On the graphs associate to finite groups

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Abstract: Let Γ be a finite group. The Mobius function graph of a group Γis denoted by $\mathcal{M}(\Gamma)$ is a simple graph whose vertex set is same as the elements of the group and any two vertices *a* and *b* are adjacent in $\mathcal{M}(\Gamma)$ if and only if $\mu(|a||b|)=\mu(|a|)\mu(|b|)$. In this paper, we discuss some properties of $\mathcal{M}(\Gamma)$.

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.Introduction

The study of algebraic structures, using the properties of graphs, becomes an exciting research topic in the last twenty years, leading to many fascinating results and questions. For example, the study zero-divisor graphs, total graph of commutative rings and commuting graph of groups has attracted many researchers towards this dimension. For basic definitions one can refer [5, 2].

We consider simple graphs which are undirected, with no loops or multiple edges. For any graph *G*, we denote the sets of the vertices and edges of *G* by $V(G)$ and $E(G)$, respectively. The degree $deg_G(v)$ of a vertex *v* in *G* isthe number of edges incident to *v*. The order of *G* is defined $|V(G)|$ and its maximum and minimum degrees will be denoted, respectively, by $\Delta(G)$ and $\delta(G)$. A graph *G* is regular if the degrees of all vertices of *G* are the same. A subset Ω of the vertices of *G* is called a clique if the induced subgraph of Ω is a complete graph. The maximum size of a clique in a graph *G* is called the clique number of *G* and denoted by $\omega(G)$.

If *u* and *v* are vertices in *G*, the $d(u, v)$ denotes the length of the shortest path between *u* and *v*. The largest distance between all pairs of the vertices of *G* is called the diameter of *G*, and is denoted by $diam(G)$. A graph *G* is defined to be split if there is a partition $V = S + K$ of its vertex set into a independent set *S* and a complete set *K*. A subset S of $V(G)$ is called an independent set of *G* if no two vertices of *S* are adjacent in *G*. A graph is said to be planar if it can be drawn in the plane so that its edges intersect only at their ends.

The Mobius function $\mu(n)$ is defined as follows: $\mu(1) = 1$. If $n = p^{a_1} p^{a_2} p^{a_3} \cdots p^{a_k} > 1$, where $p_1, \ldots, p_k$ are distinct primes, then

$$\mu(n) = \begin{cases} (-1)^k & \text{if } a_1 = a_2 = \cdots = a_k = 1 \\ 0 & \text{otherwise} \end{cases}$$

1 Properties of $\mathcal{M}(\Gamma)$

Definition 2.1. The Mobius function graph $\mathcal{M}(\Gamma)$of a finite group Γ is a simple graph whose vertex set is same as the elements of *G* and any two distinct vertices *a, b* are adjacent in $\mathcal{M}(\Gamma)$ if and only if $\mu(|a||b|) = \mu(|a|)\mu(|b|)$.

Example 2.2. Let $Z_3 = \{0, 1, 2\}$ be a group under addition modulo 3. Then the Mobius function graph of Z_3 is given below:

b ———— b
1 0 2

Fig. 2.1: $\mathcal{M}(Z_3)$

Remark 2.3. Let Γ be a finite group. Then $|x| = |x^{-1}|$ for all $x \in \Gamma$ and so $\mu(|x||x^{-1}|)=0$. Also if $\mu(|x|) = 0$ or $\mu(|x^{-1}|) = 0$, then $d(x, x^{-1}) = 1$. Also the identity *e* is adjacent to every other vertices of $\mathcal{M}(\Gamma)$and so $\Delta(\mathcal{M}(\Gamma)) = |\Gamma|-1$.

Theorem 2.4. *Let Γ be a group of order n. Then $\mathcal{M}(\Gamma)$ is connected and $diam(\mathcal{M}(\Gamma)) \le 2$.*

Proof. Let *e* be the identity element of Γ. For every $x \neq e$ in Γ, $\mu(|e||x|) = \mu(|e|)\mu(|x|)$.

Therefore the vertex e is adjacent to every other vertices of $\mathrm{M}(\Gamma)$.

Hence $\mathrm{M}(\Gamma)$is connected.

