Connected and Independent Domination of Involutory Cayley Graph

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Abstract: For the positive integer n>1 , $G(Z_n, I_v)$ is Involutory Cayley graph with vertex set $Z_n = \{1,2,3, \ldots\ldots, n-1\}$ and the edge set $E = \{ab: a, b \in Z_{n\,,} a - b \in I_v \; or \; b - a \in I_v\}$ where $I_v = \{m \in Z_n : m^2 \equiv 1 (mod\; n)\}$. In this paper, results on connected domination and independent domination of Involutory Cayley graph $G(Z_n, I_v)$ at different n values are studied and the results are illustrated.

Keywords: Involutory Cayley Graph, Connected Domination, Independent Domination.

DOI: 10.48047/ecb/2023.12.2.033

INTRODUCTION

Throughout this paper G (V, E) is the graph with |V | is number of vertices in V, N(S) is the open neighborhood of S ⊂ V and N[S] is the closed neighborhood of S ⊂ V. The concept domination is the fast-developing area in the graph theory. This topic was taken place in the 5-queen problem. The concept domination was first introduced by Berge [2]. The dominating set D ⊂ V of a graph G is defined as every vertex in V − D is adjacent to some vertex in D. The minimum cardinality of a dominating set is the domination number of G and it is denoted by γ(G). In [12] some variants of dominations are discussed and one of them, the connected domination was introduced by Sampath Kumar and Walikar in [11]. The dominating set of a graph G is said to be connected dominating set of G if the spanning subgraph < D > is connected and minimum cardinality of the connected dominating set is called a connected domination number of G and is denoted by $\gamma_c(G)$. Laskar and Hedetniemi [8] studied about connected domination in graphs.

The elementary properties of independent domination were first discussed by Berge [2]. The concept independent domination is emanated in chessboard problems. The dominating set D of a graph G is said to be an independent dominating set of G if the spanning subgraph < D > has no edges and the minimum cardinality of the independent dominating set is called independent domination number of G and is denoted by $\gamma_c(G)$. Allan and Laskar [1], Bollobas and Cockayne [3], Cockayne and Hedetniemi [4], Nowakowski and Rall [9] provided the inspiration for working on independent domination number of a graph.

Cayley introduced the Cayley graph in 1878 for finite groups. Let Γ be finite group and X be a subset of Γ and edge set $E(\Gamma, X) = \{xy/yx^{-1} or\; x^{-1}y \in X\}$ .The graph G (Γ, X) is an undirected graph without loops. Cayley graphs are discussed extensively in [5, 6, 7] as they can be used to solve rearrangement problems and parallel CPUs design. Involutory Cayley Graph was introduced by Venkata Anusha et al.[13] and defined as for a positive integer n, the involutory Cayley graph Cay (Z_n, I_v) is a graph with Z_n is the vertex set and two vertices a, b $\in Z_n$ are adjacent if and only $a-b \in I_v$ or $b-a \in I_v$ where I_v denotes the set of all involutory elements in Z_n and it is denoted by $G(Z_n, I_v)$

2.INVOLUTORY CAYLEY GRAH

Definition 2.1: For a positive integer n, the involutory Cayley graph Cay (Z_n, I_v) is a graph in which Z_n is the vertex set and I_v denotes the set of all involutory elements in Z_n. Then two vertices $a, b \in Z_n$ are adjacent if and only if $a-b$ or $b-a \in I_v$ and it is denoted by $G(Z_n, I_v)$.

Proposition 2.2 [10]: If $n = 2^\alpha$, where $\alpha \geq 3$ and I_v is the set of involutory elements of ring of integers modulo n, then $I_v = \{1, 2^{\alpha-1}-1, 2^{\alpha-1}+1, 2^\alpha - 1\}$.

Proposition 2.3 [10]: If $n = p^\alpha$, where p is a prime,$p \neq 2, \alpha \geq 1$ and I_v is the set of involutory elements of ring of integers modulo $n$then$|I_v| = \{1, n-1\}$

Proposition 2.4 [10]: If $n = p_1^{\alpha_1}.p_2^{\alpha_2}.p_3^{\alpha_3} \dots\dots.p_k^{\alpha_k}$ where each p_i is an odd prime,$\alpha_1, \alpha_2 \dots\dots\dots, \alpha_k \geq 1$ and I_v is the set of involutory elements of ring of integers modulo n, then $|I_v| = 2^k$.

