Creating the Tessellations of the Plane: An Interplay of Mathematics and Art

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Abstract
This paper investigates the 17 types of symmetry that can be employed to create wallpaper patterns. Such patterns cover the entire floor or wall (a two-dimensional plane) using the repetition of a geometric shape or small tiles with various shapes fitted together without gaps or overlaps. The resulting pattern or tiling is called a tessellation of the plane and occurs frequently in architecture and decorative art. In this paper, the author develops her procedure for tessellating a plane using different transformations such as translations, rotations, reflections, glide reflections, or any combination. With these, students can be taught a technique necessary to make their designs and drawings and thereby develop or renew their appreciation of art and mathematics, especially in algebra and geometry.
Keywords: Symmetry group, transformation, wallpaper patterns, tilling, tessellation.

1. Introduction

In modern colleges and universities, mathematics is often regarded as a science rather than an art. Various entrance exams, achievement tests, and aptitude tests indicate that students generally do not perform well in mathematics. One of the main reasons is that students often fail to appreciate mathematics before learning it, resulting in many admitting to having a "math phobia". Nonetheless, math teachers can enhance students' performance in mathematics by utilising a diverse range of teaching strategies that not only minimise math phobia. However, they may also foster a greater interest in mathematics.

Having taught Abstract Algebra and Geometry for some time, the researcher has been inspired to associate mathematics with art to foster a deeper appreciation for mathematics and make it more engaging by connecting it to artistic designs such as wallpapers, rugs, bathroom tiles, and wall carpets. By utilising the study of symmetry in group theory, the researcher has devised her techniques to design periodic patterns belonging to one of the 17 symmetry groups or wallpaper groups.

2. Objectives

The purpose of this research paper is to assist students in understanding and implementing the techniques for creating a fundamental region of each of the 17 different plane symmetry groups that can tessellate a plane. The ultimate goal is to help students cultivate their skills in creating their tessellation art. To accomplish this, the author devised an applicable procedure by examining the various lattice units with symmetries of periodic plane patterns to create these tessellations. Eventually, this will help students create their tessellations and foster an appreciation for art and mathematics, specifically algebra and geometry.

Specifically, the study has two main goals:

- To introduce the reader to various regular periodic tiling of the plane and their characteristics, such as symmetry, translation lattices, transformation; and
- To produce a tessellation of the plane using the researcher's developed procedure.

3. Limitations of the Study

Various combinations of transformations can be used to create tessellations, and there are precisely 17 symmetry groups, or wallpaper groups, that can be used to tile a flat surface. These groups are a mathematical classification that depends on the symmetries found in the two-dimensional motif. The number 17 was first established by mathematicians Evgraf Fedorov in 1891 and George Pólya in 1924. Although other proofs explain why only 17 symmetry groups exist, this paper does not discuss them.

4. Theoretical Framework

This study is built upon theoretical considerations and definitions, explained below, before delving into the more complex topic of tessellation and symmetry. In mathematics, symmetry refers to a shape or object being identical to another when rotated, flipped, or moved. A form or object is said to be symmetrical if it can be split into two identical parts. The two halves of symmetrical shapes are mirror images of one another. The line of symmetry is the path that can be taken to fold a figure into symmetrical halves.

In other words, "symmetry can be viewed when an object is flipped, turned, or slid. Four types of symmetry can be observed in various cases.

- Translational symmetry
- Rotational symmetry
- Reflexive symmetry
- Glide symmetry"

4.1. Translation Symmetry

Translational symmetry refers to the property of an object that remains unchanged in orientation when moved from one position to another. This symmetry means the object only moves or slides without rotating, resizing, or undergoing any other change. The movement is consistent in distance and direction, so all object points move similarly. In other words, a translation is simply a displacement or change in the object's location.

4.2. Rotational Symmetry

Rotational symmetry is when an object is rotated around a point in a specific direction, also known as radial symmetry. The object looks identical to its original form after being turned. "The smallest angle at which the shape coincides with itself after rotation is the angle of rotational symmetry, and the order of symmetry is how the object coincides with itself when it is in rotation. Rotation refers to spinning a shape around a centre point, where the distance from the centre to any point on the shape remains constant. Each point makes a circle around the centre point". The **centre of rotation** can be either on or outside the shape.

4.3. Reflexive Symmetry

Reflective symmetry, or mirror symmetry, occurs when one half of an object is the mirror image of the other half. A reflection is essentially flipping a shape over a line called a line of reflection, with common examples being over the y-axis or x-axis. A correctly done reflection will result in the two parts being mirror images of each other. Each point in the original shape corresponds to a corresponding point in the reflected shape that is the same distance from the line of reflection. This property allows the shape to be folded along the line of reflection so that the two halves match exactly. Human faces, for example, typically exhibit reflective symmetry.

4.4. Glide Symmetry

Glide symmetry involves the use of both translation and reflection transformations. It is characterised by a commutative property, which means that the order in which these transformations are applied does not affect the final result of the glide reflection.

Some important mathematical properties of the symmetry groups should be noted, such as that a translation does not have any fixed points, a rotation has a unique fixed point, and a reflection has a fixed line. A translation has an infinite order, while a reflection has an order of 2 and is its inverse. The inverse of a rotation r around a point P by an angle θ is a rotation around the origin by -θ. The rotational symmetry of a wallpaper pattern must be of order 2, 3, 4, or 6. In addition to these properties, being familiar with the different lattice types in a wallpaper pattern is crucial, as they serve as a guide to recognising the design and creating one's wallpaper pattern. These lattices may be classified into five (5) different kinds, namely:

1) "**Parallelogram**: has translations and half-turns, but there are neither reflections nor glide reflections".
2) "**Rectangular:** Primitive (simple)has translations, half-turns, and reflections".
3) "**Rhombic:** has translations, half-turns, reflections, and glide reflections".
4) "**Square:** has translations, rotations of 90° and 180°, reflections, and glide reflections. (Three groups: p4; p4m; p4g)"
5) "**Hexagonal**: has translations, rotations of 60°, 120°, and 180°, reflections, and glide reflections (five groups: p3; p3ml; p3lm; p6; p6m)"

