p (Probability) – Independence and π (Possibility) – Non-Interaction.

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ABSTRACT :- The present paper introduces the concept of fuzzy probability and possibility spaces. Moreover we shall discuss fuzzy events. We shall discuss fuzzy events. We shall also investigate some properties and weak relations between p (probability) – independence and π (Possibility) – non-Interaction of two fuzzy events.

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From many years fuzzy set theory suffered to be confused with probability theory. For many people, membership functions seemed to be an imitation of probability distributions. However, little by little, people have been convinced by the genuine originality of fuzzy set theory. The difference between probability theory and fuzzy set theory was made clear when Zadeh in 1977 related this latter theory to the concept of possibility.

Zadeh takes the first steps in developing the intuitive concept of possibility as a qualitative mathematical concept analogous to probability. He illustrates the colloquial use of these terms by examples, noting the difference between them. For example, a high degree of probability always implies a high degree of possibility but not conversely; also (in appropriate circumstances) probabilities add, but this is not the rule for possibilities. Zadeh puts forward various proposals for the way, possibilities should behave. For instance he proposes that the possibility of a union of events should be taken as the maximum of the possibilities of the events that compose it.

The need to study the possibility theory relates with soft data which can be inexact in several ways. In dealing with computer security consideration, it is often the case that the data which is being used is neither exact nor lends itself to exact analysis. It may not be possible to determine whether or not a piece of information which will enable an individual to over come a system security measures is available to that individual. Even if it is possible to make an exact verification of whether this information is available, it may not be within the ability of the systems to obtain this data within a reasonable cost. Often probability theory has been used to handle soft data in the security structures. However, probability theory has the inherent difficulty that there is often a difference between what is probable and what is possible. In order to provide the maximum amount of system security, it would seem that one would wish to protect against the possible as well as the probable. A system which takes into account this difference is possibility theory.

The basic concept of possibility theory is the possibility distribution. A possibility distribution arises from another closely related concept, the fuzzy restriction.

It is important to note that the concept of a possibility distribution and the concept of a probability distribution are not equivalent. The fundamental distinction between the two concepts is that while some events have high possibility, they do not necessarily also have high probabilities. In other words, an event may be totally possible but highly improbable.

The consistency of probability and possibility distributions can be stated as follows.

1. We shall now discuss fuzzy events & develop the concept of fuzzy probability & possibility space. It is then natural to investigate some properties & weak relations between p- independence & II-non interactions of two fuzzy events.

Let Ω be a set, (Ω,Σ, p) be a probability space. Let BL ([0, 1]) be a system of Borel subsets in [0, 1] & F (Ω) be a lattice of fuzzy subsets on Ω.

<u>Definition 1</u> -The mapping

$\mu_B : \Omega \rightarrow [0, 1]$ is called a Σ - m

iff $\forall \alpha \in [0,1], \{\omega / \mu_B(\omega) = \in [0,\alpha]\} \in \Sigma$ -------- I

<u>Theorem 1</u> The mapping

$\mu_B : \Omega \rightarrow [0, 1]$ is a Σ - m iff $\forall A \in BL([0,1]), \{\omega / \mu_B(\omega) = \in A) \in \Sigma$ ------- II

Proof :

Let A = [0, α] in II, then

we obtain I immediately.

Let $M = \{A / A = [0,1]; \{\omega / \mu_B(\omega) \in A\} \in \Sigma\}$ ------- III

It is obvious that M is a σ-algebra in [0, 1]. In fact the following hold :

(1) $[0, 1] \in M$, because $\{\omega / \mu_B(\omega) \in [0, 1]\} = -2 \in \Sigma$

(2) If, $A \in M$ then $A^C = [0,1] - A \subset [0,1]$ & $\{\omega / \mu_B(\omega) \in A^C\} = \{\omega / \mu_B(\omega) \in A\} \in \Sigma$

namely $A^C \in M$

(3) If $A_i \in M$, i = 1,2,...

then $\bigcup_{i=1}^{\infty} A_i \subset [0,1]$ & $\{\omega / \mu_B(\omega) \in \bigcup_{i=1}^{\infty} A_i\}$

$= \cup_{i=1}^{\infty} \{\omega / \mu_B(\omega) \in A_i\} \in \Sigma$, namely

$\cup_{i=1}^{\infty} A_i \in M$,

Secondly we can show that. $\forall\ \alpha \in [0,1]$, $[0,\alpha] \in M$

In fact it is obvious that $[0,\alpha] \subset [0,1]$ & $\{\omega / \mu_B(\omega) \in [0,\alpha]\} \in \Sigma$;

since μ_B is Σ-m.