Proposition 2.5 [10]: If $n = 2^\alpha.p_1^{\alpha_1}.p_2^{\alpha_2} \dots\dots p_k^{\alpha_k}$, where each p_i is an odd prime, $\alpha_i \geq 1, \forall\, i$ and I_v is the set of involutary elements of ring of integers modulo n, then $|I_v| = \begin{cases} 2^k, if\ \alpha = 1, \\ 2^{k+1}, if\ \alpha = 2, \\ 2^{k+2}, if\ \alpha \geq 3. \end{cases}$

3. CONNECTED DOMINATION OF INVOLUTORY CAYLEY GRAPH

In this section, results on connected domination of Involutory Cayley graph $G(Z_n, I_v)$at different n values are studied and the results are illustrated.

Definition 3.1: A dominating set $D \subseteq V$ of a graph $G(V, E)$ is called a connected dominating set of G if there exists a path between every pair of vertices in the induced sub graph $\langle D \rangle$ of G. The number $\gamma_c(G)$ is the connected domination number defined as the minimum cardinality of a connected dominating set of G.

Theorem 3.2: For the Involutory Cayley graph $G(Z_n, I_v)$when $n = p^\alpha$ or $2p^\alpha$ where p is odd prime,$\alpha \geq 1$, the connected domination number $\gamma_c\big(G(Z_n, I_v)\big) = \text{n} - 2$.

Proof: Consider the graph $G(Z_n, I_v)$with the vertex set $V = \{a_0, a_1, \dots\dots.a_{n-1}\}$.Let $n = p^\alpha$ or $2p^\alpha$ where p is odd prime and $\alpha \geq 1$. From the Proposition 2.3 and Proposition 2.5, $|I_v| =$ 1.Then the graph $G(Z_n, I_v)$ is isomorphic to the cycle C_n.Define a set $D = \{a_i = i \in Z_n : 0 \leq$

$i \leq n-3\}$ of $G(Z_n, I_v)$ .The set D is a dominating set of $G(Z_n, I_v)$, since every vertex $a_i \in V$ such that $i \neq n-3$ is adjacent to the vertex $a_{i+1} \in D$ and it is minimum. Also the induced subgraph $\langle D \rangle$ is connected in $G(Z_n, I_v)$.Therefore $\gamma_c\big(G(Z_n, I_v)\big) = |D| = n-2$.

Theorem 3.3: For the Involutory Cayley graph $G(Z_n, I_v)$when $n = 2^\alpha$,where $\alpha \geq 3$, the connected domination number $\gamma_c\big(G(Z_n, I_v)\big) = \frac{n}{2} - 2$.

Proof: Consider the graph $G(Z_n, I_v)$ with the vertex set $V = \{a_0, a_1, \dots \dots . a_{n-1}\}$.Let $n = 2^\alpha$ where $\alpha \geq 3$. Since a set $D = \{a_i = i \in Z_n : 0 \leq i \leq 2^{\alpha-1} - 3\}$ is the dominating set and for every two vertices $a_i, a_j \in Z_n (i \neq j)$ there exist a path between a_i and a_j . So that the induced subgraph $\langle D \rangle$ is connected in $G(Z_n, I_v)$.Therefore D is connected dominating set of $G(Z_n, I_v)$.Hence $\gamma_c\big(G(Z_n, I_v)\big) = \frac{\text{n}}{2} - 2$.

Theorem 3.4: For the Involutory Cayley graph $G(Z_n, I_v)$ when $n = 2^2 p^\alpha$, where p is odd prime and $\alpha \geq 1$, the connected domination number $\gamma_c\big(G(Z_n, I_v)\big) = 2(p^\alpha - 1)$.

Proof: Consider the graph $G(Z_n, I_v)$ with the vertex set $V = \{a_0, a_1, \dots \dots . a_{n-1}\}$.