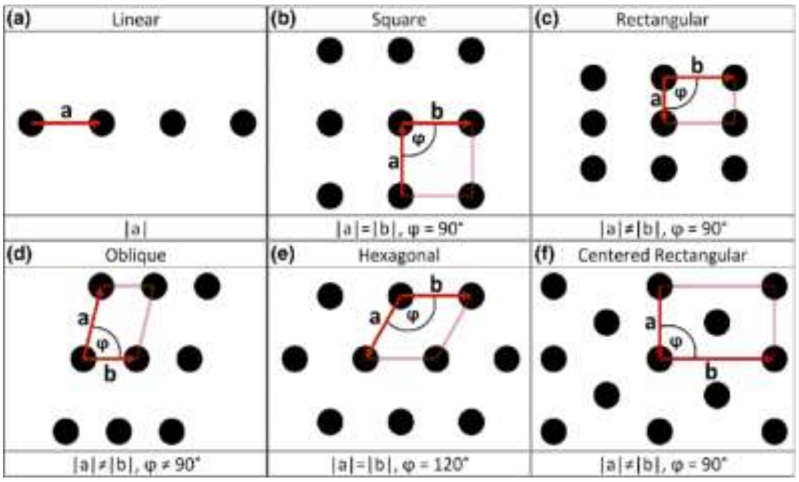


Figure 1. The FIVE lattices

The following symbols are also used to help identify the lattice units with symmetries for each of the plane's 17 types of regular tessellations. We have the diamond, triangle square, and hexagon for the two-fold, 3- fold, 4-fold, and six-fold rotation, respectively. The number **n** represents the type of rotation—a line as an axis of reflection and a perforated line as an axis of glide reflection. The plane symmetry groups are denoted by the letters ***p*** or ***c***, indicating that the lattice is either primitive or centred. ***m*** or ***mm*** indicates one or two mirror lines, while the letters ***g*** and ***gg*** stipulate one or two glide reflection lines.

Symbols used to Denote Symmetry in Lattices:

n = ROTATIONAL AXIS OF ORDER "n"
m = MIRROR LINES OR PLANES
p = PRIMITIVE LATTICE
c = CENTERED LATTICE

It can be challenging to recognise every symmetry in a wallpaper pattern. Instead, then attempting to uncover every symmetry, it is feasible to identify the pattern's symmetry group by concentrating on crucial characteristics. The concept of utilising a flow chart to determine the symmetry group was first proposed by Dorothy Washburn and Donald Crowe for use in cultural anthropology. Here is a representation of their flow chart, with the rotation order query on the left and arrows leading to the symmetry group determination on the right.

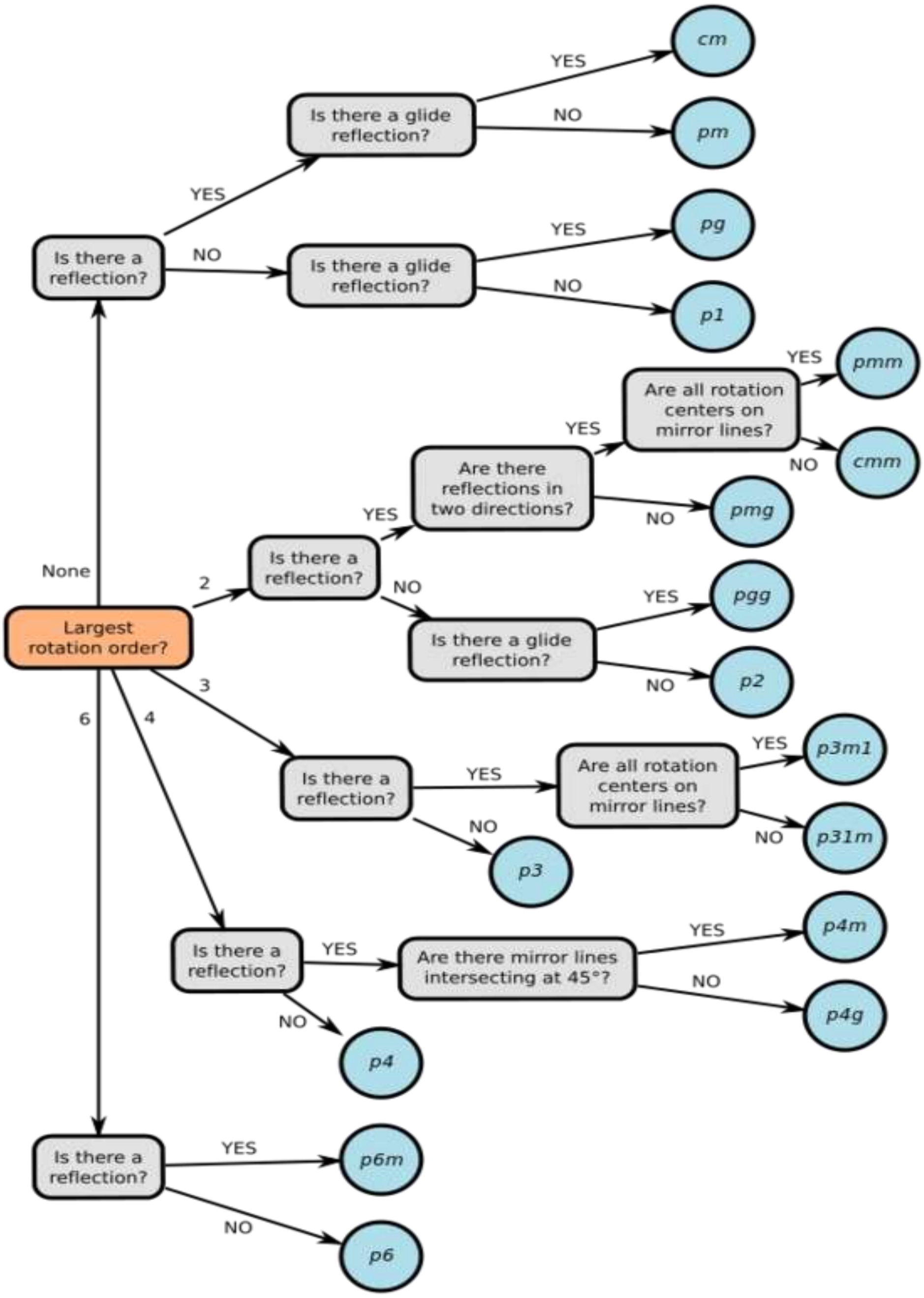


Figure 2. The Flow Chart for Identifying the Symmetry Group

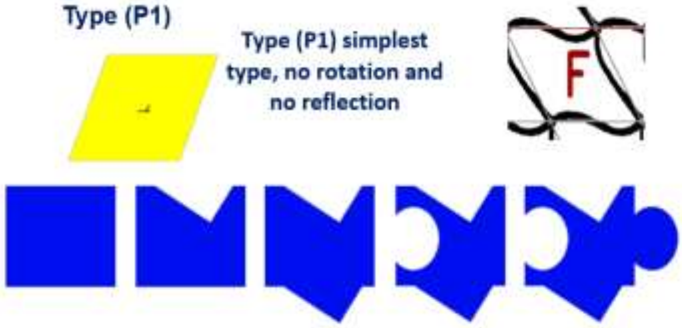
5. Methodology

The procedures used by the researcher to create the 17 types of plane tessellations are found below. When repeated through successive translations, these procedures result in the replication of a periodic pattern.

