

$$\|w\| = \|\mathfrak{y}_1 + i_2\mathfrak{y}_2\| = (|\mathfrak{y}_1|^2 + |\mathfrak{y}_2|^2)^{\frac{1}{2}}.$$

i.e., $\|w\| = (\mathfrak{d}_0^2 + \mathfrak{d}_1^2 + \mathfrak{d}_2^2 + \mathfrak{d}_3^2)^{\frac{1}{2}}$.

A bicomplex numbers

$w = \mathfrak{d}_0 + i_1\mathfrak{d}_1 + i_2\mathfrak{d}_2 + i_1 i_2\mathfrak{d}_3$ is degenerated [7] if the matrix $\begin{bmatrix} \mathfrak{d}_0 & \mathfrak{d}_1 \\ \mathfrak{d}_2 & \mathfrak{d}_3 \end{bmatrix}$ is degenerated.

Further, for $\gimel, \daleth \in \mathbb{C}_2$, it is easy to show that

(i) $0 \prec_{i_2} \gimel \prec_{i_2} \daleth \Longrightarrow \|\gimel\| \leq \|\daleth\|$

(ii) $\|\gimel + \daleth\| \leq \|\gimel\| + \|\daleth\|$

(iii) $\|\alpha\gimel\| \leq \alpha\|\gimel\|$

(iv) $\|ƛℸ\| \le \sqrt{2}\|ƛ\|\|ℸ\|$

(v) $\|ƛℸ\| = \|ƛ\|\|ℸ\|$ whenever at least one of $ƛ$ *and* $ℸ$ is degenerated [7]

(vi) $\|ƛ^{-1}\| = \|ƛ\|^{-1}$ holds for any degenerated bicomplex number.

Let $ƛ = ƛ_1 + i_2ƛ_2 \in ₵_2$ *and* $ℸ = ℸ_1 + i_2ℸ_2 \in ₵_2$,the partial order relation on $₵_2$be defined in [3] as $ƛ \preccurlyeq_{i_2} ℸ$ iff $ƛ_1 \preccurlyeq_{i_1} ℸ_1$ *and* $ƛ_2 \preccurlyeq_{i_2} ℸ_2$, where $\preccurlyeq_{i_1}$is a partial order relation in $₵_1$. Then

(1) $\Re\mathrm{e}(ƛ_1) = \Re\mathrm{e}(ℸ_1)$ and $\Im\mathrm{m}(ƛ_1) = \Im\mathrm{m}(ℸ_1)$
$\Re\mathrm{e}(ƛ_2) = \Re\mathrm{e}\,(ℸ_2\,)$and $\Im\mathrm{m}(ƛ_2) = \Im\mathrm{m}(ℸ_2)$

(2) $\Re\mathrm{e}\,(ƛ_1) < \Re\mathrm{e}(ℸ_1)$ and $\Im\mathrm{m}(ƛ_1) < \Im\mathrm{m}(ℸ_1)$
$\Re\mathrm{e}\,(ƛ_2) = \Re\mathrm{e}\,(ℸ_2)$ and $\Im\mathrm{m}(ƛ_2) = \Im\mathrm{m}(ℸ_2)$

(3) $\Re\mathrm{e}(ƛ_1) = \Re\mathrm{e}(ℸ_1)$ and $\Im\mathrm{m}(ƛ_1) = \Im\mathrm{m}(ℸ_1)$
$\Re\mathrm{e}\,(ƛ_2) < \Re\mathrm{e}(ℸ_2)$ and $\Im\mathrm{m}(ƛ_2) < \Im\mathrm{m}(ℸ_2)$

(4) $\Re\mathrm{e}\,(ƛ_1) < \Re\mathrm{e}(ℸ_1)$ and $\Im\mathrm{m}(ƛ_1) < \Im\mathrm{m}(ℸ_1)$
$\Re\mathrm{e}\,(ƛ_2) < \Re\mathrm{e}(ℸ_2)$ and $\Im\mathrm{m}(ƛ_2) < \Im\mathrm{m}(ℸ_2)$

We write $ƛ \precnsim_{i_2} ℸ$ if $ƛ \preccurlyeq_{i_2} ℸ$ and $ƛ \ne ℸ$ if any one (1), (2), (3) is satisfied and $ƛ \prec_{i_2} ℸ$ if the condition (4) is satisfied.

Definition 1.1[6]. A function $Þ_{₵_2}: 𝔖 \times 𝔖 \to ₵_2$ is a bicomplex valued metric space on a non empty set $𝔖$ if for all $Ͳ, ϰ, 𝔰 \in 𝔖$, we have

(i) $0 \preccurlyeq_{i_2} Þ_{₵_2}(Ͳ, ϰ)$;

(ii) $Þ_{₵_2}(Ͳ, ϰ) = 0$ iff $Ͳ= ϰ$;

(iii) $Þ_{₵_2}(Ͳ, ϰ) = Þ_{₵_2}(ϰ,Ͳ)$;

(iv) $Þ_{₵_2}(Ͳ, ϰ) \preccurlyeq_{i_2} Þ_{₵_2}(Ͳ, 𝔰) + Þ_{₵_2}(𝔰, ϰ)$;

Then $(𝔖, Þ_{₵_2})$ is a bicomplex valued metric space.

Definition1.2[4]**.** A bicomplex partial metric on a non-empty set $𝔖$ is a function $Þ_{bcpms}: 𝔖 \times 𝔖 \to {₵_2}^+$ such that for all $Ͳ, ϰ, 𝔰 \in 𝔖$:

1. $0 \preccurlyeq_{i_2} Þ_{bcpms}(Ͳ,Ͳ) \preccurlyeq_{i_2} Þ_{bcpms}(Ͳ, ϰ)$,
2. $Þ_{bcpms}(Ͳ, ϰ) = Þ_{bcpms}(ϰ,Ͳ)$,
3. $Þ_{bcpms}(Ͳ,Ͳ) = Þ_{bcpms}(Ͳ, ϰ) = Þ_{bcpms}(ϰ, ϰ)$ if and only if $Ͳ= ϰ$,
4. $Þ_{bcpms}(Ͳ, ϰ) \preccurlyeq_{i_2} Þ_{bcpms}(Ͳ, 𝔰) + Þ_{bcpms}(𝔰, ϰ) - Þ_{bcpms}(𝔰, 𝔰)$.

Then $(𝔖,\ Þ_{bcpms})$is a bicomplex partial metric space.

From here onwards, we denote $(𝔖, Þ_{bcpms})$ is a bicomplex partial metric space.

A bicomplex valued metric space is naturally, a bicomplex partial metric with self distance space. A bicomplex partial metric space is not required to be a bicomplex valued metric space.

Example1.3. Let $𝔖$ = { (1,2) , (3,4) , (5,6) , (7,8) } be equipped with a bicomplex partial metric space, $Þ_{bcpms}: 𝔖 \times 𝔖 \to ₵_2^+$ by which is illustrated as follows :

Clearly $(𝔖, Þ_{bcpms})$ is a bicomplex partial metric.
For example, if $Ͳ= (1,2), ϰ = (7,8), 𝔰 = (1,2)$ then

$Þ_{bcpms}(Ͳ, ϰ)$	**(1,2)**	**(3,4)**	**(5,6)**	**(7,8)**
(1,2)	**(1+i₂)(1,2)**	**(1+i₂)(3,4)**	**(1+i₂) (5,6)**	**(1+i₂)(7,8)**
(3,4)	**(1+i₂)(3,4)**	**(1+i₂)(3,4)**	**(1+i₂) (5,6)**	**(1+i₂)(7,8)**
(5,6)	**(1+i₂) (5,6)**	**(1+i₂)(5,6)**	**(1+i₂) (5,6)**	**(1+i₂)(7,8)**
(7,8)	**(1+i₂)(7,8)**	**(1+i₂)(7,8)**	**(1+i₂) (7,8)**	**(1+i₂)(7,8)**

$Þ_{bcpms}\left(\left((1,2),(7,8)\right)\right) = (1+i_2)(7,8)$

$Þ_{bcpms}\left(\left((1,2),(7,8)\right)\right) \precsim_{i_2} Þ_{bcpms}\left(\left((1,2),(3,4)\right)\right)$

$+ Þ_{bcpms}\left(\left((3,4),(7,8)\right)\right) - Þ_{bcpms}\left(\left((3,4),(3,4)\right)\right)$

$= \{(1+i_2)(3,4) + (1+i_2)(7,8)\} - (1+i_2)(3,4)$

$= (1+i_2)(7,8)$

Therefore

$Þ_{bcpms}(\daleth,\varkappa) \precsim_{i_2} Þ_{bcpms}(\daleth,\mathfrak{s})$

$+Þ_{bcpms}(\mathfrak{s},\varkappa) - Þ_{bcpms}(\mathfrak{s},\mathfrak{s}).$

Hence$(\mathfrak{S}, Þ_{bcpms})$ is a bicomplex valued partial metric space (conditions (1), (2), (3) are trivial)

Here, we note that

$Þ_{bcpms}\left(\left((1,2),(1,2)\right)\right) = (1+i_2)(1,2) \neq 0,$

This implies $(\mathfrak{S}, Þ_{bcpms})$is a not a bicomplex metric space.