Let $n = 2^2 p^\alpha$, where p is odd prime and $\alpha \geq 1$.Define a set $D = \{a_i = i \in Z_n : 0 \leq i \leq 2p^\alpha - 3\}$ in $G(Z_n, I_v)$. Since every vertex $a_i \in V$ is either in D or adjacent to some vertex in D and for any two vertices $a_i, a_j \in D, \forall\, i \neq j$, there exist a path between a_i and a_j.So that D is a minimum dominating set of $G(Z_n, I_v)$ and the induced subgraph $\langle D \rangle$ is connected in $G(Z_n, I_v)$. Hence D is connected dominating set of $G(Z_n, I_v)$ and $\gamma_c\big(G(Z_n, I_v)\big) = 2\text{p}^\alpha - 2 = 2(p^\alpha - 1)$.

Theorem 3.5: For the Involutory Cayley graph $G(Z_n, I_v)$ when $n = 2^3 p^\alpha$, where p is odd prime and $\alpha \geq 1$, the connected domination number $\gamma_c\big(G(Z_n, I_v)\big) = \frac{n}{4} - 2$.

Proof: Consider the graph $G(Z_n, I_v)$ with the vertex set $V = \{a_0, a_1, \dots \dots . a_{n-1}\}$.Let $n = 2^3 p^\alpha$,where p is odd prime and$\alpha \geq 1$.Define a set $D = \{a_i = i \in Z_n : 0 \leq i \leq 2p^\alpha - 3\}$ in $G(Z_n, I_v)$. from Proposition 2.5, $|I_v| = 8$, it implies each vertex is of degree 8. For any vertex $a_i \in D$ such that $i \neq 2p^\alpha - 3$, a_i is adjacent to the vertex $a_{i+1} \in D$and therefore the induced subgrah $\langle D \rangle$ is connected in $G(Z_n, I_v)$.Hence D is a connected dominating set with minimum cardinality and $\gamma_c\big(G(Z_n, I_v)\big) = 2\text{p}^\alpha - 2 = \frac{n}{4} - 2$.

Theorem 3.6: For the Involutory Cayley graph $G(Z_n, I_v)$ when $n = 3p$, where$p$ is an odd prime and $I_v = \{I_{v_1}, I_{v_2}, I_{v_3}, I_{v_4}\}$is a set of involutory elements, the connected domination number

$$\gamma_c(G(Z_n, I_v)) = \begin{cases} 6 & \text{if p} = 5, \\ \text{p} & \text{if p} \equiv 1(\text{mod } 3), \\ \text{p} + 2 & \text{if p} \equiv 2(\text{mod } 3). \end{cases}$$

Proof: Consider the graph $G(Z_n, I_v)$ with the vertex set $V = \{a_0, a_1, \dots \dots . a_{n-1}\}$. Let $n = 3p$ where p is an odd prime .From Proposition 2.3, $|I_v| = 4$.Let $I_v = \{I_{v_1}, I_{v_2}, I_{v_3}, I_{v_4}\}$ where$I_{v_1} < I_{v_2} < I_{v_3} < I_{v_4}$ and $I_{v_1} = 1$ and $I_{v_4} = n - 1$.

Case 1: If $p = 5$ then $I_v = \{1, 4, 11, 14\}$. So a set $D = \{a_0, a_1, a_2, a_6, a_7, a_8\}$ where $a_i = i$, $\forall i$ is a dominating set of $G(Z_n, I_v)$.Since every vertex in V is either in D or adjacent to some vertex in D, it follows that there exists a path between any two vertices in D. So the induced sub graph $\langle D \rangle$ is connected and therefore D is the connected domination set with minimum cardinality 6.

Case 2: If $p \equiv 1(mod\ 3)$,then $I_{v_2} = p + 1$.Consider a set $D = \{a_0, a_1, a_2, \dots, a_{p-1}\}$of $G(Z_n, I_v)$.Each vertex $a_i \in D$, such that , a_i is adjacent to 2 vertices in V which are not adjacent to any other vertex in D .Also the vertex a_{n-p-1} is dominated by two vertices a_0, a_{p-2} in D.and the vertex a_{n-p} is dominated by two vertices a_1, a_{p-1}.So every vertex $a_i \in V$ is in D or adjacent to some vertex in D andtherefore D is dominating set of $G(Z_n, I_v)$. For any two vertices $a_i, a_j \in D$, $for\ i \neq j$, there exist a path between a_i and a_j. So that the induced subgraph $\langle D \rangle$ is connected in $G(Z_n, I_v)$. Therefore, D is a connected dominating set with minimum cardinality$p$.