5.1. Methods and Procedures

1. Type (p1)

The essential symmetry group is (p1), which comprises only translations. Rotations, reflections, and glide reflections are absent. The two translation axes can have any orientation in relation to each other. This group's lattice is parallelogram-shaped, making a parallelogram its fundamental region and analogous to the translation groups.

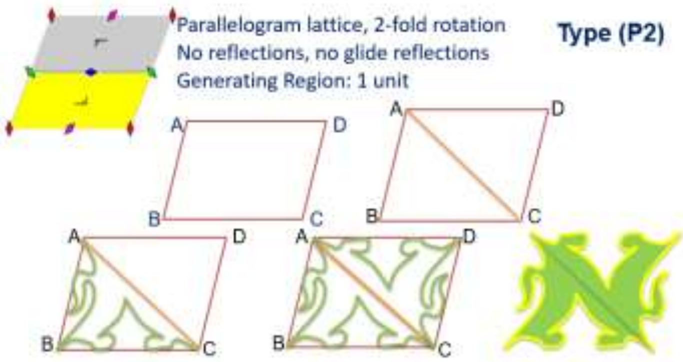


Procedure:

1. Let A, B, C, & D be lattice points of a primitive cell of a parallelogram lattice.
2. Draw a continuous curve,
   S_a, from A to B. Draw a continuous curve, S_b, from B to C, distinct from S_a. For a design, draw a figure inside the primitive cell of a parallelogram lattice = a generating region of p1.
   S_a **U** S_b **= generator of P1. (The figure can be the generator)**
3. Let α and β be linearly independent translation vectors of the parallelogram lattice, then $T_\beta\ (S_a) =\ S'_a$ & $T_\alpha\ (S_b) = S'_b$
   S_a **U** S_b **U** S'_a **U** S'_b **= U.P.P. (p1)** = *a unit of the periodic pattern* of the translation group of p1.
4. $T_{n\alpha}$ (U.P.P.), $T_{n\beta}$ (U.P.P.) all over the plane to complete the tessellation. (α, β are linearly independent translation vectors of the lattice while n ∈ Z).

2. Type (p2)

The second type, also known as P2, of the symmetry group includes a two-fold rotation but no reflections or glide reflections. To create a tessellation of this type, a continuous curve is drawn between four lattice points (A, B, C, and D) of a primitive cell of a parallelogram. Then the figure is rotated about the midpoint of A.C. and B.D., called Point X. Finally. The unit is translated across the plane to complete the tessellation. This symmetry group differs from the first group by containing 180° rotations (half-turns). "As with all symmetry groups, translations are present, but reflections and glide reflections are not. The lattice of this group is parallelogrammatic and can have two translation axes at any angle to each other. The fundamental region for this symmetry group is half of a parallelogram that serves as a fundamental region for the translation group".



Procedure:

1. Let A, B, C, & D be lattice points of a primitive cell of a parallelogram
2. Let X = midpoint of AB, Y = midpoint of BC, Z = midpoint of AC.
3. Draw a continuous curve, S_a, from A to B.

   Draw a continuous curve, S_a, from B to C, distinct from S_a such that $S_{a.}$

   Furthermore, Sb is inside the triangle A.B.C. = a generating region of p2 = ½ ABCD.

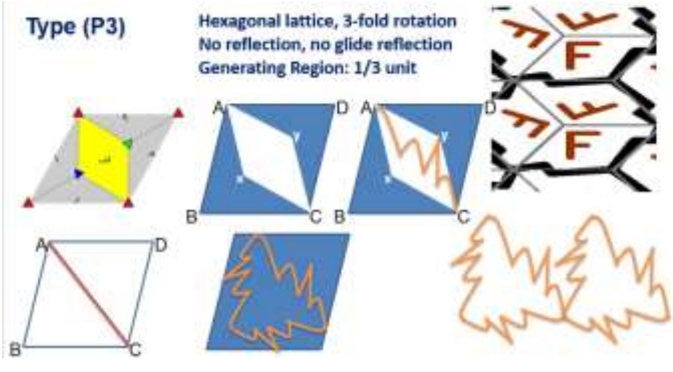
   S_a **U** S_b **= generator of p2.**
4. $R_{(X,180°)}$ $(S_a) = S'_a$ & $R_{(Y,180°)} = S'_b$

   S_a **U** S_b **U** S'_a **U** S'_b **= U.P.P. (p2) =** *a unit of the periodic pattern* of the translation group of p2.
5. $T_{n\alpha}$ (U.P.P.), $T_{n\beta}$ (U.P.P.) all over the plane to complete the tessellation.

   (α, β are linearly independent translation vectors of the lattice while n ∈ Z).

3. Type (p3)

"The third type of symmetry group, P3, is the first group that includes a 120° rotation, also known as a rotation of order 3, with a hexagonal lattice. This group does not have any reflections or glide reflections". The lattice is made up of equilateral triangles that divide the basic hexagon. The centres of the triangles and their midpoints serve as the centres of rotation. A fundamental region for this group is one-third of one of these triangles.



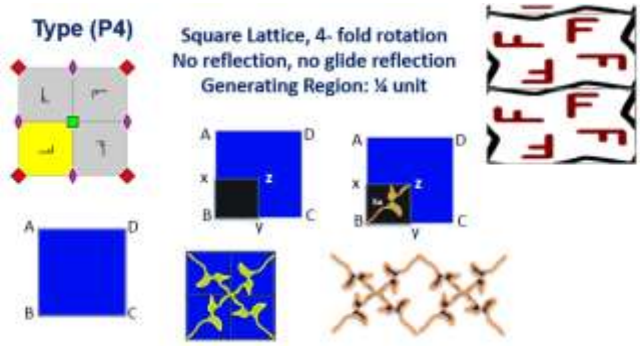
Procedure:

1. Let A, B, C, & D be lattice points of a primitive cell of a hexagonal lattice such that A.C. divides ABCD into two equilateral triangles, A.B.C. and A.D.C.
2. Let X = midpoint of triangle ABC, Y = midpoint of triangle ADC.
3. Draw a continuous curve, S, from A to C or from X to Y, or any of the two points in AXCY, such that S is inside AXCY = **a generating region of p3 = 1/3 ABCD = generator of p3**.

4. $R_{(X,120°)}$ (S) = S’ & $R_{(Y,240°)}$ (S) = S’’
   S U S' U S'' = U.P.P. (p3) = *a unit of the periodic pattern* of the translation group of p3.
5. $T_{n\alpha}$ (U.P.P.), $T_{n\beta}$ (U.P.P.) all over the plane to complete the tessellation.
   (α, β are linearly independent translation vectors of the lattice while n ∈ Z).

4. Type (p4)

The first symmetry group of type P4 has rotations of order 4, a rotation of 90 degrees, and rotations of order 2. A distance of half is between the centres of the order-4 rotations and the centres of the order-2 rotations. This group does not reflect anything. The lattice is square, and this symmetry group's fundamental region can correspond to one-fourth of the fundamental region for the translation group.