Let a and b be any two distinct vertices of $\mathrm{M}(\Gamma)$ such that $a \neq e \neq b$. If a and b are adjacent in $\mathrm{M}(\Gamma)$, then $d(a, b) = 1$. If a and b are not adjacent in $\mathrm{M}(\Gamma)$, then $a - e - b$ is path in $\mathrm{M}(\Gamma)$ and hence $diam(\mathrm{M}(\Gamma)) \leq 2$.

Theorem 2.5. *Let Γ be a group of order n. Then $gr(\mathrm{M}(\Gamma)) = 3$ or ∞*

Proof. As in the proof of Theorem 2.4, the vertex e is adjacent to all other vertices in $\mathrm{M}(\Gamma)$. If there exist two vertices $a \neq e$, $b \neq e$ in $\mathrm{M}(\Gamma)$, such that $\mu(|a|\ |b|) = \mu(|a|)\ \mu(|b|)$, then $a - e - b - a$ is a cycle in $\mathrm{M}(\Gamma)$ and so $gr(\mathrm{M}(\Gamma)) = 3$. Otherwise, $gr(\mathrm{M}(\Gamma)) = \infty$.

Theorem 2.6. *Let Γ be a group of prime order. Then $\mathrm{M}(\Gamma) \cong K_{1,|\Gamma|-1}$.*

Proof. Let $|\Gamma| = p$, where p is a prime number. Then Γ is a cyclic group generated by a for some $a \neq e$ in Γ. Note that e is adjacent to every other elements of $\mathrm{M}(\Gamma)$. Since $|\Gamma|$ is prime, $|x| = p$ for all $x \neq e$ in Γ. For any $x, y \in \Gamma - \{e\}$, $\mu(|x|\ |y|) = \mu(p^2) = 0$ and x is not adjacent to y and hence

$\mathrm{M}(\Gamma) \cong K_{1,|\Gamma|-1}$.

Theorem 2.7. *Let Γ be a finite group and $|\Gamma| = p^{k_1} \cdots p^{k_r} > 1$, where $p_1, \ldots, p_r$ are distinct primes. Then $K_{d_1,d_2,\ldots,d_r}$ is a subgraph of $\mathrm{M}(\Gamma)$ for some $d_1, \ldots, d_r \in \mathbb{Z}^+$.*

Proof. Let $\Omega_i = \{x \in \Gamma : |x| = p_i\}$ and $|\Omega_i| = d_i$ for $i = 1, 2, \ldots, r$. For any $a, b \in \Omega_i$ and $a \neq b$, by definition of Mobius function, a is not adjacent to b in $\mathrm{M}(\Gamma)$. For any $c \in \Omega_i$ and $d \in \Omega_j$, where $i \neq j$, $|c| = p_i$, $|d| = p_j$ and so $\mu(|c|)\mu(|d|) = 1 = \mu(|c|\ |d|)$ and hence c is adjacent to d. From this every elements of Ω_i is adjacent to every elements of Ω_j and so $K_{d_1,d_2,\ldots,d_r}$ is a subgraph of $\mathrm{M}(\Gamma)$.

Theorem 2.8. *Let Γ be a finite group and $|\Gamma| = p^n$, where p is prime and $n \geq 1$. Then $\mathrm{M}(\Gamma) \cong (K_m + K_t) + K_1$ for some $m, t \in \mathbb{Z}^+$.*

Proof. Let $\Omega_1 = \{x \in \Gamma : |x| = p\}$ and $\Omega_2 = \{y \in \Gamma : y \neq e,\ |y| \neq p\}$. For any, $z \in \Omega_1$ and $x \neq z$, by definition of Mobius function, x is not adjacent to z in $\mathrm{M}(\Gamma)$ and hence the subgraph induced by Ω_1 in $\mathrm{M}(\Gamma)$ is totally disconnected.

For any $y, w \in \Omega_2$ and $y \neq w$, $\mu(|y|\ |w|) = 0 = \mu(|y|)\mu(|w|)$ and so y is adjacent to w in $\mathrm{M}(\Gamma)$. Hence the subgraph of induced by Ω_2 in $\mathrm{M}(\Gamma)$ is complete.