For more literature on bicomplex partial metric spaces we refer [4]

Definition 1.4[4]. A sequence $\{\daleth_r\}$ in $(\mathfrak{S}, Þ_{bcpms})$is converges to $\daleth \in \mathfrak{S}$, if for each $0 \prec_{i_2} \epsilon \in \mathbb{C}_2^+$ there exists $\eta \in \mathbb{N}$ such that

$\daleth_r \in \mathcal{B}_{Þ_{bcpms}}(\daleth,\epsilon)$

$= \left\{\mathfrak{w} \in \mathfrak{S}: Þ_{bcpms}(\daleth,\mathfrak{w}) < \epsilon + Þ_{bcpms}(\daleth,\daleth)\right\}$

for all $r \geq \eta$, and it is denoted by $\lim_{r\to\infty}\daleth_r = \daleth$.

Lemma 1.5[4]. A sequence $\{\daleth_r\}$ in $(\mathfrak{S}, Þ_{bcpms})$ is converges to $\daleth \in \mathfrak{S}$ iff

$Þ_{bcpms}(\daleth,\daleth) = \lim_{r\to\infty} Þ_{bcpms}(\daleth,\daleth_r).$

Definition 1.6[4].A sequence $\{\daleth_r\}$ in $(\mathfrak{S}, Þ_{bcpms})$ is said to be Cauchy sequence,if for any $\epsilon > 0$ there exist $\Upsilon \in \mathbb{C}_2{}^+$and $\eta \in \mathbb{N}$ such that

$\| Þ_{bcpms}(\daleth_r,\daleth_U) - \Upsilon\| < \epsilon$ for all $r, U \geq \eta$.

Definition 1.7 [4]. A bicomplex partial metric space is complete iff every Cauchy sequence in $(\mathfrak{S}, Þ_{bcpms})$ is convergent.

Lemma 1.8[4]. Consider a sequence $\{\daleth_r\}$ in $(\mathfrak{S}, Þ_{bcpms})$.Then $\{\daleth_r\}$ is Cauchy sequence in $\mathfrak{S}$ iff $\lim_{r,U\to\infty}(\daleth_r,\daleth_U) = Þ_{bcpms}(\daleth,\daleth)$.

Definition 1.9[4]. An element $(\daleth,\varkappa) \in \mathfrak{S}\times\mathfrak{S}$ is a coupled fixed point of the mapping

$\mathfrak{F}: \mathfrak{S}\times\mathfrak{S} \to \mathfrak{S}$ if

$\mathfrak{F}(\daleth,\varkappa) = \daleth$ and $\mathfrak{F}(\varkappa,\daleth) = \varkappa.$

Theorem 1.10[4]. Consider a mapping $\mathfrak{F}: \mathfrak{S}\times\mathfrak{S} \to \mathfrak{S}$, where $\mathfrak{S}$ is a bicomplex partial metric with the following contractive condition:

$Þ_{bcpms}(\mathfrak{F}(\daleth,\varkappa),\mathfrak{F}(\curlywedge,\curlyvee))$

$\precsim_{i_2} \lambda\, Þ_{bcpms}(\mathfrak{F}(\daleth,\varkappa),\daleth)$

$+ \text{ł}\, Þ_{bcpms}(\mathfrak{F}(\curlywedge,\curlyvee),\curlywedge),$

for all $\daleth,\varkappa,\curlywedge,\curlyvee \in \mathfrak{S}$, where λ, ł are nonnegative constants with $\lambda + \text{ł} < 1$.

Then, $\mathfrak{F}$ has a unique coupled fixed point.

Theorem 1.11[4].Consider a mapping $\mathfrak{F} : \mathfrak{S}\times\mathfrak{S} \longrightarrow \mathfrak{S}$ satisfies the following contractive condition:

$(\mathfrak{F}(\daleth,\varkappa),\mathfrak{F}(\curlywedge,\curlyvee))$

$\precsim_{i_2} \lambda\, Þ_{bcpms}(\daleth,\curlywedge)$

$+\text{ł}\, Þ_{bcpms}(\varkappa,\curlyvee)$

for all $\daleth,\varkappa,\curlywedge,\curlyvee \in \mathfrak{S}$, where λ, ł≥ 0 with $\lambda + \text{ł} < 1$.

Then, $\mathfrak{F}$ admits a unique coupled fixed point.

Motivated,
by Theorem 1.10 and Theorem 1.11 ,
here we prove the existence of coupled fixed points for a self map satisfying generalized contraction condition on a bicomplex partial metric space.

Theorem 2.1.Our findings generalize those of Gunaseelan Mani *et.al*., [4].

We validate our findings using examples. As a result of our findings, we determine the existence and uniqueness of Fredhlom type of integral equations.

2. Main results

Theorem 2.1 . Consider a mapping $\mathfrak{F}: \mathfrak{S} \times \mathfrak{S} \longrightarrow \mathfrak{S}$, where $(\mathfrak{S}, \text{Þ}_{bcpms})$ is a bicomplex partial metric space is such that:

$$(\mathfrak{F}(⅂,\varkappa), \mathfrak{F}(⅄,\text{Y})) \preceq_{i_2} \beta\, M(⅂,\varkappa,⅄,\text{Y}) + L\, N(⅂,\varkappa,⅄,\text{Y}) \tag{2.1.1}$$

for all $⅂,\varkappa,⅄,\text{Y} \in \mathfrak{S}, 0 \leq \beta < 1, L > 0$,

where $M(⅂,\varkappa,⅄,\text{Y})$

$$= \max\left\{\begin{matrix} \dfrac{\text{Þ}_{bcpms}(⅂,⅄) + \text{Þ}_{bcpms}(\varkappa,\text{Y})}{2}, \\ \dfrac{\text{Þ}_{bcpms}(⅂,\mathfrak{F}(⅂,\varkappa)) + \text{Þ}_{bcpms}(\varkappa,\mathfrak{F}(\varkappa,⅂))}{2}, \\ \dfrac{\text{Þ}_{bcpms}(⅄,\mathfrak{F}(⅄,\text{Y})) + \text{Þ}_{bcpms}(\text{Y},\mathfrak{F}(\text{Y},⅄))}{2}, \\ \dfrac{\text{Þ}_{bcpms}(⅄,\mathfrak{F}(⅂,\varkappa)).\text{Þ}_{bcpms}(⅄,\mathfrak{F}(⅄,\text{Y}))}{1 + \text{Þ}_{bcpms}(⅂,⅄) + \text{Þ}_{bcpms}(\varkappa,\text{Y})} \end{matrix}\right\}$$

and

$N(⅂,\varkappa,⅄,\text{Y})$

$$= \min\left\{\begin{matrix} \text{Þ}_{bcpms}(\mathfrak{F}(⅂,\varkappa),⅄), \text{Þ}_{bcpms}(\mathfrak{F}(⅄,\text{Y}),⅂), \\ \text{Þ}_{bcpms}(\mathfrak{F}(⅂,\varkappa),\text{Y}), \text{Þ}_{bcpms}(\mathfrak{F}(⅄,\text{Y}),⅄) \end{matrix}\right\}$$

Then $\mathfrak{F}$ allows a unique coupled fixed point in $\mathfrak{S}$.

Proof. Let $(⅂_0, \varkappa_0) \in \mathfrak{S} \times \mathfrak{S}$ be an arbitrary.

We construct sequences $\{⅂_n\}$ and $\{\varkappa_n\}$ in $\mathfrak{S}$ such that

$⅂_{n+1} = \mathfrak{F}(⅂_n, \varkappa_n),\ \varkappa_{n+1} = \mathfrak{F}(\varkappa_n,⅂_n)$

for all $n \geq 0$

$\text{Þ}_{bcpms}(⅂_n,⅂_{n+1})$

$= \text{Þ}_{bcpms}(\mathfrak{F}(⅂_{n-1},\varkappa_{n-1}), \mathfrak{F}(⅂_n,\varkappa_n))$

$$\preceq_{i_2} \beta \max\left\{\begin{matrix} \dfrac{\text{Þ}_{bcpms}(⅂_{n-1},⅂_n) + \text{Þ}_{bcpms}(\varkappa_{n-1},\varkappa_n)}{2}, \\ \dfrac{\text{Þ}_{bcpms}(⅂_{n-1},\mathfrak{F}(⅂_{n-1},\varkappa_{n-1})) + \text{Þ}_{bcpms}(\varkappa_{n-1},\mathfrak{F}(\varkappa_{n-1},⅂_{n-1}))}{2} \\ \dfrac{\text{Þ}_{bcpms}(⅂_n,\mathfrak{F}(⅂_n,\varkappa_n)) + \text{Þ}_{bcpms}(\varkappa_n,\mathfrak{F}(\varkappa_n,⅂_n))}{2}, \\ \dfrac{\text{Þ}_{bcpms}(⅂_n,\mathfrak{F}(⅂_{n-1},\varkappa_{n-1})).\text{Þ}_{bcpms}(⅂_n,\mathfrak{F}(⅂_n,\varkappa_n))}{1 + \text{Þ}_{bcpms}(⅂_{n-1},⅂_n) + \text{Þ}_{bcpms}(\varkappa_{n-1},\varkappa_n)} \end{matrix}\right\}$$