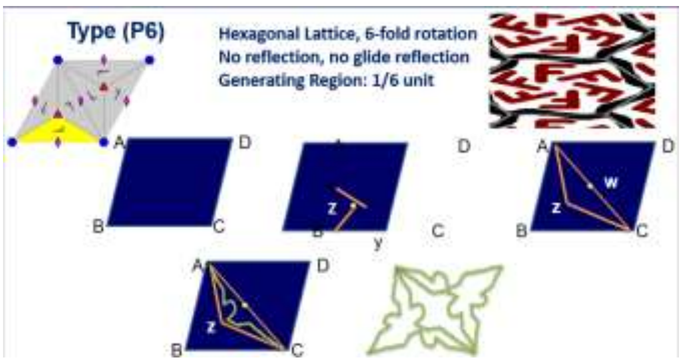


Procedure:

1. Let A, B, C & D be lattice points of a primitive cell of a square lattice.
2. Let X = midpoint of AB, Y = midpoint of BC, Z = midpoint of Ac = midpoint of BD.
3. Draw a continuous curve, S_a, from B to Z, or from X to Y, or any of the two points in BXZY, such that S_a is inside BXZY = a generating region of p4 = ¼ ABCD.
   S_a **= generator of p4.**
4. R (Z, 90°) = S_b, R (Z, 180°) (S_a) = S_c R (Z, 270°) (S_a) = S_d
   S_a **U** S_b **U** S_c **U** S_d **= U.P.P. (p4)** = *a unit of the periodic pattern* of the translation group of p4.
5. $T_{n\alpha}$ (U.P.P.), $T_{n\beta}$ (U.P.P.) all over the plane to complete the tessellation.
   (α, β are linearly independent translation vectors of the lattice while n ∈ Z).

5. Type (p6)

The presence of rotations of order six and orders 2 and 3 distinguishes the sixth type, designated P6, from the other five types. However, there are no reflections or glide reflections. A fundamental region for the symmetry group can be found by taking a sixth of an equilateral triangle from the hexagonal lattice of this group.

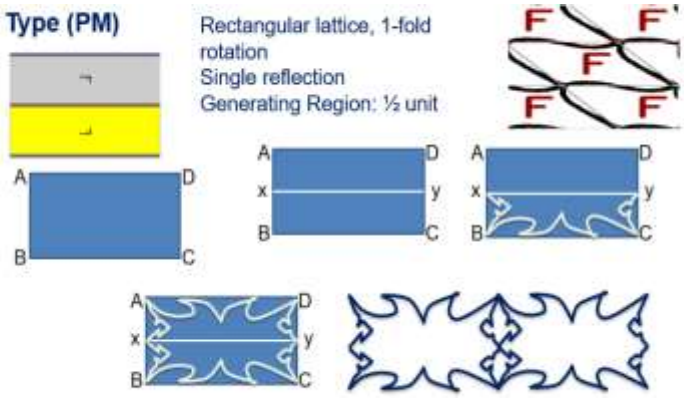


Procedure:

1. Let A, B, C, & D be lattice points of a primitive cell of a hexagonal lattice.
2. Let X = midpoint of AB, Y = midpoint of BC, Z = midpoint of AY = midpoint of XC, W = AC ∩ BD.
3. Draw a continuous curve, S_a, from X to Y or any of the two points in the BXZY such that S_a is inside BXZY = **a generating region of p6** = 1/6 ABCD
   [Option 2: Draw S_a from A to C, or any of the two points on triangle A.Z.C., such that S_a is inside the triangle A.Z.C. = a generating region of p6 = 1/6 ABCD] S_a = generator of p6.
4. $R_{(Z,60°)} = S_b$, R (Z, 120°) $(S_a) = S_c$, $R_{(Z,180°)}$ $(S_a) = S_d$, $R_{(Z,240°)}$ $(S_a) = S_e$ $R_{(Z,300°)}$ $(S_a) = S_f$
   [Option 2: $R_{(Z,120°)}$ $(S_a) = S_b$, $R_{(Z,240°)}$ $(S_a) = S_c$, $R_{(Z,180°)}$ $(S_a$ U S_b U $S_c) =$ $(S_a$ U S_b U $S_c)$']
   S_a **U** S_b **U** S_c **U** S_d **U** S_e **U** S_f **= U.P.P. (p6)** = *a unit of the periodic pattern* of the translation group of p6.
   [Option 2: $(S_a$ **U** S_b **U** $S_c)$ **U** $(S_a$ **U** S_b **U** $S_c)'$ **=** U.P.P. (p6) = a unit of the periodic pattern of the translation group of p6.
5. $T_{n\alpha}$ (U.P.P.), $T_{n\beta}$ (U.P.P.) all over the plane to complete the tessellation.
   (α, β are linearly independent translation vectors of the lattice while n ∈ Z).

6. Type (pm)

The first category of reflections, Type (pm), lacks rotational or gliding reflections. Concerning one translation axis, the axes of reflection are perpendicular while parallel to the other. A fundamental region for the translation group is a rectangle, and the lattice is rectangular. By dividing a fundamental region for the translation group in half along an axis of reflection, one can obtain a fundamental region for the symmetry group.



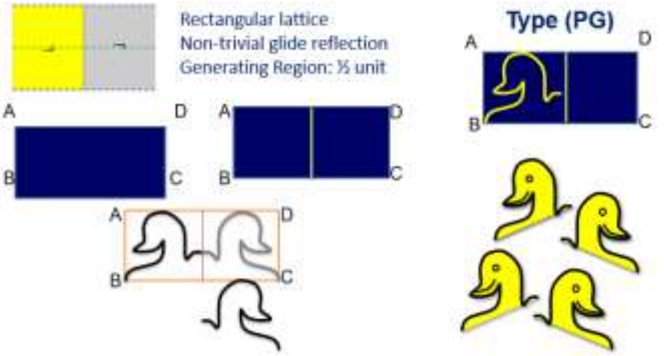
Procedure:

1. Let A, B, C & D be latticing points of a primitive cell of a rectangular lattice.
2. Let X = midpoint of AB, Y = midpoint of Dc.
3. Draw a continuous curve, **S**, from B to C, or any of the two points in BXYC, such that S is inside BXYC = **a generating region of pm** = ½ ABCD.
   S = generator of pm.
4. M_{xy} (S) = S'. **S U S' = U.P.P. (pm)** = a unit of the periodic pattern of the translation group of pm.

5. $T_{n\alpha}$ (U.P.P.), $T_{n\beta}$ (U.P.P.) all over the plane to complete the tessellation.
   (α, β are linearly independent translation vectors of the lattice while n ∈ Z).