We have thus proved that M is a σ-algebra & BL ([0,1]) ⊂ M

So A ∈ BL ([0,1]), $\{\omega / \mu_B(\omega) \in A\} \in \Sigma$

Definition 2: Let $\beta = \{B/B \in F(\Omega)$; **<u>μ_B is a Σ-m}</u>**

If then B ∈ β, then B is called a fuzzy event on Ω.

Theorem 2 : All fuzzy events (Ω, Σ, P, β) constitute a fuzzy σ-algebra in Ω.

Proof :

Corresponding to the three conditions of fuzzy σ-algebra, we now prove the following propositions as follows:

(1) Ω ∈ β, since

$\{\omega / \mu_\Omega(\omega) \in [0,\alpha] = \left\{ \begin{matrix} \Omega \in \Sigma;\ \alpha=1 \\ \emptyset \in \Sigma;\ \alpha \in [0,1] \end{matrix} \right\}$

(2) If, B ∈ β, it is obvious from theorem (1) that for any $\alpha \in [0,1]$

$\{\omega / \mu_{B^c} \in [0,\alpha]\} = \{\omega / 1 - \mu_B(\omega) \in [0,\alpha]\}$

$= \{\omega / \mu_B(\omega) \in [1,\alpha,1]\} \in \Sigma$

namely $B^C \in \beta$

(3) If $B_i \in \beta$; i = 1, 2,........., then

for any $\alpha \in [0,1]$

$\{\omega / \mu_{\cup_{i=1}^{\infty} B_i}(\omega) \in [0,\alpha]\}$

$= \{\omega / Sup \{\mu_{B_i}(\omega) \in [0,\alpha]\}$

$= \cap_{i=1}^{\infty} \{\omega / \mu_{B_1}(\omega) \in [0,\alpha]\} \in \Sigma$

namely. $\cup_{i=1}^{\infty} Bi \in \beta$

Corollary 1:

1^0 $\emptyset \in \beta$

2^0 of $B_i \in \beta$; i = 1, 2, ... then

$\cap_{i=1}^{\infty} B_C \in \beta$

THEOREM 3:

Let (Ω, Σ, P) be a probability space [6], B ∈ F(Ω). The following statements are equivalent to each other.

1. B is fuzzy event on Ω.

2. μ_B is Σ - m.

3. $\forall\ \alpha \in [0,1]$; $B_\alpha \in \Sigma$; where B_α is a strong α - cut of B,

i.e. $B_\alpha = \{\omega \,/\, \mu_B\,(\omega) \geq \alpha\,\}$

4. $\forall\ \alpha \in [0,1]$; $B_{\alpha\omega} \in \Sigma$; where $B_{\alpha\omega}$ is the weak α - cut of B. i.e.

$B_{\alpha\omega} = \{\omega \,/\, \mu_B\,(\omega) > \alpha\,\}$

5. $\forall_c \in (-\infty, \infty)$, $\{\omega \,/\, \mu_B\,(\omega) > c\,\} \in \Sigma$

6. $\forall_c \in (-\infty, \infty)$, $\{\omega \,/\, \mu_B\,(\omega) < c\,\} \in \Sigma$

Proof :

Since Σ is a σ -algebra, then theorem is clear from definitions & set operations.

Theorem 4 -

If A;B ∈ β then

$A\oplus B \in \beta$ & $A \otimes B \in \beta$, where

$A\oplus B{:}_{A\oplus B}\,(\omega) = \min.(1, \mu_A(\omega) + \mu_B(\omega))$

$A \otimes B : \mu_{A \otimes B}\,(\omega)\ \ \max(0, \mu_A(\omega) + \mu_B(\omega) - 1)$

Proof :