$$+L \min\left\{\begin{matrix} \text{Þ}_{bcpms}(\mathfrak{F}(⅂_{n-1},\varkappa_{n-1}),⅂_n), \text{Þ}_{bcpms}(\mathfrak{F}(⅂_n,\varkappa_n),⅂_{n-1}) \\ \text{Þ}_{bcpms}(\mathfrak{F}(⅂_{n-1},\varkappa_{n-1}),\varkappa_n), \text{Þ}_{bcpms}(\mathfrak{F}(⅂_n,\varkappa_n),⅂_n) \end{matrix}\right\}$$

$$= \beta \max\left\{\begin{matrix} \dfrac{\text{Þ}_{bcpms}(⅂_{n-1},⅂_n) + \text{Þ}_{bcpms}(\varkappa_{n-1},\varkappa_n)}{2}, \\ \dfrac{\text{Þ}_{bcpms}(⅂_{n-1},⅂_n) + \text{Þ}_{bcpms}(\varkappa_{n-1},\varkappa_n)}{2}, \\ \dfrac{\text{Þ}_{bcpms}(⅂_n,⅂_{n+1}) + \text{Þ}_{bcpms}(\varkappa_n,\varkappa_{n+1})}{2}, \\ \dfrac{\text{Þ}_{bcpms}(⅂_n,⅂_n).\text{Þ}_{bcpms}(⅂_n,⅂_{n+1})}{1 + \text{Þ}_{bcpms}(⅂_{n-1},⅂_n) + \text{Þ}_{bcpms}(\varkappa_{n-1},\varkappa_n)} \end{matrix}\right\}$$

$$+ L \min\left\{\begin{matrix} \text{Þ}_{bcpms}(⅂_n,⅂_n), \text{Þ}_{bcpms}(⅂_{n+1},⅂_{n-1}), \\ \text{Þ}_{bcpms}(⅂_n,\varkappa_n), \text{Þ}_{bcpms}(⅂_{n+1},⅂_n) \end{matrix}\right\}$$

$$= \beta \max\left\{\frac{r_n}{2}, \frac{r_n}{2}, \frac{r_{n+1}}{2}, 0\right\} + L\,(0)$$

$$= \beta \max\left\{\frac{r_n}{2}, \frac{r_{n+1}}{2}\right\} \tag{2.1.2}$$

where

$r_n = \text{Þ}_{bcpms}(⅂_{n-1},⅂_n) + \text{Þ}_{bcpms}(\varkappa_{n-1},\varkappa_n)$

Similarly,

$\text{Þ}_{bcpms}(\varkappa_n,\varkappa_{n+1})$

$= \text{Þ}_{bcpms}(\mathfrak{F}(\varkappa_{n-1},⅂_{n-1}), \mathfrak{F}(\varkappa_n,⅂_n))$

$$\preceq_{i_2} \beta \max\left\{\begin{matrix} \dfrac{\text{Þ}_{bcpms}(\varkappa_{n-1},\varkappa_n) + \text{Þ}_{bcpms}(⅂_{n-1},⅂_n)}{2}, \\ \dfrac{\text{Þ}_{bcpms}(\varkappa_{n-1},\mathfrak{F}(\varkappa_{n-1},⅂_{n-1})) + \text{Þ}_{bcpms}(⅂_{n-1},\mathfrak{F}(⅂_{n-1},\varkappa_{n-1}))}{2} \\ \dfrac{\text{Þ}_{bcpms}(\varkappa_n,\mathfrak{F}(\varkappa_n,⅂_n)) + \text{Þ}_{bcpms}(⅂_n,\mathfrak{F}(⅂_n,\varkappa_n))}{2}, \\ \dfrac{\text{Þ}_{bcpms}(\varkappa_n,\mathfrak{F}(\varkappa_{n-1},⅂_{n-1})).\text{Þ}_{bcpms}(\varkappa_n,\mathfrak{F}(\varkappa_n,⅂_n))}{1 + \text{Þ}(\varkappa_{n-1},\varkappa_n) + \text{Þ}(⅂_{n-1},⅂_n)} \end{matrix}\right\}$$

$$+L \min\left\{\begin{matrix} \text{Þ}_{bcpms}(\mathfrak{F}(\varkappa_{n-1},⅂_{n-1}),\varkappa_n), \text{Þ}_{bcpms}(\mathfrak{F}(\varkappa_n,⅂_n),\varkappa_{n-1}) \\ \text{Þ}_{bcpms}(\mathfrak{F}(\varkappa_{n-1},⅂_{n-1}),⅂_n), \text{Þ}_{bcpms}(\mathfrak{F}(\varkappa_n,⅂_n),\varkappa_n) \end{matrix}\right\}$$

$$= \beta \max\left\{\begin{matrix} \dfrac{\text{Þ}_{bcpms}(\varkappa_{n-1},\varkappa_n) + \text{Þ}_{bcpms}(⅂_{n-1},⅂_n)}{2}, \\ \dfrac{\text{Þ}_{bcpms}(\varkappa_{n-1},\varkappa_n) + \text{Þ}_{bcpms}(⅂_{n-1},⅂_n)}{2}, \\ \dfrac{\text{Þ}_{bcpms}(\varkappa_n,\varkappa_{n+1}) + \text{Þ}_{bcpms}(⅂_n,⅂_{n+1})}{2}, \\ \dfrac{\text{Þ}_{bcpms}(\varkappa_n,\varkappa_n).\text{Þ}_{bcpms}(\varkappa_n,\varkappa_{n+1})}{1 + \text{Þ}_{bcpms}(\varkappa_{n-1},\varkappa_n) + \text{Þ}_{bcpms}(⅂_{n-1},⅂_n)} \end{matrix}\right\}$$

$$+L\min\begin{Bmatrix}Þ_{bcpms}(\varkappa_n,\varkappa_n), Þ_{bcpms}(\varkappa_{n+1},\varkappa_{n-1}),\\ Þ_{bcpms}(\varkappa_n,ꓔ_n), Þ_{bcpms}(\varkappa_{n+1},\varkappa_n)\end{Bmatrix}$$

$$= \beta\max\left\{\frac{r_n}{2},\frac{r_n}{2},\frac{r_{n+1}}{2},0\right\}+L(0), \qquad (2.1.3)$$

on combining (2.1.2) and (2.1.3),

we have

$$Þ_{bcpms}(ꓔ_n,ꓔ_{n+1}) + Þ_{bcpms}(\varkappa_n,\varkappa_{n+1})$$

$$\precsim_{i_2} 2\beta\max\left\{\frac{r_n}{2},\frac{r_{n+1}}{2}\right\} \qquad (2.1.4)$$

i.e., $r_{n+1} \precsim_{i_2} 2\beta\max\left\{\frac{r_n}{2},\frac{r_{n+1}}{2}\right\}$.

If $r_{n+1} > r_n$ then from (2.1.4), we get

$$\|r_{n+1}\| \leq 2\beta\left\|\frac{r_{n+1}}{2}\right\| = \beta\|r_{n+1}\|,$$

which is a contradiction since $\beta < 1$.

Therefore

$$\|r_{n+1}\| \leq \beta\|r_n\|. \qquad (2.1.5)$$

Similarly, we can show that

$$\|r_n\| \leq \beta\|r_{n-1}\|. \qquad (2.1.6)$$

Hence from (2.1.5) and (2.1.6), we can conclude that

$$\|r_n\| \leq \beta\|r_{n-1}\|, \text{ for all } n\in\mathbb{N} \qquad (2.1.7)$$

Thus, $\|r_n\| \leq \beta\|r_{n-1}\| \leq \beta^2\|r_{n-2}\| \leq \beta^3\|r_{n-3}\|$

$$\leq \cdots \leq \beta^n\|r_0\|. \qquad (2.1.8)$$

If $r_0 = 0$ in (2.1.8)

then $Þ_{bcpms}(ꓔ_1,ꓔ_0) = 0$, $Þ_{bcpms}(\varkappa_1,\varkappa_0) = 0$

$\Longrightarrow ꓔ_0 = ꓔ_1, \varkappa_0 = \varkappa_1$,

so $(ꓔ_0,\varkappa_0)$ follows as a fixed point of $\mathfrak{F}(ꓔ,\varkappa)$,

Now suppose $r_0 > 0$

For $m > n$, we now show that $\{ꓔ_n\}$ and $\{\varkappa_n\}$ are Cauchy sequence in $\mathfrak{S}$.