7. TYPE pg

The first symmetry group, type (pg), has glide reflections but neither rotations nor reflections. One of the translation axes and the glide reflection are perpendicular. Since the lattice of this group is rectangular, a rectangle can be selected as the basic region for the translation group. When an axis of glide reflection splits one of these fundamental regions' half rectangles, the other half rectangle becomes a fundamental region for the symmetry group.

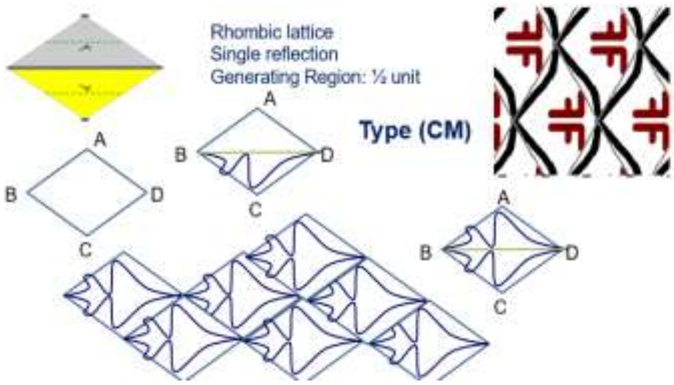


Procedure:

1. Let A, B, C, & D be latticing points of a primitive cell of a rectangular lattice.
2. Let X = midpoint of AB, Y = midpoint of DC, Z = midpoint of AD, Z' = midpoint of XY.
3. Draw a continuous curve, S_a, from A to X. Draw a continuous curve, S_b, from A to Y such that S_a and S_b are inside AXYD = **a generating region of pg** = ½ ABCD. S_a **U** S_b **= generator of pg**
4. $(S_a\ U\ S_b)\cdot T_{DY}(S_a \text{ U } S_b) = S'_a \text{ U } S'_b$
   S_a **U** S_b **U** S'_a **U** S'_b **= U.P.P. (pg)** = a unit of the periodic pattern of the translation group of pg.
5. $T_{n\propto}(U.P.P.), T_{n\beta}(U.P.P.)$ all over the plane to complete the tessellation. (α, β are linearly independent translation vectors of the lattice while n ∈ Z).

8. TYPE cm

Reflections and gliding reflections with parallel axes can be found in the symmetry group Type (cm). It does not have any rotations like the preceding groups did. Any angle between the translations in this group is allowed, but the axes of the reflections cut across the angle created by the translations. The translation group's primary region is a rhombus as a result. The half of the rhombus is a fundamental region for the symmetry group.



Procedure:

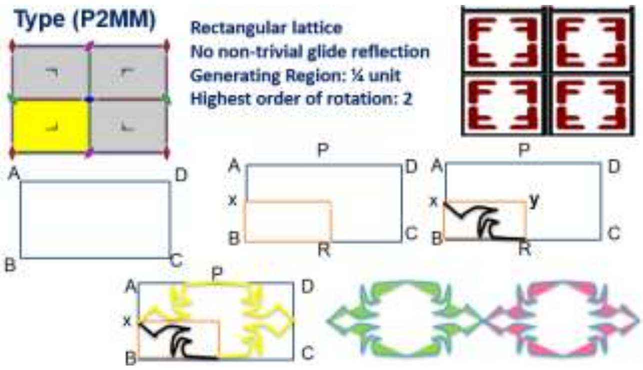
1. Let A, B, C, & D be latticing points of a primitive cell of a rhombic lattice such that B.D. divides ABCD into two congruent triangles.
2. Draw a continuous curve, **S**, inside the triangle B.C.D. = **a generating region of cm** = ½ ABCD. **S = generator of cm**
3. $M_{BD}(S) = S'$

   S ∪ S' = U.P.P. (cm) = a unit of the periodic pattern of the translation group of cm.
4. $T_{n\propto}(U.P.P.), T_{n\beta}(U.P.P.)$ all over the plane to complete the tessellation. (α, β are linearly independent translation vectors of the lattice while n ∈ Z).

9. TYPE pmm

(p2mm)

Type(pmm) Perpendicular axes of reflection are present in this symmetry group. No rotations or reflections of the glide exist. Since the lattice is rectangular, it is possible to select a rectangle for the translation group's fundamental region and a quarter rectangle of it for the symmetry group's fundamental region.



Procedure:

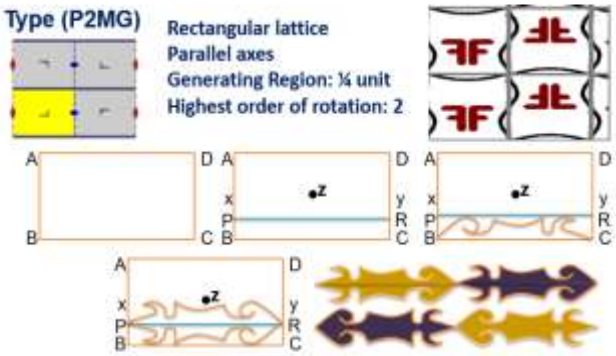
1. Let A, B, C, & D be latticing points of a primitive cell of a rectangular lattice.
2. Let X = midpoint of AB, Y = midpoint of DC, P = midpoint of AD, R = midpoint of BC, Z = midpoint of XY = midpoint of PR.
3. Draw a continuous curve, S_a, from X to R or from B to Z, or from any of the two points of XZRB such that S_a is inside XZRB = **a generating region of pmm** = ¼ ABCD. S_a **= generator of pmm**
4. $M_{XZ}(S_a) = S_b$, $M_{ZR}(S_a) = S_c$, $M_{ZY}(S_c) = S_d$

   S_a **U** S_b **U** S_c **U** S_d **= U.P.P. (pmm)** = a unit of the periodic pattern of the translation group of pmm.
5. $T_{n\propto}(U.P.P.), T_{n\beta}(U.P.P.)$ all over the plane to complete the tessellation. (α, β are linearly independent translation vectors of the lattice while n ∈ Z).

10. TYPE pmg

(p2mg)

A symmetry group of type (pmg) has a rotation of order two as well as a reflection, but the rotational centres do not coincide with the axes of reflection. This group has a rectangular lattice, and one can obtain a fundamental region for this group by taking a quarter-rectangle of a fundamental region for the translation group.