For any $x_1 \in \Omega_1$ and $y_1 \in \Omega_2$, $\mu(|x_1|\ |y_1|) = 0 = \mu(|x_1|)\mu(|y_1|)$ and so x_1 is adjacent to y_1 in $\mathrm{M}(\Gamma)$. Hence every elements in Ω_1 is adjacent to every elements of Ω_2. From the above argument, we have $\mathrm{M}(\Gamma) = (K_{|\Omega_1|} + K_{|\Omega_2|}) + K_1$.

A split graph is a graph in which the vertices can be partitioned into a clique and an independent set. Let G be a connected graph. Then G is a split graph if and only if C_4, C_4 and C_5 are forbidden subgraphs of G.

Corollary 2.9. *Let Γ be a finite group and $|\Gamma| = p^n$, where p is prime and $n > 1$. Then $\mathrm{M}(\Gamma)$ is a split graph.*

Proof. In view of Theorem 2.8, Ω_1 is an independent set, $\Omega_2 \cup \{e\}$ is a clique set and hence $\mathrm{M}(\Gamma)$ is a split graph.

Theorem 2.10. *Let Γ be a finite group. Then $\mathrm{M}(\Gamma)$ is a split graph if and only if $|\Gamma| = p^n$, where p is prime and $n \geq 1$.*

Proof. Suppose $|\Gamma| = p^n$, where p is prime and $n \geq 1$. If $n = 1$, then $\mathrm{M}(\Gamma) \cong K_{1,p-1}$ is a split graph. If $n > 1$, then by Corollary 2.9, $\mathrm{M}(\Gamma)$ is split. Conversely, let us assume that $\mathrm{M}(\Gamma)$ is split graph. If there exist two distinct primes p, q divides $|\Gamma|$, then either $p \geq 3$ or $q \geq 3$. Without loss of gen- erality, assume that $q \geq 3$. Then by Cauchy's Theorem, $|a| = p$, $|b| = |b^{-1}| = q$ for some $a, b \in \Gamma$. From this we get, $a - b - e - b^{-1}$ is forbidden subgraph of $\mathrm{M}(\Gamma)$ and so $\mathrm{M}(\Gamma)$ is not a split graph, a contradiction. Hence $|\Gamma| = p^n$ for some prime p and $n \in \mathbb{Z}^+$.

Theorem 2.11. *Let G be a cyclic group with $|G| = p_1p_2 \ldots p_r$, where $p_1, p_2, \ldots, p_r$ are distinct primes. Then $\mathrm{M}(\Gamma)$ has at least $\varphi(|G|)$ pendent vertices, where φ is Euler function.*

Proof. Let $\Omega = \{x \in G : |x| = |G|\}$. Then $|\Omega| = \varphi(p_1p_2 \ldots p_r)$. For any $x, y \in \Omega$ and $x \neq y$, $\mu(|x|) = (-1)^r = \mu(|y|)$, $\mu(|x|\ |y|) = 0$, $\mu(|x|)\mu(|y|) = (-1)^{2r} = 1$ and so x is not adjacent to y. For any $w \in G - \Omega$ and $w \neq e$, $|w|$ divides $|G|$, $\mu(|w|\ |a|) = 0$, $\mu(|w|)\mu(|a|) \neq 0$ for all $a \in \Omega$. Hence every elements of $G - \Omega$ is not adjacent to every elements of Ω in $\mathrm{M}(\Gamma)$ and e is only element adjacent to every elements of Ω. Hence $deg(x) = 1$ for all $x \in \Omega$

Theorem 2.12. *Let G be a finite group. If $|G|$ is odd integer, then $\mathrm{M}(\Gamma)$ is not a complete graph.*

Proof. Since $|G$ is odd, there exist an odd prime p such that p divides $|G|$. By Cauchy's Theorem, $|a| = p$ for some $a \in G$. Also $|a^{-1}| = p$

and by definition of Mobius function, $\mu(|a||a^{-1}|) = 0 \neq \mu(|a|)\mu(a^{-1}) = 1$. Hence a is not adjacent to a^{-1} in $\mathrm{M}(\Gamma)$ and $\mathrm{M}(\Gamma)$ is not complete.