First we assert that $\mu_A(\omega) + \mu_B(\omega)$ is Σ - m. In order to prove this assertion, it is sufficient to show that for any $c \in (-\infty, \infty)$, we have $\{\omega \,/ \mu_A(\omega) + \mu_B(\omega) > c\}$. Now let$\{\gamma'_n\}$ be sequence of all rational numbers. It is clear that $\{\omega \,/ \mu_A(\omega) + \mu_B(\omega) > c\} = \cup_{i=1}^{\infty}\ \{\{\omega \,/ \mu_A(\omega) > \gamma_i\} \cap \{\omega \,/ \mu_B(\omega) > c - \gamma_i\,\}$

Since $\forall_i = 1,2,\ldots;\ \{\omega / \mu_A(\omega) > \gamma_i\} \in \Sigma$;

$\{\omega / \mu_B(\omega) > c - \gamma_i\} \in \Sigma$ & Σ is a σ-algebra, we have

$\{\omega / \mu_A(\omega) + \mu_B(\omega) > c\} \in \Sigma$

Now,

$= \{\omega / \mu_{A \oplus B}(\omega) > c\}$

$= \{\omega / \mu_\Omega(\omega) > c\} \cap \{\omega / \mu_A(\omega) + \mu_B(\omega) > c\}$

& $\{\omega / \mu_{A \otimes B}(\omega) > c\} = \{\omega / \mu_\emptyset(\omega) > c\} \cup \{\omega / \mu_A(\omega) + \mu_B(\omega) > 1 + c\}$;

We have

$\{\omega / \mu_{A \oplus B}(\omega) > c\} \in \Sigma$ &

$\{\omega / \mu_{A \otimes B}(\omega) > c\} \in \Sigma$

namely

$A \oplus B \in \beta$ & $A \otimes B \in \beta$

<u>Theorem 5</u> -

If A, B $\in \beta$; then

AB $\in \beta$; & $\widehat{A + B} \in \beta$;

where A.B: $\mu_{A.B}(\omega) = \mu_A(\omega).\ \mu_B(\omega)$

A.B: $\mu_{\widehat{A+B}}(\omega) = \mu_A(\omega) + \mu_B(\omega) - \mu_A(\omega).\mu_B(\omega)$

Proof :

First we show that $\mu_A(\omega) - \mu_B(\omega)$ is a Σ - m. In fact let (γ_i) be the sequence of all rational numbers, it is clear that for any $c \in (-\infty, \infty)$, it $\{\omega / \mu_A(\omega) - \mu_B(\omega) > c\} = \bigcup_{i=1}^{\infty} \{\omega / \mu_A(\omega) > \gamma_i\} \cap \{\omega / \mu_B(\omega) < \gamma_i - c\}$

Since Σ is a σ-algebra, $\mu_A(\omega) - \mu_B(\omega)$ is a Σ - m.

Furthermore, we can prove that if $\mu_A(\omega)$ is Σ - m, then so is

$\mu_{AA}(\omega)$. In fact for any $\alpha \in [0,1]$, we have $\sqrt{\alpha} \in [0,1]$ & $(A.A)_\alpha = A_{\sqrt{\alpha}}$. It is clear from theorem (3) that $(A.A)_\alpha \in \Sigma$ namely is $\mu_{AA}(\omega)$ is Σ – m.

Now

$\mu_{AB}(\omega) = [\frac{1}{4}\{\mu_A(\omega) + \mu_B(\omega))^2 - \mu_A(\omega) - \mu_B(\omega)\}]^2$

& $(\mu_A(\omega) + \mu_B(\omega))^2 - (\mu_A(\omega) + \mu_B(\omega))^2$ is a Σ - m, we have that $\mu_{A.B}(\omega)$ is a Σ – m i.e. A.B. ∈ β. of Course μ_{A+B} is a Σ- m, too.

Definition 3 - If B is a fuzzy event in the probability space (Ω, Σ, P) then the mathematical expectation of μ_B i.e. E[μ_B] is called the fuzzy probability measure of B, denoted by P(B)

Theorem 6 - Fuzzy probability measure possesses the following properties:

1. $\forall B \in \beta; 0 \leq \tilde{P}(B) \leq 1$

2. $\tilde{P}(\Omega) = 1$

3. A, B ∈ β; $A \subset B \Longrightarrow \tilde{P}(A) \leq \tilde{P}(B)$

4. If there are such countable fuzzy events B_n, for any i ≠ j; $B_i \cap B_j = \Phi$

then $\tilde{P}(\bigcup_{n=1}^{\infty} B_n) = \sum_{n=1}^{\infty} \tilde{P}(B_n)$

Proof :

Properties 1,2,3 are obvious by definition. So we only prove property 4.