Procedure:

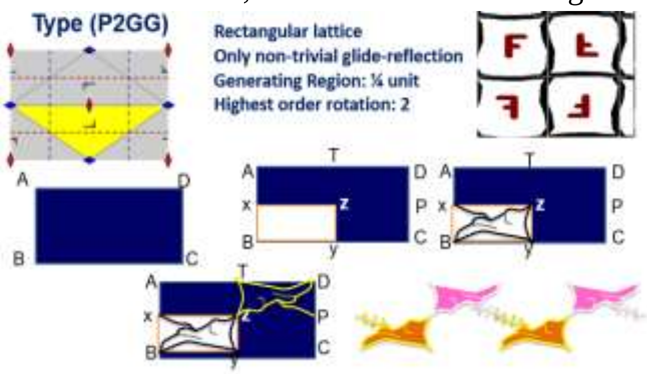
1. Let A, B, C, & D be lattice points of a primitive cell of a rectangular lattice.
2. Let X = midpoint of AB, Y = midpoint of DC, Z = midpoint of XY, P = midpoint of XB, R = midpoint of YC.
3. Draw a continuous curve, S_a, from B to C or from any of the two points of PRBC, such that S_a is inside PRBC = **a generating region of pmg** = ¼ ABCD. S_a = **generator of pmg**
4. $M_{PR}(S_a) = S_b$, $R_{(Z,180^\circ)}(S_a) = S_d$, $R_{(Z,180^\circ)}(S_b) = S_{c.}$

   S_a **U** S_b **U** S_c **U** S_d = **U.P.P. (pmg)** = a unit of the periodic pattern of the translation group of pmg.
5. $T_{n\propto}(U.P.P.), T_{n\beta}(U.P.P.)$ all over the plane to complete the tessellation. (α, β are linearly independent translation vectors of the lattice while n ∈ Z).

11. TYPE pgg

(p2gg)

There are no reflections in type (pgg), although glide reflections and half-turns exist. The centres of rotation are not on the glide reflections' perpendicular axes. A quarter rectangle of a fundamental area for the translation group can be employed as a fundamental region for the symmetry group in this form of a lattice, which is likewise rectangular.



Procedure:

1. Let A, B, C, & D be latticing points of a primitive cell of a rectangular lattice.
2. Let X = midpoint of AB, Y = midpoint of BC, T = midpoint of AD, Z = midpoint of TY.
3. Draw a continuous curve, S_a, from X to Y or any of the two points in XZYB such that S_a is inside

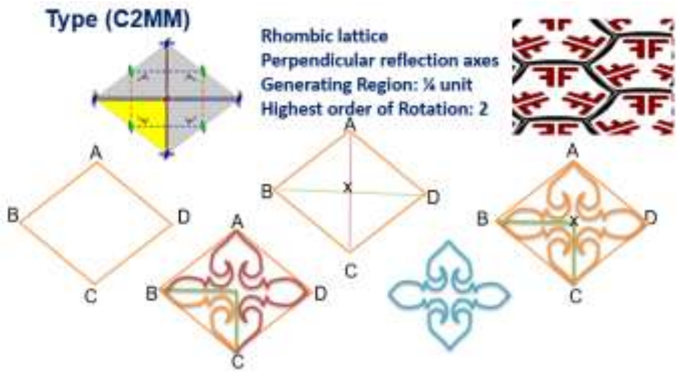
   XZYB = **a generating region of pgg** = ¼ ABCD.

   S_a = **generator of pgg**

4. $M_{PR}(S_a) \cdot T_{1/2\propto}(S_a) = S_b,\ M_{ST}(S_a) \cdot T_{1/2\propto}(S_a) =$
$S_c,\ \ R_{(Z,180^\circ)}(S_a) = S_d$.
   S_a **U** S_b **U** S_c **U** S_d **= U.P.P. (pgg)** = a unit of the periodic pattern of the translation group of pgg.
5. $T_{n\propto}(U.P.P.), T_{n\beta}(U.P.P.)$ all over the plane to complete the tessellation. (α, β are linearly independent translation vectors of the lattice while n ∈ Z).

12. TYPE cmm
(c2mm)

Although type (cmm) and group 6 (pmm) both have perpendicular axes of reflection, type (cmm) additionally possesses rotations of order 2. These half-turn centres are situated where the reflection and glide reflection axes intersect. This group's lattice is rhombic, and the symmetry group's fundamental region shares one-fourth of the fundamental region for the translation group.

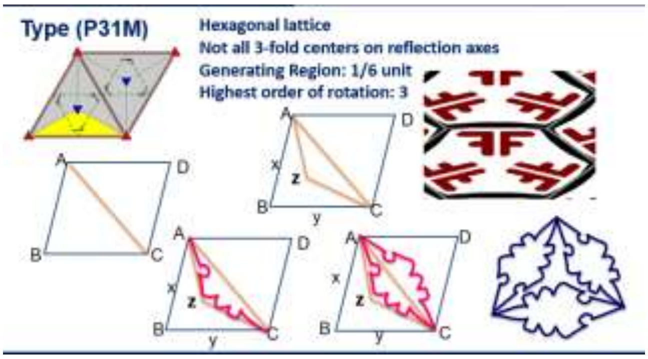


Procedure:

1. Let A, B, C, & D be lattice points of a primitive cell of a rhombic lattice.
2. Let X = midpoint of AC, = midpoint of BD.
3. Draw a continuous curve, **S**, from B to C, or any of the two points of triangle B.X.C., such that S is inside the triangle B.X.C. = **the generating region of cmm** = ¼ ABCD.
   S = generator of cmm
4. $R_{(Y,180^\circ)}(S) = S'$& let S U S' = S_a
5. $M_{XC}(S_a) = S_b,\ M_{BX}(S_a) = S_d,\ \ M_{XD}(S_b) = S_{c.}$
   S_a **U** S_b **U** S_c **U** S_d **= U.P.P. (cmm)** = a unit of the periodic pattern of the translation group of cmm.
6. $T_{n\propto}(U.P.P.), T_{n\beta}(U.P.P.)$ all over the plane to complete the tessellation. (α, β are linearly independent translation vectors of the lattice while n ∈ Z).

13. TYPE p31m
(p31m)

The types group contains reflections and rotations of order three (p31m). An angle of 60 degrees separates the reflection axes from one another. The rotational axes of some of the centres of rotation and not of others are different. A fundamental area for the symmetry group is one-sixth of an equilateral triangle of the hexagonal lattice, which is a fundamental region for the lattice.



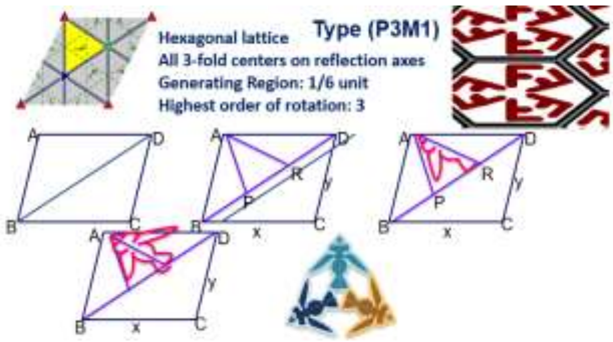
Procedure:

1. Let A, B, C, & D be latticing points of a primitive cell of a hexagonal lattice.
2. Let $X = midpoint\ of\ AB, Y = midpoint\ of\ BC,\ P = AY\ \cap\ XC.$
3. Draw the continuous curve, **S**, from A to C, or from any of the two points in triangle A.Z.C., such that S is inside the triangle AZC=**a generating region of p31m**= $1/6\ ABCD$. $\quad S = generator\ of\ p31m$
4. $M_{AC}(S) = S'$. Let S U $S' = S_{\alpha}$.
5. $R_{(P,120^0)}(S_{a)} = S_b,\ R_{(P,240^0)}(S_a) = S_c$
   $S_a\ U\ S_b\ U\ S_c = U.P.P\ (p31m) =$ *a unit of a periodic pattern of the translation group of p31m.*
6. $T_{n\alpha}(U.P.P), T_{n\beta}(U.P.P.)$ all over the plane to complete the tessellation. (α, β are linearly independent translation vectors of the lattice while n ∈ Z).