Case 3: If $p \equiv 2\ (mod\ 3)$,then $I_{v_2} = p - 1$.Consider two disjoint subsets $D_1 = \{a_0, a_1, a_2, \dots, a_{p-3}\}$ and $D_2 = \{a_{2p-1}, a_{2p-2}, a_{2p-3}, a_{2p-4}\}$of V. For each vertex $a_i \in D_1$ such that $i \neq 0\ or\ p - 3, a_i$ is adjacent to two vertices in V which are not adjacent to any other vertex in D_1 and a_0, a_{p-3} are adjacent to 3 vertices in V. Denote $D = D_1 \cup D_2$. For any vertex $a_i \in V$, a_i is either in D or adjacent to some vertex in D.Therefore D is the dominating set of $G(Z_n, I_v)$. Every vertex $a_i \in D_1$ such that $i \neq p - 3$ is adjacent to the vertex $a_{i+1} \in D_1$, the vertex $a_{p-3} \in D_1$ is adjacent to the vertex $a_{2p-1} \in D_2$ and the vertices in D_2 are connected. For any two vertices $a_i, a_j \in$ D for $i \neq j$, there exist a path between a_i and a_j. The induced subgraph $\langle D \rangle$ is connected. Therefore, D is the connected dominating set with minimum cardinality$|D| = |D_1| + |D_2| = p - 2 + 4 = p + 2$.

4. INDEPENDENT DOMINATION OF INVOLUTORY CAYLEY GRAPH

Definition 4.1: Let D be the dominating set of a graph $G(V, E)$. Then D is called an independent dominating set of G if no two vertices of D are adjacent to each other that means the induced sub graph $\langle D\rangle$ has no edges (i.e., a null graph) in G.The independent domination number $\gamma_i(G)$ is the minimum cardinality of an independent dominating set of G.

Theorem 4.2: For the Involutory Cayley graph $G(Z_n, I_v)$when $n = p^\alpha$(or),$2p^\alpha$ where p is odd prime and $\alpha \geq 1$, the independent domination number

$$\gamma_i\big(G(Z_n, I_v)\big) = \begin{cases} \dfrac{n}{3} & if \quad p = 3, \\ \left[\dfrac{n}{3}\right] + 1 & if \quad p \neq 3. \end{cases}$$

Proof: Consider the graph$G(Z_n, I_v)$with vertex set $V = \{a_0 = 0, a_1 = 1, \dots a_{n-1} = n - 1\}$.

Let $n = p^\alpha$, where p is an odd prime and $\alpha \geq 1$.

Case 1: If $p = 3$, then the graph $G(Z_n, I_v)$ is a cycle of length n.Therefore, the set $D = \{a_i = i \in Z_n : i \equiv 0(mod 3)\}$ of $G(Z_n, I_v)$ forms a dominating set with cardinality $\frac{n}{3}$ and the induced subgraph$\langle D\rangle$has no edges. Hence the independent domination number$\gamma_i\big(G(Z_n, I_v)\big) = \frac{n}{3}$.

Case 2: If $p \neq 3$ then either $n \equiv 1(mod 3)$ or $n \equiv 2(mod 3)$.If $n \equiv 1(mod 3)$

The set $D = \{a_i = i \in Z_n : i \equiv 0(mod 3)\} \cup \{a_{n-2}\}$is a dominating set with minimum cardinality $\left[\frac{n}{3}\right] + 1$ and induced subgraph $\langle D\rangle$ has no edges.

Therefore, the independent domination number$\gamma_i\big(G(Z_n, I_v)\big) = \left[\frac{n}{3}\right] + 1$.