For all $n \geq m$, we have

$$\begin{aligned}Þ_{bcpms}(ꓔ_n,ꓔ_m) &\precsim_{i_2} Þ_{bcpms}(ꓔ_n,ꓔ_{n+1}) + Þ_{bcpms}(ꓔ_{n+1},ꓔ_m) - Þ_{bcpms}(ꓔ_{n+1},ꓔ_{n+1})\\ &\precsim_{i_2} Þ_{bcpms}(ꓔ_n,ꓔ_{n+1}) + Þ_{bcpms}(ꓔ_{n+1},ꓔ_m)\\ &\precsim_{i_2} Þ_{bcpms}(ꓔ_n,ꓔ_{n+1}) + Þ_{bcpms}(ꓔ_{n+1},ꓔ_{n+2}) + Þ_{bcpms}(ꓔ_{n+2},ꓔ_m)\\ &\dots\\ &\precsim_{i_2} Þ_{bcpms}(ꓔ_n,ꓔ_{n+1}) + Þ_{bcpms}(ꓔ_{n+1},ꓔ_{n+2}) + \cdots + Þ_{bcpms}(ꓔ_{m-1},ꓔ_m) \qquad (2.1.9)\end{aligned}$$

Similarly,

$$\begin{aligned}Þ_{bcpms}(\varkappa_n,\varkappa_m) &\precsim_{i_2} Þ_{bcpms}(\varkappa_n,\varkappa_{n+1}) + Þ_{bcpms}(\varkappa_{n+1},\varkappa_m) - Þ_{bcpms}(\varkappa_{n+1},\varkappa_{n+1})\\ &\precsim_{i_2} Þ_{bcpms}(\varkappa_n,\varkappa_{n+1}) + Þ_{bcpms}(\varkappa_{n+1},\varkappa_m)\\ &\precsim_{i_2} Þ_{bcpms}(\varkappa_n,\varkappa_{n+1}) + Þ_{bcpms}(\varkappa_{n+1},\varkappa_{n+2}) + Þ_{bcpms}(\varkappa_{n+2},\varkappa_m)\\ &\dots\\ &\precsim_{i_2} Þ_{bcpms}(\varkappa_n,\varkappa_{n+1}) + Þ_{bcpms}(\varkappa_{n+1},\varkappa_{n+2}) + \dots + Þ_{bcpms}(\varkappa_{m-1},\varkappa_m) \qquad (2.1.10)\end{aligned}$$

Based on (2.1.9) and (2.1.10), it follows that

$$Þ_{bcpms}(ꓔ_n,ꓔ_m) + Þ_{bcpms}(\varkappa_n,\varkappa_m)$$

$$\precsim_{i_2} Þ_{bcpms}(ꓔ_n,ꓔ_{n+1}) + Þ_{bcpms}(\varkappa_n,\varkappa_{n+1})$$

$$+ Þ_{bcpms}(ꓔ_{n+1},ꓔ_{n+2}) + Þ_{bcpms}(\varkappa_{n+1},\varkappa_{n+2}) + \cdots$$

$$+Þ_{bcpms}(ꓔ_{m-1},ꓔ_m) + Þ_{bcpms}(\varkappa_{m-1},\varkappa_m)$$

$$\Longrightarrow \left\|Þ_{bcpms}(ꓔ_n,ꓔ_m) + Þ_{bcpms}(\varkappa_n,\varkappa_m)\right\|$$

$$\leq \|r_n + r_{n+1} + r_{n+2} + \cdots + r_{m-1}\|$$

$$\leq \beta^n\|r_0\| + \beta^{n+1}\|r_0\| + \beta^{n+2}\|r_0\| + \ldots + \beta^{m-1}\|r_0\|$$

$$\leq \beta^n(1+\beta+\beta^2+\ldots+\beta^{m-n-1})\|r_0\|$$

$$= \frac{\beta^n}{1-\beta}\|r_0\| \to 0 \text{ as } n\to\infty,$$

Therefore $\{ꓔ_n\}$ *and* $\{\varkappa_n\}$ are Cauchy sequences in $\mathfrak{S}$.

Based on completeness bicomplex partial metric space $(\mathfrak{S}, Þ_{bcpms})$, there exists $ꓔ,\varkappa\in\mathfrak{S}$

such that

$\{\daleth_n\} \to \daleth$ *and* $\{\varkappa_n\} \to \varkappa$ as $n \to \infty$, and

$$Þ_{bcpms}(\daleth,\daleth) = \lim_{m,n\to\infty} Þ_{bcpms}(\daleth,\daleth_n)$$
$$= \lim_{m,n\to\infty} Þ_{bcpms}(\daleth_n,\daleth_m)$$
$$= 0 \qquad (2.1.11)$$

Similarly,

$$Þ_{bcpms}(\varkappa,\varkappa) = \lim_{m,n\to\infty} Þ_{bcpms}(\varkappa,\varkappa_n)$$
$$= \lim_{m,n\to\infty} Þ_{bcpms}(\varkappa_n,\varkappa_m)$$
$$= 0 \qquad (2.1.12)$$

We now show that $\daleth = \mathfrak{F}(\daleth,\varkappa)$ and $\varkappa = \mathfrak{F}(\varkappa,\daleth)$

Suppose that $\daleth \neq \mathfrak{F}(\daleth,\varkappa)$ and $\varkappa \neq \mathfrak{F}(\varkappa,\daleth)$

Therefore

$0 \prec_{i_2} Þ_{bcpms}(\daleth,\mathfrak{F}(\daleth,\varkappa))$ and

$0 \prec_{i_2} Þ_{bcpms}(\varkappa,\mathfrak{F}(\varkappa,\daleth))$

Then

$$Þ_{bcpms}(\daleth,\mathfrak{F}(\daleth,\varkappa)) \preccurlyeq_{i_2} Þ_{bcpms}(\daleth,\daleth_{n+1}) + Þ_{bcpms}(\daleth_{n+1},\mathfrak{F}(\daleth,\varkappa)) - Þ_{bcpms}(\daleth_{n+1},\daleth_{n+1})$$
$$\preccurlyeq_{i_2} Þ_{bcpms}(\daleth,\daleth_{n+1}) + Þ_{bcpms}(\daleth_{n+1},\mathfrak{F}(\daleth,\varkappa))$$
$$\preccurlyeq_{i_2} Þ_{bcpms}(\daleth,\daleth_{n+1}) + Þ_{bcpms}(\mathfrak{F}(\daleth_n,\varkappa_n),\mathfrak{F}(\daleth,\varkappa))$$

$$\Rightarrow \|Þ_{bcpms}(\daleth,\mathfrak{F}(\daleth,\varkappa))\| \le \|Þ_{bcpms}(\daleth,\daleth_{n+1})\|$$
$$+\beta \max\left\{\begin{array}{c}\dfrac{\|Þ_{bcpms}(\daleth_n,\daleth) + Þ_{bcpms}(\varkappa_n,\varkappa)\|}{2},\\ \dfrac{\|Þ_{bcpms}(\daleth_n,\mathfrak{F}(\daleth_n,\varkappa_n)) + Þ_{bcpms}(\varkappa_n,\mathfrak{F}(\varkappa_n,\daleth_n))\|}{2},\\ \dfrac{\|Þ_{bcpms}(\daleth,\mathfrak{F}(\daleth,\varkappa))) + Þ_{bcpms}(\varkappa,\mathfrak{F}(\varkappa,\daleth))\|}{2},\\ \left\|\dfrac{Þ_{bcpms}(\daleth,\mathfrak{F}(\daleth_n,\varkappa_n)).Þ_{bcpms}(\varkappa,\mathfrak{F}(\varkappa_n,\daleth_n))}{1 + Þ_{bcpms}(\daleth_n,\daleth) + Þ_{bcpms}(\varkappa_n,\varkappa)}\right\|\end{array}\right\}$$
$$+L \min\left\{\begin{array}{c}\|Þ_{bcpms}(\mathfrak{F}(\daleth_n,\varkappa_n),\daleth\|, \|Þ_{bcpms}(\mathfrak{F}(\daleth,\varkappa),\daleth_n\|\\ \|Þ_{bcpms}(\mathfrak{F}((\daleth_n,\varkappa_n),\varkappa)\|, \|Þ_{bcpms}(\mathfrak{F}(\daleth,\varkappa),\daleth)\|\end{array}\right\} \qquad (2.1.13)$$

Letting $n \to \infty$, using (2.1.11) and (2.1.12) in (2.1.13), we get

$$\|Þ_{bcpms}(\daleth,\mathfrak{F}(\daleth,\varkappa))\| \le \beta\|Þ_{bcpms}(\daleth,\mathfrak{F}(\daleth,\varkappa)) + Þ_{bcpms}(\varkappa,\mathfrak{F}(\varkappa,\daleth))\|$$