14. TYPE p3m1

(p3m1)

This group resembles the previous one with reflections and rotations of order 3. Although the reflection axes are all at an angle of 60 degrees to one another in this group, none of the centres of rotation are on the reflection axis. Like the preceding group, the fundamental region for the symmetry group is one-sixth of an equilateral triangle of the hexagonal lattice.



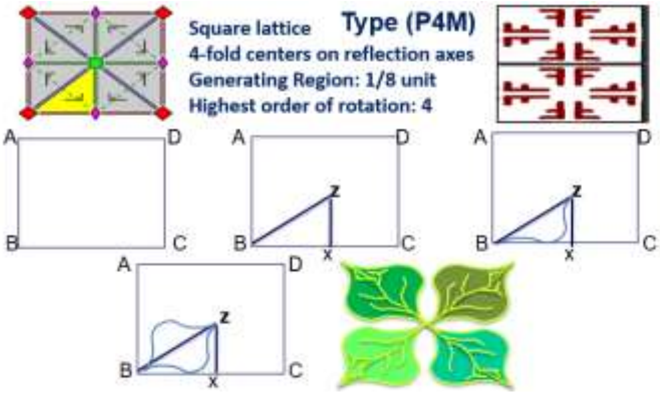
Procedure:

1. Let A, B, C, & D be lattice points of a primitive cell of a hexagonal lattice.
2. Let $X = midpoint\ of\ BC, Y = midpoint\ of\ CD$
   $$P = AX\ \cap\ BD, R = AY\ \cap\ BD.$$
3. Draw the continuous curve, **S**, from A to P, or from any of the two points in triangle A.P.R., such that S is inside the triangle APR= **a generating region of p3m1**= $1/6\ ABCD$. $\quad S = generator\ of\ p3m1$
4. $M_{AP}(S) = S_{,}$. Let S U $S' = S_{\alpha}$.

5. $R_{(P,120^0)}(S_{a)} = S_b$, $R_{(P,240^0)}(S_a) = S_c$
   $S_a\ U\ S_b\ U\ S_c = U.P.P\ (p3m1) =$ *a unit of a periodic pattern of the translation group of p3m1.*
6. $T_{n\alpha}(U.P.P), T_{n\beta}(U.P.P.)$ all over the plane to complete the tessellation. (α, β are linearly independent translation vectors of the lattice while n ∈ Z).

15. TYPE p4m
(p4mm)

The axes of reflection in this group are at a 45° angle to one another, and all of the rotational centres are located on the axes of reflection, which sets it apart from the group (p4) in terms of reflections and order-4 rotations. A triangle that makes up one-eighth of the square lattice's fundamental region for the translation group also serves as the symmetry group's fundamental region.



Procedure:

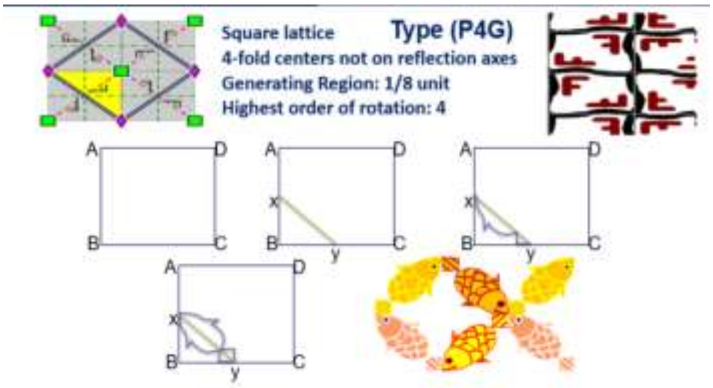
1. Let A, B, C, & D be latticing points of a primitive cell of a square lattice.
2. Let
   $X = midpoint\ of\ BC, Y = midpoint\ of\ AD, Z = midpoint\ of\ YX = midpoint\ of\ BD$
3. Draw the continuous curve, **S**, from Z to B or from any of the two points in triangle B.Z.X., such that S is inside the triangle B.X.Z. = **a generating region of p4m=** $1/8\ ABCD$.

$S = generator\ of\ p4m$

4. $M_{BZ}(S) = S'$, Let $S\ U\ S' = S_a$.
5. $R_{(Z,90^0)}(S_{a)} = S_b$, $R_{(Z,180^0)}(S_a) = S_c$ $R_{(Z,270^0)}\ (S_a) = S_d$.
   $S_\alpha\ U\ S_b\ U\ S_c\ U\ S_d = U.P.P\ (p4m) =$ *a unit of a periodic pattern of the translation group of p4m.*
6. $T_{n\alpha}(U.P.P), T_{n\beta}(U.P.P.)$ all over the plane to complete the tessellation. (α, β are linearly independent translation vectors of the lattice while n ∈ Z).

16. TYPE p4g
(p4gm)

Due to the presence of reflections and rotations of orders 2 and 4, this group resembles the one before it (p4m). The axes of reflection are perpendicular, which makes a difference because none of the centres of rotation are located on the axes of reflection. The symmetry group's fundamental region is an eighth of the square fundamental region for the translation group, which is a square in the lattice.