Theorem 4.3: For the Involutory Cayley graph $G(Z_n, I_v)$when $n = 2^\alpha$, where $\alpha \geq 3$, the independent domination number $\gamma_i\big(G(Z_n, I_v)\big) = \begin{cases} \frac{n+4}{3} & if \quad \frac{n}{2} \equiv 1(mod 3), \\ \frac{n+2}{3} & if \quad \frac{n}{2} \equiv 2(mod 3). \end{cases}$

Proof: Consider the graph $G(Z_n, I_v)$ with vertex set $V = \{a_0 = 0, a_1 = 1, \dots a_{n-1} = n-1\}$.

Let $n = 2^\alpha$, where $\alpha \geq 3$.

Case 1: Suppose $\frac{n}{2} \equiv 1\ (mod 3)$. Then $2^\alpha - 1 \equiv 0 (mod 3)$. Consider a set $D_1 = \{a_i = i \in Z_n : i \equiv 0(mod 3), 0 \leq i \leq 2^{\alpha-1}\}$ of $G(Z_n, I_v)$. Let $a_i \in D_1$. Then a_i is adjacent to 4 vertices of Vwhich are not adjacent to any vertex of D_1.It is true for all $a_i \in D_1$, therefore $|D_1| = \frac{2^{\alpha-1}-1}{3}$ and $|N[D_1]| = \frac{5}{3}(2^{\alpha-1} - 1)$. Again consider another set $D_2 = V - N[D_1]$. Then $|D_2| = n - \frac{5}{3}(2^{\alpha-1} - 1)$. Denote $D = D_1 \cup D_2$.Then D is the dominating set of $G(Z_n, I_v)$and induced subgraph $\langle D \rangle$ has no edges. Therefore, D is the independent dominating set of $G(Z_n, I_v)$ and

$$\gamma_i\big(G(Z_n, I_v)\big) = |D| = |D_1| + |D_2| = \frac{2^{\alpha-1} - 1}{3} + n - \frac{5}{3}(2^{\alpha-1} - 1) = \frac{n+4}{3}.$$

Case 2: Suppose $\frac{n}{2} \equiv 2(mod 3)$. Consider a set$D_1 = \{a_i = i \in Z_n : i \equiv 0(mod 3), 0 \leq i < 2^{\alpha-1} - 1\}$. Then $|D_1| = \frac{2^{\alpha-1}+1}{3}$. Let $a_i \in D_1$ such that $a_i \neq 2^{\alpha-1} - 2$ isadjacent to 4 vertices which are not adjacent to any vertex of D_1 and the vertex $a_{2^{\alpha-1}-2}$ is adjacent to two vertices in V which are not adjacent to any other vertex of D_1. Therefore $|N[D_1]| = \frac{5}{3}(2^{\alpha-1} - 2) + 3(1) = \frac{5\times2^{\alpha-1}-1}{3}$. Consider another set $D_2 = V - N[D_1]$, then $|D_2| = n - \left(\frac{5\times2^{\alpha-1}-1}{3}\right)$. Let $D = D_1 \cup D_2$. Set D is the dominating set of $G(Z_n, I_v)$and induced subgraph $\langle D \rangle$ has no edges.Therefore D is the independent dominating set of $G(Z_n, I_v)$ and the independent domination number $\gamma_i\big(G(Z_n, I_v)\big) = |D| = |D_1| + |D_2| = \frac{2^{\alpha-1}+1}{3} + n - \left(\frac{5\times2^{\alpha-1}-1}{3}\right) = \frac{n+2}{3}$.

Theorem 4.4 : For the Involutory Cayley graph $G(Z_n, I_v)$ when $n = 2^2 p^\alpha$,where p is an odd prime and $\alpha \geq 1$, the independent domination number$\gamma_i\big(G(Z_n, I_v)\big) = \begin{cases} \frac{n}{3} & if \quad \frac{n}{2} \equiv 0(mod 3), \\ \frac{n+4}{3} & if \quad \frac{n}{2} \equiv 1(mod 3), \\ \frac{n+2}{3} & if \quad \frac{n}{2} \equiv 2(mod 3). \end{cases}$

Proof: Consider the graph $G(Z_n, I_v)$ with vertex set $V = \{a_0 = 0, a_1 = 1, \dots a_{n-1} = n-1\}$.