Similarly,

$$Þ_{bcpms}(\varkappa,\mathfrak{F}(\varkappa,\daleth)) \preccurlyeq_{i_2} Þ_{bcpms}(\varkappa,\varkappa_{n+1}) + Þ_{bcpms}(\varkappa_{n+1},\mathfrak{F}(\varkappa,\daleth)) - Þ_{bcpms}(\varkappa_{n+1},\varkappa_{n+1})$$
$$\preccurlyeq_{i_2} Þ_{bcpms}(\varkappa,\varkappa_{n+1}) + Þ_{bcpms}(\varkappa_{n+1},\mathfrak{F}(\varkappa,\daleth))$$
$$\preccurlyeq_{i_2} Þ_{bcpms}(\varkappa,\varkappa_{n+1}) + Þ_{bcpms}(\mathfrak{F}(\varkappa_n,\daleth_n),\mathfrak{F}(\varkappa,\daleth))$$
$$\|Þ_{bcpms}(\varkappa,\mathfrak{F}(\varkappa,\daleth))\| \le \|Þ_{bcpms}(\varkappa,\varkappa_{n+1})\|$$
$$+\beta \max\left\{\begin{array}{c}\dfrac{\|Þ_{bcpms}(\varkappa_n,\varkappa) + Þ_{bcpms}(\daleth_n,\daleth)\|}{2},\\ \dfrac{\|Þ_{bcpms}(\varkappa_n,\mathfrak{F}(\varkappa_n,\daleth_n)) + Þ_{bcpms}(\daleth_n,\mathfrak{F}(\daleth_n,\varkappa_n))\|}{2}\\ \dfrac{\|Þ_{bcpms}(\varkappa,\mathfrak{F}(\varkappa,\daleth))) + Þ_{bcpms}(\daleth,\mathfrak{F}(\daleth,\varkappa))\|}{2},\\ \left\|\dfrac{Þ_{bcpms}(\varkappa,\mathfrak{F}(\varkappa_n,\daleth_n)).Þ_{bcpms}(\daleth,\mathfrak{F}(\daleth_n,\varkappa_n))}{1 + Þ_{bcpms}(\varkappa_n,\varkappa) + Þ_{bcpms}(\daleth_n,\daleth)}\right\|\end{array}\right\}$$
$$+L \min\left\{\begin{array}{c}\|Þ_{bcpms}(\mathfrak{F}(\varkappa_n,\daleth_n),\varkappa)\|, \|Þ_{bcpms}(\mathfrak{F}(\varkappa,\daleth),\varkappa_n\|\\ \|Þ_{bcpms}(\mathfrak{F}((\varkappa_n,\daleth_n),\daleth))\|, \|Þ_{bcpms}(\mathfrak{F}(\varkappa,\daleth),\varkappa)\|\end{array}\right\} \qquad (2.1.14)$$

$$\Rightarrow \lim_{n\to\infty}\|Þ_{bcpms}(\varkappa,\mathfrak{F}(\varkappa,\daleth))\|$$
$$\le \beta\,\|Þ_{bcmps}(\daleth,\mathfrak{F}(\daleth,\varkappa)) + Þ_{bcmps}(\varkappa,\mathfrak{F}(\varkappa,\daleth))\|$$

Hence from (2.1.13) and (2.1.14), it follows that

$$\|Þ_{bcpms}(\daleth,\mathfrak{F}(\daleth,\varkappa))\| + \|Þ_{bcpms}(\varkappa,\mathfrak{F}(\varkappa,\daleth))\|$$
$$\le \beta[\|Þ_{bcpms}(\daleth,\mathfrak{F}(\daleth,\varkappa))\| + \|Þ_{bcpms}(\varkappa,\mathfrak{F}(\varkappa,\daleth))\|]$$

which is a contradiction, since $\beta < 1$,

Thus $\|Þ_{bcpms}(\daleth,\mathfrak{F}(\daleth,\varkappa))\| + \|Þ_{bcpms}(\varkappa,\mathfrak{F}(\varkappa,\daleth))\| = 0$

$\Longrightarrow \|Þ_{bcpms}(\daleth,\mathfrak{F}(\daleth,\varkappa))\| = 0$

and $\|Þ_{bcpms}(\varkappa,\mathfrak{F}(\varkappa,\daleth))\| = 0.$

Therefore $\daleth = \mathfrak{F}(\daleth,\varkappa), \varkappa = \mathfrak{F}(\varkappa,\daleth)$,

which ensure that $\mathfrak{F}$ has a coupled fixed point $(\daleth,\varkappa)$.

Next, suppose that $\mathfrak{F}$ has more than one coupled fixed point such that $(\curlywedge,\curlyvee) \neq (\daleth,\varkappa)$

Hence $\curlywedge = \mathfrak{F}(\curlywedge,\curlyvee)$, $\curlyvee = \mathfrak{F}(\curlyvee,\curlywedge)$.

Based on the inequality (2.1.1), we have

$$\text{Þ}_{bcpms}(\top,\curlywedge) = \text{Þ}_{bcpms}\big(\mathfrak{F}(\top,\varkappa),\mathfrak{F}(\curlywedge,\curlyvee)\big)$$

$$\preccurlyeq_{i_2} \beta \max\left\{\begin{array}{c}\frac{\text{Þ}_{bcpms}(\top,\curlywedge)+\text{Þ}_{bcpms}(\varkappa,\curlyvee)}{2},\\ \frac{\text{Þ}_{bcpms}(\top,\mathfrak{F}(\top,\varkappa))+\text{Þ}_{bcpms}(\varkappa,\mathfrak{F}(\varkappa,\top))}{2}\\ \frac{\text{Þ}_{bcpms}(\curlywedge,\mathfrak{F}(\curlywedge,\curlyvee))+\text{Þ}_{bcpms}(\curlyvee,\mathfrak{F}(\curlyvee,\curlywedge))}{2},\\ \frac{\text{Þ}_{bcpms}(\curlywedge,\mathfrak{F}(\top,\varkappa)).\text{Þ}_{bcpms}(\curlywedge,\mathfrak{F}(\curlywedge,\curlyvee))}{1+\text{Þ}_{bcpms}(\top,\curlywedge)+\text{Þ}_{bcpms}(\varkappa,\curlyvee)}\end{array}\right\}$$

$$+\ L\min\left\{\begin{array}{c}\text{Þ}_{bcpms}(\mathfrak{F}(\top,\varkappa),\curlywedge),\text{Þ}_{bcpms}(\mathfrak{F}(\curlywedge,\curlyvee),\top),\\ \text{Þ}_{bcpms}(\mathfrak{F}(\top,\varkappa),\curlyvee),\text{Þ}_{bcpms}(\mathfrak{F}(\curlywedge,\curlyvee),\curlywedge)\end{array}\right\}$$

$$= \beta\max\left\{\frac{\text{Þ}_{bcpms}(\top,\curlywedge)+\text{Þ}_{bcpms}(\varkappa,\curlyvee)}{2},0,0,0\right\}+L(0)$$

Therefore

$$\|\text{Þ}_{bcpms}(\top,\curlywedge)\| \le \beta\left\|\frac{\text{Þ}_{bcpms}(\top,\curlywedge)+\text{Þ}_{bcpms}(\varkappa,\curlyvee)}{2}\right\| \quad (2.1.15)$$

Similarly,

$$\text{Þ}_{bcpms}(\varkappa,\curlyvee) = \text{Þ}_{bcpms}\big(\mathfrak{F}(\varkappa,\top),\mathfrak{F}(\curlyvee,\curlywedge)\big)$$

$$\preccurlyeq_{i_2} \beta \max\left\{\begin{array}{c}\frac{\text{Þ}_{bcpms}(\varkappa,\curlyvee)+\text{Þ}_{bcpms}(\top,\curlywedge)}{2},\\ \frac{\text{Þ}_{bcpms}(\varkappa,\mathfrak{F}(\varkappa,\top))+\text{Þ}_{bcpms}(\top,\mathfrak{F}(\top,\varkappa))}{2}\\ \frac{\text{Þ}_{bcpms}(\curlyvee,\mathfrak{F}(\curlyvee,\curlywedge))+\text{Þ}_{bcpms}(\curlywedge,\mathfrak{F}(\curlywedge,\curlyvee))}{2},\\ \frac{\text{Þ}_{bcpms}(\curlywedge,\mathfrak{F}(\top,\varkappa)).\text{Þ}_{bcpms}(\curlywedge,\mathfrak{F}(\curlywedge,\curlyvee))}{1+\text{Þ}_{bcpms}(\top,\curlywedge)+\text{Þ}_{bcpms}(\varkappa,\curlyvee)}\end{array}\right\}$$

$$+L\min\left\{\begin{array}{c}\text{Þ}_{bcpms}(\mathfrak{F}(\varkappa,\top),\curlyvee),\text{Þ}_{bcpms}(\mathfrak{F}(\curlyvee,\curlywedge),\varkappa),\\ \text{Þ}_{bcpms}(\mathfrak{F}(\varkappa,\top),\curlywedge),\text{Þ}_{bcpms}(\mathfrak{F}(\curlyvee,\curlywedge),\curlyvee)\end{array}\right\}$$

$$= \beta\max\left\{\frac{\text{Þ}_{bcpms}(\varkappa,\curlyvee)+\text{Þ}_{bcpms}(\top,\curlywedge)}{2},0,0,0\right\}+L(0)$$

$$\|\text{Þ}_{bcpms}(\varkappa,\curlyvee)\| \le \beta\left\|\frac{\text{Þ}_{bcpms}(\varkappa,\curlyvee)+\text{Þ}_{bcpms}(\top,\curlywedge)}{2}\right\| \quad (2.1.16)$$

Hence from (2.1.15) and (2.1.16), it follows

$$\|\text{Þ}_{bcpms}(\top,\curlywedge)\|+\|\text{Þ}_{bcpms}(\varkappa,\curlyvee)\| \le \beta\,[\|\text{Þ}_{bcpms}(\top,\curlywedge)\|+\|\text{Þ}_{bcpms}(\varkappa,\curlyvee)\|],$$

a contradiction, since $\beta < 1$

Hence $\mathfrak{F}$ has a unique coupled fixed point.