Under the given conditions

$\mu\bigcup_{n=1}^{M} B_n(\omega) = \sum_{n=1}^{M} \mu B_n(\omega)$;m = 1,2,......

$\mu\bigcup_{n=1}^{\infty} B_n(\omega) = \sum_{n=1}^{M} \mu B_n(\omega)$.

Then

$\tilde{P}(\bigcup_{n=1}^{\infty} B_n) = E[\mu\bigcup_{n=1}^{\infty} B_n]$

$E[\sum_{n=1}^{\infty} \mu B_n] = \sum_{n=1}^{\infty} E[\mu_{B_n}] = \sum_{n=1}^{\infty} \tilde{P}(B_n)$

Definition 4 -

The mapping Π : Ω → [0,1] is called a possibility distribution on Ω of ∀ω ∈ Ω,

$\Pi(\omega) \geq P(\omega)$

Definition 5 - If B ∈ β; then $\bigvee_{\omega \in \Omega}(\pi(\omega) \wedge \mu_B(\omega))$ is called possibility measure of fuzzy event B, denoted by Π(B).

Definition 6 - By a fuzzy probability and possibility space, we mean a

5-tuple (Ω, Σ, β, P, Π) in which (Ω, Σ, P) is a probability space, β is the family of all fuzzy events in tuple (Ω, Σ, P) & Π is a possibility distribution.

Remark If A, B ∈ β, A ⊂ B; then

$\Pi(A) \leq \Pi(B)$

3. P-independence & Π -non interaction between two fuzzy events:

Definition 7 -

Let A, B ∈ B, If

$\tilde{P}(A.B) = \tilde{P}(A).\ \tilde{P}(B)$

then A & B are said to be independent to each other under the probability distribution P, or simply P-independent.

Definition 8 - Let A, B ∈ β, If

$(\Pi(A \cap B) = (\Pi(A) \wedge \Pi(B)$

then A & B are said to non interaction to each other under possibility

distribution Π, or simply Π - non interaction.

Following are the immediate consequences:

Theorem 8 -

If $\mu_A(\omega)$ = constt. then ∀B ∈ β; A & B are P-independent

Theorem 9 - If A, B ∈ β, then the following statements are equivalent to

each other.

1. A & B are P-independent.

2.A & A^C are P-independent.

3. A^C & B^C are P-independent.

Definition 9 - Let Ω be a finite set. $\{\omega_1, \omega_{2,\ldots\ldots\ldots}\, \omega_n\}$. The probability distribution

P is said to be undegenerative,

if for any i ∈ {1,2,..........n}; $P(\omega_i) > 0$.

In order to understand the properties of P-independence & Π-non interaction and their relations, let us consider at present the simplest case in which Ω contains two or three elements. Let Ω be the set $\{\omega_1, \omega_2)$, Σ consists of all subsets of Ω. The probability P, possibility distribution Π, fuzzy sets A & B are as follows:

Ω	ω_1	ω_2
P	X	1 - X
Π	Π_1	Π_2
A	a_1	a_2
B	b_1	b_2

where $\mu_A(\omega_1)= a_i$; $\mu_B(\omega_i)= b_i$; $i = 1,2$; $0 < x < 1$. Without loss of generality we assume that $a_1 > a_2$.

Theorem -10: A & B are Π - non interaction,

Iff $\Pi_1 \wedge \Pi_2 \wedge a_1 \wedge b_1 \leq \Pi(A \cap B)$.

Proof : By definition 5, we have

$\Pi(A) \wedge \Pi(B) = [(\Pi_1 \wedge a_1) \vee (\Pi_2 \wedge a_2)]\ [(\Pi_1 \wedge b_1) \vee (\Pi_2 \wedge b_2)]$

$= (\Pi_1 \wedge \Pi_2 \wedge a_1 \wedge b_2) \vee$

$(\Pi_1 \wedge \Pi_2 \wedge a_2 \wedge b_1) \vee \Pi (A \cap B)$

since $\Pi_1 \wedge \Pi_2 \wedge a_2 \wedge b_1 \leq \Pi_1 \wedge a_1 \wedge b_1 \leq \Pi (A \cap B)$ the theorem is true.