Let $n = 2^2 p^\alpha$,where p is an odd prime and $\alpha \geq 1$.

Case 1: Suppose $\frac{n}{2} \equiv 0(mod3)$. Consider a set $D_1 = \{a_i = i \in Z_n : i \equiv 0(mod3), 1 \leq i \leq 2p^\alpha - 1\}$. Then $|D_1| = \frac{2}{3}(p^\alpha)$.If $a_i \in D_1$, then a_i is adjacent to 4 vertices in V which are not adjacent to any vertex of D_1. It is true for all vertices $a_i \in D_1$ and $|N[D_1]| = 5\left(\frac{2p^\alpha}{3}\right) = \frac{10}{3}p^\alpha$. Consider second set $D_2 = V - N[D_1]$, then $|D_2| = n - \frac{10}{3}p^\alpha$. Let $D = D_1 \cup D_2$.Then D is the dominating set of $G(Z_n, I_v)$and induced subgraph $\langle D \rangle$ has no edges in $G(Z_n, I_v)$.Therefore D is the independent dominating set of $G(Z_n, I_v)$ and $\gamma_i\big(G(Z_n, I_v)\big) = |D| = |D_1| + |D_2| = \frac{2p^\alpha}{3} + n - \frac{10}{3}(p^\alpha) = \frac{3n-2n}{3} = \frac{n}{3}$.

Case 2: Suppose $\frac{n}{2} \equiv 1(mod3)$. Choose a set $D_1 = \{a_i = i \in Z_n : i \equiv 0(mod3), 0 \leq i < 2p^\alpha - 1\}$. Then $|D_1| = \frac{2p^\alpha-1}{3}$. If $a_i \in D_1$, then $a_i$is adjacent to 4 vertices in V which are not adjacent to any vertex of D_1. And $|N[D_1]| = \frac{5}{3}(2p^\alpha - 1)$. Choose another set $D_2 = V - N[D_1]$. Then $|D_2| = n - \frac{5}{3}(2p^\alpha - 1)$. Denote $D = D_1 \cup D_2$. It is clear that D is the dominating set of $G(Z_n, I_v)$and induced subgraph $\langle D \rangle$ has no edges in $G(Z_n, I_v)$.Therefore D is the independent dominating set of $G(Z_n, I_v)$ and $\gamma_i\big(G(Z_n, I_v)\big) = |D| = |D_1| + |D_2| = \frac{2p^\alpha-1}{3} + n - \frac{5}{3}(2p^\alpha - 1) = \frac{n+4}{3}$.

Case 3: Suppose $\frac{n}{2} \equiv 2(mod3)$. Choose a set $D_1 = \{a_i = i \in Z_n : i \equiv 0(mod3), 1 \leq i \leq 2p^\alpha - 1\}$. Then $|D_1| = \frac{2p^\alpha+1}{3}$. For $a_i \in D_1$ such that $i \neq 2p^\alpha - 2$, a_i is adjacent to four vertices in V which are not adjacent to any other vertex of D_1. It is true for all $a_i \in D_1$, and $a_{p^\alpha+2}$ is adjacent to two vertices in V which is not adjacent to any other vertex of D_1 and $|N[D_1]| = 5\left(\frac{2p^\alpha-2}{3}\right) + 3$. Choose another set $D_2 = V - N[D_1]$. Then $|D_2| = n - 5\left(\frac{2p^\alpha-2}{3}\right) + 3$.Let $D = D_1 \cup D_2$. Then D is the dominating set of $G(Z_n, I_v)$ and induced subgraph $\langle D \rangle$ has no edges in $G(Z_n, I_v)$.Therefore D is the independent dominating set of $G(Z_n, I_v)$ and $\gamma_i\big(G(Z_n, I_v)\big) = |D| = |D_1| + |D_2| = \frac{2p^\alpha+1}{3} + n - \frac{5}{3}(2p^\alpha - 2) + 3 = \frac{3n-8p^\alpha+3}{3} = \frac{3n-2n+2}{3} = \frac{n+2}{3}$.