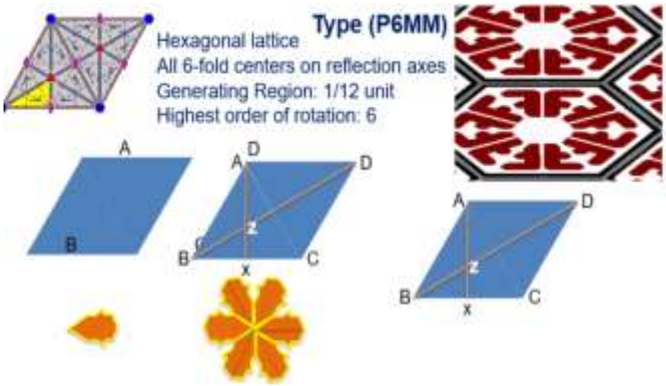


Procedure:

1. Let A, B, C, & D be lattice points of a primitive cell of a square lattice.
2. Let $X = midpoint\ of\ AB, Y = midpoint\ of\ BC,.\ Z = midpoint\ of\ AC = midpoint\ of\ BD$
3. Draw the continuous curve, **S**, from X to Y, or from any of the two points in triangle X.B.Y., such that S is inside the triangle X.B.Y. = **a generating region of p4g=** $1/8\ ABCD — S = generator\ of\ p4g.$
4. $M_{XY}(S) = S'$, Let $S\ U\ S' = S_a$.
5. $R_{(Z,\,90^0)}(S_{a)} = S_b,\ R_{(Z,\,180^0)}(S_a) = S_c \quad R_{(Z,\,270^0)}\ (S_a) = S_d$.
   $S_\alpha\ U\ S_b\ U\ S_c\ U\ S_d = U.P.P\ (p4m) =$ *a unit of a periodic pattern of the translation group of p4m.*
6. $T_{n\alpha}(U.P.P), T_{n\beta}(U.P.P.)$ all over the plane to complete the tessellation. (α, β are linearly independent translation vectors of the lattice while n ∈ Z).

17. TYPE p6m
(p6mm)

The most complicated group, type (p6m), includes rotations of orders 2, 3, and 6 and reflections. Reflection axes come together at each centre of rotation. Six reflection axes that meet and are inclined at 30° to one another are located at the centres of the order 6-rotations. Hexagons make up the lattice. One-twelfth of an equilateral triangle for the lattice is a fundamental region for the symmetry group.



Procedure:

1. Let A, B, C, & D be lattice points of a primitive cell of a hexagonal lattice.
2. Let $X = midpoint\ of\ BC, Z = BD\ \cap\ AX.$

3. Draw the continuous curve, **S**, from B to X, or from any of the two points in triangle B.X.Z., such that S is inside the triangle B.X.Z. = **a generating region of p6m=** $1/12\ ABCD$.

$$S = generator\ of\ p6m.$$

4. $M_{BX}(S) = S'$. Let S U $S' = S_a$.
5.

$R_{(B,60^0)}(S_\alpha) = S_b,\ R_{(B,120^0)}(S_\alpha) = S_c\ R_{(B,180^0)}(S_\alpha) = S_d, R_{(B,240^0)}(S_\alpha) = S_e\ R_{(B,300^0)}(S_\alpha) = Sf$

$S_\alpha\ U\ S_b\ U\ S_c\ U\ S_d\ U\ S_e\ U\ S_f = U.P.P\ (p6m) =$ *a unit of the periodic pattern of the translation group of p6m.*

6. $T_{n\alpha}(U.P.P), T_{n\beta}(U.P.P.)$ all over the plane to complete the tessellation. (α, β are linearly independent translation vectors of the lattice while n ∈ Z).

6. Summary and Conclusions

In summary, here is a way to organise the 17 wallpaper symmetry groups that may provide another tool for determining the symmetry group of a wallpaper pattern.

Table 1. The Characteristics of the 17 Wallpaper Symmetry Groups

Type	Lattice	Rotation	Order of Rotation	Reflection / Glide Reflection	Special Characteristics
p1	Parallelogram	No	none	No / No	Only translations
p2	Parallelogram	Yes	2	No / No	Four types of rotation
p4	Square	Yes	none	No / No	
p4m	Square	Yes	none	Yes / Yes	Reflection axes intersect at 45 degrees
p4g	Square	Yes	none		Reflection axes are perpendicular
pm	Rectangular	No	2	Yes / No	Reflection axes are parallel
pg	Rectangular	No	2	No /Yes	
pmg	Rectangular	Yes	2	Yes / Yes	Reflection axes are parallel
pgg	Rectangular	Yes	2	No /Yes	Yes
pmm	Rectangular	Yes	4	Yes / No	Reflection axes are perpendicular
p3	Hexagonal	Yes	4 +	No / No	
p3m1	Hexagonal	Yes	4 *	Yes / Yes	Reflection axes intersect at 60 degrees, and three types of centres of rotation
p31m	Hexagonal	Yes	3	Yes / Yes	Reflection axes intersect at 60 degrees; centres of rotation are reflection axes, two types of centres

Type	Lattice	Rotation	Order of Rotation	Reflection / Glide Reflection	Special Characteristics
					of rotation
p6	Hexagonal	Yes	3 *	No / No	
p6m	Hexagonal	Yes	3 +	Yes / Yes	Reflection axes intersect at 30 degrees
cm	Rhombic	No	6	Yes / Yes	Reflection axes are parallel
cmm	Rhombic	Yes	6	Yes / Yes	Reflection axes are perpendicular

+ = all rotation centres lie on reflection axes * = not all rotation centres lie on reflection axes

- "***p1*** No reflections and no glide-reflections.
- ***pg*** No reflections; with glide-reflections.
- ***pm*** with reflections, any glide-reflection axis is also a reflection axis.
- ***Cm*** With reflections, some glide-reflection axis is not a reflection axis.

With 2-fold rotations but no 4-fold

- ***p2*** No reflections and no glide-reflections.
- ***pgg*** No reflections; with glide-reflections.
- ***pmm*** with reflections, any glide-reflection axis is also a reflection axis.
- ***cmm*** with reflections, some glide-reflection axis is not reflection axis but is parallel to a reflection axis.
- ***pmg*** with reflections, some glide-reflection axis is not reflection axis and is not parallel to any reflection axis.

With 4-fold rotations

- ***p4*** No reflections.
- ***p4m*** with reflections, 4-fold rotation centres lie on reflection axes.
- ***p4g*** with reflections, 4-fold rotation centres do not lie on reflection axes.

With 3-fold rotations but no 6-fold

- ***p3*** No reflections.
- ***p3m1*** with reflections, any rotation centre lies on a reflection axis.
- ***p31m*** with reflections, some rotation centre is not on any reflection axis.

With 6-fold rotations

- ***p6*** No reflections.
- ***p6m*** with reflections."

The author offers a straightforward and accessible method for creating captivating designs by studying the 17 types of plane wallpaper or tessellation groups that display symmetries in periodic patterns. While achieving tessellations akin to Escher's work requires practice, experimentation, patience, and inspiration, this paper presents abstract designs that demand less artistic skill but are equally enjoyable. Exploring Escher's designs with optical illusions, geometric impossibilities, and Arabic mosaics can be both fascinating and educational. Identifying and categorising symmetries in periodic patterns can be a fun endeavour. Once readers start noticing them, they will realise how prevalent these ornamental designs are in our surroundings. Understanding the symmetries in-plane symmetry groups allows one to appreciate the beauty of wallpapers, carpets, floor tiles, window grills, and even strangers' clothing, discerning which of the 17 types of designs they belong to. However, indulging in this pastime might attract curious glances from others.

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