Corollary 1 : If $a_1 = a_2$, then A & B are Π - non interaction.

In this case, we have

$\Pi_1 \wedge \Pi_2 \wedge a_1 \wedge b_2 \leq \Pi_2 \wedge a_2 \wedge b_2 \leq \Pi (A \cap B)$

Corollary 2 : If $b_1 \geq b_2$; then A & B are Π -non- interaction.

Proof : In this case, we have

$\Pi_1 \wedge \Pi_2 \wedge a_1 \wedge b_2 \leq \Pi_1 \wedge a_1 \wedge b_1 \leq \Pi (A \cap B)$

Corollary 3 : If $\Pi_1 \geq a_2$ then A & B are Π - non interaction.

Proof : In this case, we have

$\Pi_1 \wedge \Pi_2 \wedge a_1 \wedge b_2 = \Pi_1 \wedge \Pi_2 \wedge b_2 \leq \Pi (A \cap B)$

Corollary 4 : ____If $a_2 \vee b_1 < \Pi_1 \wedge \Pi_2 \wedge a_1 \wedge b_2$;

then A & B are not Π -non-interaction.

Proof : In this case it is clear that $\Pi_1 \wedge \Pi_2 \wedge a_1 \wedge b_2 > \Pi(A \cap B)$

Theorem 11 : If $a_1 = a_2$ or $b_1 = b_2$, then A & B are P-independent, where

P is any probability distribution on Ω.

Proof is the special case of theorem 8.

Theorem 12 : If $a_1 \neq a_2$ $b_1 \neq b_2$, then A & B are not P-independent where P is any undegenerative probability distribution on Ω.

Proof :

We have

$P(A)P(B)-P(A.B) = [xa_1 + (1-x)a_2][xb_1 + (1-x)b_2]-[xa_1b_1 + (1-x)a_2b_2]$

$= x(1-x)(a_1 - a_2)(b_1 - b_2)$

Under the conditions of $a_1 \neq a_2$ & $b_1 \neq b_2$; the function P(A) P (B) - P(A.B) has only two zero points, i.e. x=0, x=1. So, under an undegenerative probability distribution P; i.e. $o < x < 1$, we have $P(A.B) \neq P(A)P(B)$.

Proved.

Theorem13: If $a_1=a_2$ or $b_1=b_2$, then for any possibility distribution probability distribution P, A&B are Π -independent. If $a_1>a_2$; $b_1>b_2$ then A&B are non-interaction under any possibility distribution Π but are not independent under any undegenerative probability distribution P.

This is an immediate consequence of theorem 11, 12 & corollary 1,2.

Let Ω be the set $(\omega_1, \omega_2, \omega_3)$; Σ consists of all subsets in Ω. The probability P, possibility distribution Π, fuzzy sets A & B as follows:

Ω	ω_1	ω_2	ω_3
P	x	y	1 – x-y
Π	Π_1	Π_2	Π_3
A	a_1	a_2	a_3
B	b_1	b_2	b_2

where $x \geq 0$; $y \geq 0$, $x + y \leq 1$. Without loss of generality, we

assume that $a_1 \geq a_2 \geq a_3$.

Theorem 14 - A& B are Π -non interaction iff the following inequalities hold.

$\Pi_1 \wedge \Pi_2 \wedge a_1 \wedge b_2 \leq \Pi\ (A \cap B)$

$\Pi_1 \wedge \Pi_3 \wedge a_1 \wedge b_3 \leq \Pi\ (A \cap B)$

$\Pi_2 \wedge \Pi_3 \wedge a_2 \wedge b_3 \leq \Pi\ (A \cap B)$

Proof :

From definition (5), we have

$\Pi(A) \wedge \Pi(B) = [(\Pi_1 \wedge a_1) \vee (\Pi_2 \wedge a_2) \vee (\Pi_3 \wedge a_3)]$

$\wedge [(\Pi_1 \wedge b_1) \vee (\Pi_2 \wedge b_2) \vee (\Pi_3 \wedge b_3)]$

$= (\Pi_1 \wedge \Pi_2 \wedge a_1 \wedge b_2) \vee (\Pi_1 \wedge \Pi_3 \wedge a_1 \wedge b_3)$

$\vee (\Pi_2 \wedge \Pi_3 \wedge a_2 \wedge b_3) \vee (\Pi_3 \wedge \Pi_2 \wedge a_3 \wedge b_2) \vee \Pi(A \cap B)$

$= (\Pi_1 \wedge \Pi_2 \wedge a_1 \wedge b_2) \vee (\Pi_1 \wedge \Pi_2 \wedge a_1 \wedge b_3)$

$\vee (\Pi_2 \wedge \Pi_3 \wedge a_2 \wedge b_3) \vee \Pi(A \cap B)$

& thus the theorem is true.