Theorem 4.5: For the Involutory Cayley graph $G(Z_n, I_v)$ $when\ n = 2^3p^\alpha$ where p is prime and $\alpha \geq 1$, the independent domination number $\gamma_i\big(G(Z_n, I_v)\big) = \begin{cases} \frac{n}{3} & if \quad \frac{n}{4} \equiv 0(mod3), \\ \frac{n+8}{3} & if \quad \frac{n}{4} \equiv 1(mod3), \\ \frac{n+4}{3} & if \quad \frac{n}{4} \equiv 2(mod3). \end{cases}$

Proof: Consider the graph $G(Z_n, I_v)$ with vertex set $V = \{a_0 = 0, a_1 = 1, \dots a_{n-1} = n-1\}$.

Let $n = 2^3p^\alpha$,where p is odd prime and $\alpha \geq 1$.

Case 1: Let $\frac{n}{4} \equiv 0(mod3)$. Choose a subset D_1 of V such that

$D_1 = \{a_i = i \in Z_n : i \equiv 0(mod3), 0 \leq i < 2p^\alpha - 1\}$.Then $|D_1| = \frac{2p^\alpha}{3}$. If $a_i \in D_1$, then by the definition of involutory Cayley graph, a_i is adjacent to 8 vertices in V which are not adjacent to any other vertex of D_1.It is true for all $a_i \in D_1$ and $|N[D_1]| = 9\left(\frac{2p^\alpha}{3}\right) = \frac{18p^\alpha}{3} = 6p^\alpha$.

Choose another set $D_2 = V - N[D_1]$. Then clearly $|D_2| = n - 6p^\alpha$. Denote $D = D_1 \cup D_2$.

Then D is the dominating set of $G(Z_n, I_v)$and induced sub graph $\langle D \rangle$ has no edges in $G(Z_n, I_v)$.

Therefore D is an independent dominating set of $G(Z_n, I_v)$ and $\gamma_i\big(G(Z_n, I_v)\big) = |D| = |D_1| + |D_2| = \frac{2p^\alpha}{3} + n - 6p^\alpha = \frac{3n-16p^\alpha}{3} = \frac{3n-2n}{3} = \frac{n}{3}$.

Case 2: Let $\frac{n}{4} \equiv 1(mod3)$. Choose a set D_1 of V such that $D_1 = \{a_i = i \in Z_n : i \equiv 0(mod3), 0 \leq i < 2p^\alpha - 1\}$.Then $|D_1| = \frac{2p^\alpha-1}{3}$. For any vertex $a_i \in D_1, a_i$ is adjacent to 8 vertices in V which are not adjacent to any other vertex of D_1 and $|N[D_1]| = 9\left(\frac{2p^\alpha-1}{3}\right) = 3(2p^\alpha - 1)$.

Consider another set $D_2 = V - N[D_1]$. Then $|D_2| = n - 6p^\alpha + 3$. Denote $D = D_1 \cup D_2$.

Then set D is the dominating set of $G(Z_n, I_v)$and induced subgraph $\langle D \rangle$ has no edges in $G(Z_n, I_v)$. Therefore $D$is an independent dominating set of $G(Z_n, I_v)$ and $\gamma_i\big(G(Z_n, I_v)\big) = |D| = |D_1| + |D_2| = \frac{2p^\alpha-1}{3} + n - 6p^\alpha - 3 = \frac{3n-16p^\alpha+8}{3} = \frac{3n-2n+8}{3} = \frac{n+3}{3}$.