Corollary2.2. If a mapping $\mathfrak{F}:\mathfrak{S}\times\mathfrak{S}\longrightarrow\mathfrak{S}$ is a bicomplex partial metric, is such that

$$\text{Þ}_{bcpms}\big(\mathfrak{F}(\top,\varkappa),\mathfrak{F}(\curlywedge,\curlyvee)\big) \preccurlyeq_{i_2} \beta\, M(\top,\varkappa,\curlywedge,\curlyvee) \quad (2.2.1)$$

Where M is defined as in the Theorem 2.1, then $\mathfrak{F}$ has a unique coupled fixed point in $\mathfrak{S}$.

Proof. If we choose $L = 0$, in Theorem- 2.1 then the proof follows.

Corollary 2.3. Corollary 2.2also continuous to be true if (2.2.1) is replaced with

$$\text{Þ}_{bcpms}\big(\mathfrak{F}(\top,\varkappa),\mathfrak{F}(\curlywedge,\curlyvee)\big) \preccurlyeq_{i_2} \text{ƛ}\,\text{Þ}_{bcpms}(\top,\curlywedge) + \text{ł}\,\text{Þ}_{bcpms}(\varkappa,\curlyvee),$$

for all $\top,\varkappa,\curlywedge,\curlyvee \in \mathfrak{S}$ and $\text{ƛ},\text{ł}$ is in $[0,1)$

with $\text{ƛ}+\text{ł} < 1$.

Proof. Proof follows by choosing $\text{ƛ} = \text{ł} = \frac{\beta}{2}$, $\beta < 1$ and

$M(\top,\varkappa,\curlywedge,\curlyvee) = \text{Þ}_{bcpms}(\top,\curlywedge)+\text{Þ}_{bcpms}(\varkappa,\curlyvee)$

for all $\top,\varkappa,\curlywedge,\curlyvee \in \mathfrak{S}$ in Theorem 2.1.

Corollary 2.4. Corollary 2.2 also continuous to be true if (2.2.1) is replaced by

$$\big(\mathfrak{F}(\top,\varkappa),\mathfrak{F}(\curlywedge,\curlyvee)\big) \preccurlyeq_{i_2} \text{ƛ}\,[\text{Þ}_{bcpms}(\mathfrak{F}(\top,\varkappa),\top)] + \text{ł}\,[\text{Þ}_{bcpms}(\mathfrak{F}(\curlywedge,\curlyvee),\curlywedge)]$$

for all $\top,\varkappa,\curlywedge,\curlyvee \in \mathfrak{S}$with $\text{ƛ}+\text{ł} < 1$.

Proof. Proof follows by choosing $\text{ƛ} = \text{ł} = \frac{\beta}{2}$, $\beta < 1$ and

$M(\top,\varkappa,\curlywedge,\curlyvee) = [\text{Þ}_{bcpms}(\mathfrak{F}(\top,\varkappa),\top)]+[\text{Þ}_{bcpms}(\mathfrak{F}(\curlywedge,\curlyvee),\curlywedge)]$

in Theorem- 2.1.

Remark 2.5.

(i) Corollary 2.3 is an enhancement of Theorem 1.10 of Gunaseelan Mani *et.al.*,[4]

(ii) Corollary 2.4 is an enhancement of Theorem 1.11 of Gunaseelan Mani *et.al.*,[4]

Example 2.6. Let $\mathfrak{S} = [0,1]$ be equipped with the partial order $\preccurlyeq_{i_2}$define $\text{Þ}_{bcmps}:\mathfrak{S}\times\mathfrak{S}\longrightarrow\mathbb{C}_2$

by $Þ_{bcpms}(ꓕ,\varkappa) = (1+i_2)\max\{ꓕ,\varkappa\}$ ⍰ $ꓕ,\varkappa \in \mathfrak{S}$
Clearly, $(\mathfrak{S}, Þ_{bcpms})$is a bicomplex partial metric space.
We define $\mathfrak{F}: \mathfrak{S} \times \mathfrak{S} \longrightarrow \mathfrak{S}$ by

$$\mathfrak{F}(ꓕ,\varkappa) = \begin{cases} 0, & ꓕ< \varkappa \\ \frac{ꓕ^2 - \varkappa^2}{4}, & ꓕ\geq \varkappa \end{cases}$$

Let $ꓕ,\varkappa,ʎ,ⵖ \in \mathfrak{S}$such that $ꓕ\leq ʎ, \varkappa \geq ⵖ$.

To verify the inequality (2.1.1), we have the following cases

Case (i): Let$ꓕ\geq \varkappa$, since $ꓕ\leq ʎ, \varkappa \geq ⵖ$,

$\Longrightarrow ʎ \geq ꓕ \geq \varkappa \geq ⵖ$.

Now

$$Þ_{bcpms}(\mathfrak{F}(ꓕ,\varkappa), \mathfrak{F}(ʎ,ⵖ)) = (1+i_2)\, max\left\{\frac{ꓕ^2-\varkappa^2}{4}, \frac{ʎ^2-ⵖ^2}{4}\right\}$$

$$= (1+i_2)\left(\frac{ʎ^2-ⵖ^2}{4}\right)$$

$$\preccurlyeq_{i_2} \left(\frac{ʎ-ⵖ}{2}\right)(1+i_2)\left(\frac{ʎ+ⵖ}{2}\right)$$

$$= \left(\frac{ʎ-ⵖ}{2}\right)\frac{1}{2}[(1+i_2)\, max\{ꓕ,ʎ\} + (1+i_2)\, max\{\varkappa,ⵖ\}]$$

$$= \left(\frac{ʎ-ⵖ}{2}\right)\left[\frac{Þ_{bcpms}(ꓕ,ʎ) + Þ_{bcpms}(\varkappa,ⵖ)}{2}\right]$$

$$\preccurlyeq_{i_2} \frac{ʎ}{2}\ \mathrm{M}(ꓕ,\varkappa,ʎ,ⵖ)$$

$$\preccurlyeq_{i_2} \frac{1}{2}\ \mathrm{M}(ꓕ,\varkappa,ʎ,ⵖ)$$

$$\preccurlyeq_{i_2} \beta\, \mathrm{M}(ꓕ,\varkappa,ʎ,ⵖ) + \mathrm{L}\, \mathrm{N}(ꓕ,\varkappa,ʎ,ⵖ)$$

Case (ii): When $ꓕ< \varkappa,\ ʎ\geq ⵖ$ *and* $ꓕ>ⵖ$

$$Þ_{bcpms}(\mathfrak{F}(ꓕ,\varkappa), \mathfrak{F}(ʎ,ⵖ)) = Þ_{bcpms}\left(0, \frac{ʎ^2-ⵖ^2}{4}\right)$$

$$= (1+i_2)\max\left\{0, \frac{ʎ^2-ⵖ^2}{4}\right\}$$

$$= (1+i_2)\frac{ʎ^2-ⵖ^2}{4}$$

$$= (1+i_2)\left(\frac{ʎ+ⵖ}{2}\right)\left(\frac{ʎ-ⵖ}{2}\right)$$

$$\preccurlyeq_{i_2} \left(\frac{ʎ-ⵖ}{2}\right)(1+i_2)\left(\frac{ʎ+ⵖ}{2}\right)$$

$$= \left(\frac{ʎ-ⵖ}{2}\right)\frac{1}{2}[(1+i_2)\max\{ꓕ,ʎ\} + (1+i_2)\max\{\varkappa,ⵖ\}]$$

$$= \left(\frac{ʎ-ⵖ}{2}\right)\left[\frac{Þ_{bcpms}(ꓕ,ʎ)+Þ_{bcpms}(\varkappa,ⵖ)}{2}\right]$$

$$\preccurlyeq_{i_2} \frac{ʎ}{2}\ \mathrm{M}(ꓕ,\varkappa,ʎ,ⵖ)$$

$$\preccurlyeq_{i_2} \frac{1}{2}\ \mathrm{M}(ꓕ,\varkappa,ʎ,ⵖ)$$

$$\preccurlyeq_{i_2} \beta\, \mathrm{M}(ꓕ,\varkappa,ʎ,ⵖ) + \mathrm{L}\, \mathrm{N}(ꓕ,\varkappa,ʎ,ⵖ)$$

Case(iii): When $ꓕ\geq \varkappa,\ ʎ<ⵖ$ does not arise,

since $ꓕ\geq ⵖ, \varkappa \geq ʎ$.

Case(iv): When $ꓕ< \varkappa, ʎ<ⵖ$.

Then $\mathfrak{F}(ꓕ,\varkappa) = 0$ and $\mathfrak{F}(ʎ, ⵖ) = 0$ and

$Þ_{bcpms}(\mathfrak{F}(ꓕ, \varkappa), \mathfrak{F}(ʎ, ⵖ)) = 0$ so that inequality (2.1.1) holds.