Corollary 5 - If $b_1 \geq b_2 \geq b_3$, then A & B are Π -non interaction.

Proof :

In this case, we have

$\Pi_1 \wedge \Pi_2 \wedge a_1 \wedge b_2 \leq \Pi \wedge a_1 \wedge b_1 \leq \Pi(A \cap B)$

$\Pi_1 \wedge \Pi_3 \wedge a_1 \wedge b_3 \leq \Pi_1 \wedge a_1 \wedge b_1 \leq \Pi(A \cap B)$

& $\Pi_2 \wedge \Pi_3 \wedge a_2 \wedge b_3 \leq \Pi_2 \wedge a_2 \wedge b_2 \leq \Pi(A \cap B)$. Proved.

Corollary 6 -

If $b_1 \geq \max \{b_2, b_3\}$,

$\Pi_1 \geq \max. \{ \Pi_2, \Pi_3\}$

then A & B are Π - non interaction.

Proof :

In this case we have

$\Pi_1 \wedge \Pi_2 \wedge a_1 \wedge b_2 \leq \Pi_1 \wedge a_1 \wedge b_1 \leq \Pi(A \cap B)$

& $\Pi_1 \wedge \Pi_3 \wedge a_1 \wedge b_3 \leq \Pi_1 \wedge a_1 \wedge b_1 \leq \Pi(A \cap B)$

$\Pi_2 \wedge \Pi_3 \wedge a_2 \wedge b_3 \leq \Pi_1 \wedge a_1 \wedge b_1 \leq \Pi(A \cap B)$. Proved.

Corollary 7 -

If $b_i \geq a_i$; $\forall i = 1,2,3$; then A & B are Π -non interaction.

Proof : In this case $B \supseteq A$; $\Pi(B) \geq \Pi(A)$

Then $\Pi(A \cap B) = \Pi(A) = \Pi(A) \wedge \Pi(B)$. Proved.

Definition 10 - By the judge function of P - independence for A & B we

mean the function Δ (x,y);

$\Delta\ (x,y) = x(x-1)(a_1 - a_3)(b_1 - b_3) +$

$y\ (y-1)\ (a_2 - a_3)(b_2 - b_3)\ +$

$xy\ [(a_1 - a_3)(b_2 - b_3) + (a_2 - a_3)(b_1 - b_3)]$

Theorem 15 - The fuzzy events A & B are P-independent iff Δ (x,y) = 0.

Proof : Theorem follows from the fact that

$\tilde{P}(A)\ \tilde{P}(B) - \tilde{P}(A.B) \equiv \Delta\ (x,y)$.

Proved.

Theorem 16 - If $a_1 = a_3$, then A & B are independent to each other under

any probability distribution.

This theorem is the special case of theorem (8).

Theorem 17 - Let $a_1 = a_2 > a_3$. Then iff $b_1 \wedge b_2 < b_3 < b_1 \vee b_2$;

A & B can be independent to each other under some undegenerative probability distribution P. In this case, the probability distribution P(x, y, 1-x-y) satisfies the following linear equation

$x\,(b_1 - b_3) + y(b_2 - b_3) = 0$.

Proof :

From the given condition, we have

$\Delta(x,y) = (a_1 - a_3)[x(x-1)(b_1 - b_3) + y(y-1)(b_2 - b_3) + xy(b_1 + b_2 - 2b_3)]$

$= (a_1 - a_3)(x + y - 1)[x(b_1 - b_3) + y(b_2 - b_3)]$

Since P is undegenerative, the sufficient & necessary condition for $\Delta(x,y) = 0$ is that $x\,(b_1 - b_3) + y\,(b_2 - b_3) = 0$.

This equation has positive roots iff $(b_1 - b_3)(b_2 - b_3) < 0$

or, in other terms, $b_1 \wedge b_2 < b_3 < b_1 \vee b_2$

Proved.