Case 3: Let $\frac{n}{4} \equiv 2(mod3)$. Now consider a set D_1 of V such that

$D_1 = \{a_i = i \in Z_n: i \equiv 0(mod3), 0 \le i < 2p^\alpha - 1\}$.Then $|D_1| = \frac{2p^\alpha+1}{3}$. For the vertex $a_i \in D_1$ such that $i \ne 2p^\alpha - 2$, a_i is adjacent to 8 vertices in V which are not adjacent to any other vertex of D_1 and the vertex $a_{2p^\alpha-2}$ is adjacent to 4 vertices in Vwhich are not adjacent to any other vertex of D and $|N[D_1]| = 9\left(\frac{2p^\alpha-2}{3}\right) + 5(1) = 6p^\alpha - 1$. Again, consider a set $D_2 = V - N[D_1]$. Then $|D_2| = n - 6p^\alpha + 1$. If $D = D_1 \cup D_2$, then D is the dominating set of $G(Z_n, J_v)$ and induced subgraph $\langle D \rangle$ has no edges in $G(Z_n, J_v)$.Therefore set $D$is an independent dominating set of $G(Z_n, I_v)$ and $\gamma_i\big(G(Z_n, I_v)\big) = |D| = |D_1| + |D_2| = \frac{2p^\alpha+1}{3} + n - 6p^\alpha + 1 = \frac{3n-16p^\alpha+4}{3} = \frac{3n-2n+4}{3} = \frac{n+4}{3}$.

Theorem 4.6: For the Involutory Cayley graph $G(Z_n, I_v)$when $n = 3p^\alpha$, where $p > 3$, p is prime and $\alpha \ge 1$, the independent domination number $\gamma_i\big(G(Z_n, I_v)\big) = p^\alpha$.

Proof: Consider the graph $G(Z_n, I_v)$ with vertex set $V = \{a_0 = 0, a_1 = 1, \dots a_{n-1} = n - 1\}$.

Choose a set $D = \{a_i = i \in Z_n: i \equiv 0(mod3)\}$ of $G(Z_n, I_v)$.Then D is a dominating set of $G(Z_n, I_v)$and for any $a_i \in D, D - a_i$ is not a dominating set.Also the induced subgraph $\langle D \rangle$ has no edges in $G(Z_n, I_v)$.Therefore, D is an independent dominating set of $G(Z_n, I_v)$ and $\gamma_i\big(G(Z_n, I_v)\big) = \frac{n}{3} = p^\alpha$.

REFERENCES

1. Allan.R.B. and Laskar.R.C., - *On domination and independent domination number of a graph*,Discrete Math.23(1978), 73-76.
2. Berge. C, - *The Theory of Graphs and its Applications*, Methuen, London (1962).
3. Bollobas. B. and Cockayne. E.J. - *Graph-theoretic parameters concerning domination, independence and irredundance*, J. Graph Theory, 3 (1979) 241-249.
4. Cockayne.E.J., Hedetniemi.S.T., - *Independence graphs*, Congr. Number. X (1974), 471-491.
5. Chen, G. and Lau, F.C.M, *Comments on a new family of Cayley graph interconnection, IEEE Transactions on Parallel and Distributed Systems* 8(12) (1997), 1299–1300.

6. Curran, S.J. and Gallian, J.A., *Hamiltonian cycles and paths in Cayley graphs and digraphs—a survey, Discrete Mathematics*, 156(1) (1996), 01–18.
7. Klotz, W. and Sander, T., *Integral Cayley graphs over abelian groups,* Electron. J. Combin 17(1) (2010).
8. Laskar.R and Hedetniemi.S.T. - *Connected domination in graphs*, Graph Theory and Combinatorics, Academic Press, London (1984) 209-218.
9. Nowakowski. R. and Rall. D. - *Associative graph products and their independence, domination and coloring numbers*, Discussiones Mathematicae Graph Theory. 16 (1) 53-79.
10. Prameela Rani. C and Siva Parvathi. M - *Characterization of the set of Involutory elements of* (Z_n, $\oplus_n$, $\odot_n$), Advances in Mathematics: Scientific Journal 10(2021), no.1, 583–588.
11. Sampath Kumar.E. and Walikar H.B., - *The Connected domination number of a graph*, J.Math.Phys.Sci.,13(1979), 607-6
12. Syed Asma Kausar and Siva Parvathi, M. - Independent, Connected and Connected Total Domination in an Undirected Graph $\mathbf{G}_{m,n}$, Journal of Computer and Mathematical Sciences, Vol.10(5),1077-1081, May 2019.
13. Venkata Anusha, M., Siva Parvathi, M. - *Properties of the Involutory Cayley graph of* (Z_n,$\oplus_n$,$\odot_n$), AIP Conference Proceedings 2246, 020065 (2020).