From the above four cases,we validated the inequality (2.1.1) with $\beta = \frac{1}{2}$ and for any $\mathrm{L} \geq 0$. Therefore, all the conditions of Theorem 2.1, are satisfied and $\mathfrak{F}$ has a unique coupled fixed point $(0,0)$.

Example 2.7. Let $\mathfrak{S} = \{0,5,7\}$ and we define

$Þ_{bcpms}: \mathfrak{S} \times \mathfrak{S} \rightarrow \mathbb{C}_2$by

$$Þ_{bcpms}(ꓕ,\varkappa) = \begin{cases} (1+i_2)2 & if(ꓕ,\varkappa) \in A_1 \\ (1+i_2)4 & if(ꓕ,\varkappa) \in A_2 \\ 0 & if\ (ꓕ,\varkappa) \in A_3 \\ (1+i_2)7 & if(ꓕ,\varkappa) \in A_4 \\ (1+i_2)5 & if(ꓕ,\varkappa) \in A_5 \end{cases}$$

where $A_1 = \{(0,0),(5\ ,7)\}$, $A_2 = \{(5,0)\}$,
$A_3 = \{(0,5),(0,7),(5,5)\}$, $A_4 = \{(7,0)\}$,
$A_5 = \{(7,5\ )(7,7)\}$

Clearly $(\mathfrak{S}, Þ_{bcpms)}$ is a bicomplex partial metric space but it is not a bicomplex valued metric space,

since $Þ_{bcpms}(7,7) = (1+i_2)\ 5$

We define $\mathfrak{F}: \mathfrak{S} \times \mathfrak{S} \longrightarrow \mathfrak{S}$ by

$$\mathfrak{F}(ꓕ,\varkappa) = \begin{cases} 5\ \ if(ꓕ,\varkappa) \in B_1 \\ 7\ \ if\ (ꓕ,\varkappa) \in B_2 \end{cases}$$

where

$B_1 = \{(0,0),(0,5),(0,7),(5,0),(5,7),(7,5),(5,5),(7,7)\}$
$B_2 = \{(7,0)\}$

We now verify the inequality (2.1.1) with

$\beta = \frac{5}{6}$ and $\mathrm{L} = \frac{1}{2}$

when

$$(\daleth, \varkappa) \in \left\{ \begin{matrix} (0,0), (0,5), (0,7), \\ (5,0), (5,7), (7,5), (5,5), (7,7) \end{matrix} \right\}$$

and $(\curlywedge, \curlyvee) \in \{(7,0)\}$

since in remaining cases

we have $Þ_{bcpms}(\mathfrak{F}(\daleth, \varkappa), \mathfrak{F}(\curlywedge, \curlyvee)) = 0$.

Case 1: When $(\daleth, \varkappa) = (0,0), (\curlywedge, \curlyvee) = (7,0)$

$$\begin{aligned} &Þ_{bcpms}(\mathfrak{F}(\daleth, \varkappa), \mathfrak{F}(\curlywedge, \curlyvee)) \\ &= Þ_{bcpms}(\mathfrak{F}(0,0), \mathfrak{F}(7,0)) \\ &= Þ_{bcpms}(5,7) = (1 + i_2)2 \\ &\preccurlyeq_{i_2} \frac{5}{6}(1 + i_2)(4.5) \\ &= \beta\, \mathrm{M}(\daleth, \varkappa, \curlywedge, \curlyvee) + \mathrm{L}\, \mathrm{N}(\daleth, \varkappa, \curlywedge, \curlyvee) \end{aligned}$$

Case 2: When $(\daleth, \varkappa) = (0,5), (\curlywedge, \curlyvee) = (7,0)$

$$\begin{aligned} &Þ_{bcpms}(\mathfrak{F}(\daleth, \varkappa), \mathfrak{F}(\curlywedge, \curlyvee)) \\ &= Þ_{bcpms}(\mathfrak{F}(0,5), \mathfrak{F}(7,0)) \\ &= Þ_{bcpms}(5,7) = (1 + i_2)2 \\ &\preccurlyeq_{i_2} \frac{5}{6}(1 + i_2)(3.5) \\ &= \beta\, \mathrm{M}(\daleth, \varkappa, \curlywedge, \curlyvee) + \mathrm{L}\, \mathrm{N}(\daleth, \varkappa, \curlywedge, \curlyvee) \end{aligned}$$

Case 3: When $(\daleth, \varkappa) = (0,7), (\curlywedge, \curlyvee) = (7,0)$

$$\begin{aligned} &Þ_{bcpms}(\mathfrak{F}(\daleth, \varkappa), \mathfrak{F}(\curlywedge, \curlyvee)) \\ &= Þ_{bcpms}(\mathfrak{F}(0,7), \mathfrak{F}(7,0)) \\ &= Þ_{bcpms}(5,7) = (1 + i_2)2 \\ &\preccurlyeq_{i_2} \frac{5}{6}(1 + i_2)(7) \\ &= \beta\, \mathrm{M}(\daleth, \varkappa, \curlywedge, \curlyvee) + \mathrm{L}\, \mathrm{N}(\daleth, \varkappa, \curlywedge, \curlyvee) \end{aligned}$$

Case 4: When $(\daleth, \varkappa) = (5,0), (\curlywedge, \curlyvee) = (7,0)$

$$\begin{aligned} &Þ_{bcpms}(\mathfrak{F}(\daleth, \varkappa), \mathfrak{F}(\curlywedge, \curlyvee)) \\ &= Þ_{bcpms}(\mathfrak{F}(5,0), \mathfrak{F}(7,0)) \\ &= Þ_{bcpms}(5,7) = (1 + i_2)2 \\ &\preccurlyeq_{i_2} \frac{5}{6}(1 + i_2)(7) \\ &= \beta\, \mathrm{M}(\daleth, \varkappa, \curlywedge, \curlyvee) + \mathrm{L}\, \mathrm{N}(\daleth, \varkappa, \curlywedge, \curlyvee) \end{aligned}$$

Case 5: When $(\daleth, \varkappa) = (7,0), (\curlywedge, \curlyvee) = (7,0)$

$$\begin{aligned} &Þ_{bcpms}(\mathfrak{F}(\daleth, \varkappa), \mathfrak{F}(\curlywedge, \curlyvee)) \\ &= Þ_{bcpms}(\mathfrak{F}(7,0), \mathfrak{F}(7,0)) \\ &= Þ_{bcpms}(7,7) = (1 + i_2)5 \\ &\preccurlyeq_{i_2} \frac{5}{6}(1 + i_2)(2.5) + \frac{1}{2}(1 + i_2)(7) \end{aligned}$$

Case 6: When $(\daleth, \varkappa) = (5,5), (\curlywedge, \curlyvee) = (7,0)$

$$\begin{aligned} &Þ_{bcpms}(\mathfrak{F}(\daleth, \varkappa), \mathfrak{F}(\curlywedge, \curlyvee)) \\ &= Þ_{bcpms}(\mathfrak{F}(5,5), \mathfrak{F}(7,0)) \\ &= Þ_{bcpms}(5,7) = (1 + i_2)2 \\ &\preccurlyeq_{i_2} \frac{5}{6}(1 + i_2)(2.5) \\ &= \beta\, \mathrm{M}(\daleth, \varkappa, \curlywedge, \curlyvee) + \mathrm{L}\, \mathrm{N}(\daleth, \varkappa, \curlywedge, \curlyvee) \end{aligned}$$

Case 7: When $(\daleth, \varkappa) = (5,7), (\curlywedge, \curlyvee) = (7,0)$

$$\begin{aligned} &Þ_{bcpms}(\mathfrak{F}(\daleth, \varkappa), \mathfrak{F}(\curlywedge, \curlyvee)) \\ &= Þ_{bcpms}(\mathfrak{F}(5,7), \mathfrak{F}(7,0)) \\ &= Þ_{bcpms}(5,7) = (1 + i_2)2 \\ &\preccurlyeq_{i_2} \frac{5}{6}(1 + i_2)(4.5) \\ &= \beta\, \mathrm{M}(\daleth, \varkappa, \curlywedge, \curlyvee) + \mathrm{L}\, \mathrm{N}(\daleth, \varkappa, \curlywedge, \curlyvee) \end{aligned}$$

Case 8: When $(\daleth, \varkappa) = (7,5), (\curlywedge, \curlyvee) = (7,0)$

$$\begin{aligned} &Þ_{bcpms}(\mathfrak{F}(\daleth, \varkappa), \mathfrak{F}(\curlywedge, \curlyvee)) \\ &= Þ_{bcpms}(\mathfrak{F}(7,5), \mathfrak{F}(7,0)) \\ &= Þ_{bcpms}(5,7) = (1 + i_2)2 \\ &\preccurlyeq_{i_2} \frac{5}{6}(1 + i_2)(2.5) \\ &= \beta\, \mathrm{M}(\daleth, \varkappa, \curlywedge, \curlyvee) + \mathrm{L}\, \mathrm{N}(\daleth, \varkappa, \curlywedge, \curlyvee) \end{aligned}$$

Case 9: When $(\daleth, \varkappa) = (7,7), (\curlywedge, \curlyvee) = (7,0)$

$$\begin{aligned} &Þ_{bcpms}(\mathfrak{F}(\daleth, \varkappa), \mathfrak{F}(\curlywedge, \curlyvee)) \\ &= Þ_{bcpms}(\mathfrak{F}(7,7), \mathfrak{F}(7,0)) \\ &= Þ_{bcpms}(5,7) = (1 + i_2)2 \\ &\preccurlyeq_{i_2} \frac{5}{6}(1 + i_2)(6) \\ &= \beta\, \mathrm{M}(\daleth, \varkappa, \curlywedge, \curlyvee) + \mathrm{L}\, \mathrm{N}(\daleth, \varkappa, \curlywedge, \curlyvee) \end{aligned}$$

Therefore, all the conditions of Theorem 2.1 are satisfied and $\mathfrak{F}$ has a unique coupled fixed point (5,5).