<u>Theorem 18</u> - Let $a_1 > a_2 = a_3$. Then iff $b_2 \wedge b_3 < b_1 < b_2 \vee b_3$; A & B can be independent to each other under some undegenerative probability distribution P & in this case, the probability distribution P (x, y, 1-x-y) satisfies the following linear equation $(1 - x - y)(b_3 - b_1) + y(b_2 - b_1) = 0$.

Proof : By theorem (9), A & B are P-independent iff A^C & B^C are

P-independent. Since $1-a_3 = 1-a_2 > 1-a_1$; it is obvious from theorem

(17), that A^C & B^C are P-independent iff

$(1 - b_2) \wedge (1 - b_3) < (1 - b_1) < (1 - b_2) \vee (1 - b_3)$

& $(1-x-y)\,[(1-b_3)-(1-b_1)] + y\,[(1-b_2)-(1-b_1)] = 0$

or, in other words,

$b_2 \wedge b_3 < b_1 < b_2 \vee b_3$

i.e., $(1-x-y)(b_3 - b_1) + y\,(b_2 - b_1) = 0$

Proved.

<u>Theorem 19</u> - Let $a_1 > a_2 > a_3$ & $\prod_{1 \le i \le j \le 3} (b_i - b_j) \neq 0$. Then iff either $b_2 > b_1 \vee b_3$ or $b_2 < b_1 \wedge b_3$; A & B can be independent to each other under some undegenerative probability distribution P (x, y, 1-x-y).

Proof : Using the point (x, y) to represent the probability distribution P (x, y, 1-x- y); we see that the undegenerative probability distribution P is nothing but a point (x, y) located inside the triangle T (figure - 1). All the vertices of T are zero points of $\Delta(x,y)$. On each side of T; the changing values of Δ (x,y) remain certainly positive or negative. In fact.

ι_1 : $\Delta(x,y) = y(y - 1)(a_2 - a_3)(b_2 - b_3)$

$\text{sgn}(\Delta(x,y)) = -\text{sgn}(b_2 - b_3)$

ι_2 : $\Delta(x,y) = x(x-1)(a_1 - a_3)(b_1 - b_3)$

$\text{sgn}(\Delta(x,y)) = -\text{sgn}(b_1 - b_3)$

ι_3 : $\Delta(x,y) = x(x-1)(a_1 - a_2)(b_1 - b_2)$

$\text{sgn}(\Delta(x,y)) = -\text{sgn}(b_1 - b_2)$

Under the given condition, the equation $\Delta(x,y) = 0$ is obviously hyperbolic type. It has real roots inside T iff the function sgn(Δ(x.y)) takes different values on ι_1, ι_2 & ι_3; i.e.

$$\left|\sum_{1\le i\le j\le 3} \text{sgn}(bi - bj)\right| = 1$$

or, in other words $b_2 \neq b_1 \vee b_3$.

Proved

The locus of the point (x, y) is a segment of hyperbola. Attach here (fig. 1).

Theorem 20 - If $\forall$ i≠j; $(a_i - a_j)(b_i - b_j) > 0$; then A & B non interaction under any possibility distribution but are not independent under any undegenerative probability distribution P.

Proof : Theorem is a combination of corollary (5) & theorem (19).
It shows that the relation between P - independence & Π - non interaction is indeed very weak.

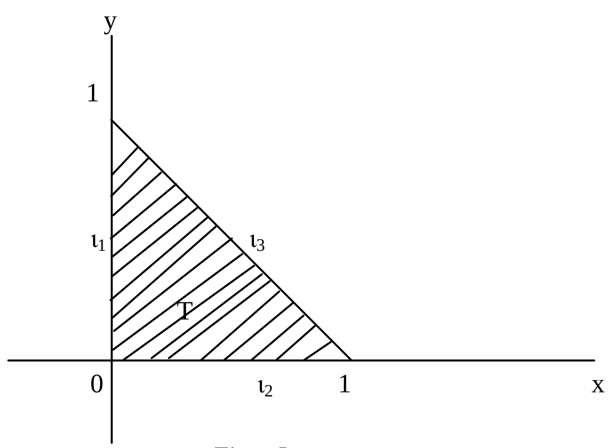

Fig. - I

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