3. Applications to integral equations

We study the existence of the following system of integral equations.

$$\text{Ͳ}(\ltimes) = \mu(\ltimes) + \int_a^b \big(k_1(\ltimes,\rtimes) + k_2(\ltimes,\rtimes)\big)\big(\mathfrak{F}_1(\rtimes,\text{Ͳ}(\rtimes)) + \mathfrak{F}_2(\rtimes,\varkappa(\rtimes))\big)d\rtimes$$

$$\varkappa(\ltimes) = \mu(\ltimes) + \int_a^b \big(k_1(\ltimes,\rtimes) + k_2(\ltimes,\rtimes)\big)\big(\mathfrak{F}_1(\rtimes,\text{Ͳ}(\rtimes)) + \mathfrak{F}_2(\rtimes,\varkappa(\rtimes))\big)d\rtimes$$

where $\ltimes \in [a,b], k_1, k_2 \in \big(([a,b]\times[a,b]),\mathbb{R}\big)$ such that $k_1(\ltimes,\rtimes) \geq 0$ and $k_2(\ltimes,\rtimes) \leq 0$

Let $\mathfrak{S} = C([a,b],\mathbb{R})$ be the class of all real valued continuous functions on $[a,b]$.

We define a partial order $\preccurlyeq_{i_2}$ on $\mathbb{C}_2^{\;+}$ as $x \preccurlyeq_{i_2} y$ if and only if $x \leq y$.

Define $\mathfrak{F}: \mathfrak{S}\times\mathfrak{S} \longrightarrow \mathfrak{S}$ by

$$\mathfrak{F}(\text{Ͳ},\varkappa)(\rtimes) = \int_a^b k_1(\ltimes,\rtimes)[\mathfrak{F}_1(\rtimes,\text{Ͳ}(\rtimes)) + \mathfrak{F}_2(\rtimes,\varkappa(\rtimes))]d\rtimes + \int_a^b k_2(\ltimes,\rtimes)[\mathfrak{F}_1(\rtimes,\varkappa(\rtimes)) + \mathfrak{F}_2(\rtimes,\text{Ͳ}(\rtimes))]d\rtimes + \mu(\ltimes)$$

Now $(\text{Ͳ}(\rtimes),\varkappa(\rtimes))$ is a solution of system of integral equation iff $(\text{Ͳ}(\rtimes),\varkappa(\rtimes))$ is a coupled fixed point of $\mathfrak{F}$.

Define $\text{Þ}_{bcpms}: \mathfrak{S}\times\mathfrak{S} \longrightarrow \mathbb{C}_2$ by

$$\text{Þ}_{bcmps}(\text{Ͳ},\varkappa) = |\text{Ͳ} - \varkappa| + 2 + i_2|\text{Ͳ} - \varkappa| + 2$$

for all $\text{Ͳ},\varkappa \in \mathfrak{S}$.

Theorem 3.1. Assume the following hypothesis:

1. The mapping $\mathfrak{F}_1 : [a,b]X\mathbb{R} \to \mathbb{R}, \mathfrak{F}_2: [a,b] \to \mathbb{R}$ are continuous.
2. There exists $\text{Ͳ},\varkappa \in \mathbb{R}$, $\text{Ͳ} < \varkappa$ such that $0 \preccurlyeq \mathfrak{F}_1(\ltimes,\text{Ͳ}) - \mathfrak{F}_1(\ltimes,\varkappa) \preccurlyeq_{i_2} \text{Ͳ} - \varkappa$ and $-(\text{Ͳ} - \varkappa) \preccurlyeq_{i_2} \mathfrak{F}_2(\ltimes,\text{Ͳ}) - \mathfrak{F}_2(\ltimes,\varkappa) \preccurlyeq_{i_2} 0$
3. $\int |k_1(\ltimes,\rtimes) - k_2(\ltimes,\rtimes)| d\rtimes \leq \frac{\beta}{2}$, $\beta \in [0,1)$,

then the integral equation (3.1.1) has a unique solution in $\mathfrak{S}$.

Proof. Consider

$$\text{Þ}_{bcmps}\big(\mathfrak{F}(\text{Ͳ},\varkappa),\mathfrak{F}(\curlywedge,\curlyvee)\big) = (1+i_2)(|\mathfrak{F}(\text{Ͳ},\varkappa) - \mathfrak{F}(\curlywedge,\curlyvee)| + 2)$$

$$\preccurlyeq_{i_2} (1+i_2) \left| \begin{array}{c} \int_a^b k_1(\ltimes,\rtimes)\,[\mathfrak{F}_1(\rtimes,\text{Ͳ}(\rtimes)) + \mathfrak{F}_2(\rtimes,\varkappa(\rtimes))]d\rtimes \\ + \int_a^b k_2(\ltimes,\rtimes)\,[\mathfrak{F}_1(\rtimes,\varkappa(\rtimes)) + \mathfrak{F}_2(\rtimes,\text{Ͳ}(\rtimes))]d\rtimes + \mu(\rtimes) \\ - \int_a^b k_1(\ltimes,\rtimes)\,[\mathfrak{F}_1(\rtimes,\curlywedge(\rtimes)) + \mathfrak{F}_2(\rtimes,\curlyvee(\rtimes))]d\rtimes \\ - \int_a^b k_1(\ltimes,\rtimes)\,[\mathfrak{F}_1(\rtimes,\curlywedge(\rtimes)) + \mathfrak{F}_2(\rtimes,\curlyvee(\rtimes))]d\rtimes - \mu(\rtimes) \end{array} \right| + 2$$

$$\preccurlyeq_{i_2} (1+i_2) \left| \begin{array}{c} \int_a^b k_1(\ltimes,\rtimes)\left[\left|\begin{array}{c}\mathfrak{F}_1(\rtimes,\text{Ͳ}(\rtimes)) - \mathfrak{F}_1(\rtimes,\curlywedge(\rtimes)) \\ +\mathfrak{F}_2(\rtimes,\varkappa(\rtimes)) - \mathfrak{F}_2(\rtimes,\curlyvee(\rtimes))\end{array}\right|\right]d\rtimes \\ - \int_a^b k_2(\ltimes,\rtimes)\left[\left|\begin{array}{c}\mathfrak{F}_1(\rtimes,\curlyvee(\rtimes)) - \mathfrak{F}_1(\rtimes,\varkappa(\rtimes)) \\ +\mathfrak{F}_2(\rtimes,\curlywedge(\rtimes)) - \mathfrak{F}_2(\rtimes,\text{Ͳ}(\rtimes))\end{array}\right|\right]d\rtimes \end{array} \right| + 2$$

$$\preccurlyeq_{i_2} (1+i_2)\int_a^b \begin{array}{c} |k_1(\ltimes,\rtimes) - k_2(\ltimes,\rtimes)|d\rtimes \\ |\text{Ͳ}(\rtimes) - \curlywedge(\rtimes) + \varkappa(\rtimes) - \curlyvee(\rtimes)| \end{array} + 2$$

$$\preccurlyeq_{i_2} \frac{\beta}{2}[|\text{Ͳ}(\rtimes) - \curlywedge(\rtimes)| + 2 + i_2|\varkappa(\rtimes) - \curlyvee(\rtimes)| + 2]$$

$$= \frac{\beta}{2}\left[\text{Þ}_{bcpms}(\text{Ͳ},\curlywedge) + \text{Þ}_{bcpms}(\varkappa,\curlyvee)\right]$$

$$= \beta\left[\frac{\text{Þ}_{bcpms}(\text{Ͳ},\curlywedge) + \text{Þ}_{bcpms}(\varkappa,\curlyvee)}{2}\right]$$

$$\preccurlyeq_{i_2} \beta\, M(\text{Ͳ},\varkappa,\curlywedge,\curlyvee) + L\, N(\text{Ͳ},\varkappa,\curlywedge,\curlyvee)$$

for all $\text{Ͳ},\varkappa,\curlywedge,\curlyvee \in \mathfrak{S}, \beta \in [0,1)$

Hence all the hypotheses of Theorem 2.1 holds . Therefore by Theorem 2.1, the $\mathfrak{F}$has unique coupled fixed point in $\mathfrak{S}$, hence the integral equation (3.1.1) has a unique solution in $\mathfrak{S}$.

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