Corollary 2.13. *Let Γ be a finite group. If $\mathrm{M}(\Gamma)$ is complete, then $|G|$ is even, but the converse is not ture.*

Proof. Proof follows from Theorem 2.12. For example, let G be Kelin 4-group. Then $\mathrm{M}(\Gamma) \cong K_{1,3}$ is not complete.

Theorem 2.14. *Let Γ be a finite group. Then $\mathrm{M}(\Gamma)$ is complete if and only if Γ is cyclic and $|\Gamma| = 2^k$ for some $k \in \mathbb{Z}^+$.*

Proof. Suppose $\mathrm{M}(\Gamma)$ is complete. Then by Corollary 2.13, $|\Gamma|$ is even. If there is an odd prime p such that p divides $|\Gamma|$, then there exist an element$a \in \Gamma$ such that $|a| = p = |a^{-1}|$ and so a is not adjacent to a^{-1}, which is a contradiction. Hence $|\Gamma| = 2^k$ for some $k \in \mathbb{Z}^+$. As in the proof of Theorem 1.8 , $\mathrm{M}(\Gamma) \cong (\overline{K_{|\Omega 1|}} + K_{|\Omega 2|}) + K_1$. If $|\Omega_1| > 1$, then $\mathrm{M}(\Gamma)$ is not complete, a contradiction. Hence $|\Omega_1| = 1$ and so Γ is cyclic.

Converse follows easily by the definition of Mobius function.

Theorem 2.15. *Let Γ be a finite group. If at least two distinct primes divides $|\Gamma|$, then $\mathrm{M}(\Gamma)$ never be unicyclic.*

Proof. Let p and q be two distinct primes divides $|\Gamma|$.

Then by Cauchy'sTheorem, there exist distinct $x, y \in \Gamma$ such that $|x| = p$ and $|y| = q$. Withoutoss of generality, let us assume that $p < q$. Then $q \geq 3$, $y \neq y^{-1}$ and $|y^{-1}| = q$.

Note that e is adjacent to every other vertices of $\mathrm{M}(\Gamma)$. By definition Mobius function, x is adjacent to both y and y^{-1} in $\mathrm{M}(\Gamma)$. Hence $x - e - y - x$ and $x - e - y^{-1} - x$ are two distinct cycles in $\mathrm{M}(\Gamma)$ and so $\mathrm{M}(\Gamma)$ is not unicyclic.

Theorem 2.16. *Let Γ be a finite group. If $\mathrm{M}(\Gamma)$ is complete bipartite, then $\mathrm{M}(\Gamma) = \overline{K_t} + K_1$ for some positive integer t.*

Proof. Suppose $\mathrm{M}(\Gamma)$ is complete bipartite. If there exist two distinct primes p, q divides $|\Gamma|$, then $|a| = p$ and $|b| = q$ for some $a, b \in G$ and so $a - e - b - a$ is a 3-cycle in $\mathrm{M}(\Gamma)$, a contradiction. Hence $|G| = p^m$ for some prime p and $m \in \mathbb{Z}^+$. By Theorem 2.8, $\mathrm{M}(\Gamma) \cong (\overline{K_m} + K_t) + K_1$ for some $m, t \in \mathbb{Z}^+$.

If $t \geq 1$, then $\mathrm{M}(\Gamma)$ contains three cycle, a contradiction. Thus $t = 0$, $\mathrm{M}(\Gamma) = \overline{K_m} + K_1$.

In the following, we classify all finite p-groups for which the Mobius functiongraph is planar

Theorem 2.17. *Let Γ be a finite group and $|\Gamma| = p^n$ where p is prime and $n \in \mathbb{Z}^+$. Then $\mathrm{M}(\Gamma)$ is planar if and only if Γ is isomorphic to one of the following group:* $\mathbb{Z}_2$, $\mathbb{Z}_3$, $\mathbb{Z}_4$, $\mathbb{Z}_5$ *or* $\mathbb{Z}_2 \times \mathbb{Z}_2$.

Proof. Suppose $\mathrm{M}(\Gamma)$ is planar. By Theorem 2.8, $\mathrm{M}(\Gamma) \cong (\overline{K_\ell} + K_t) + K_1$ for some $\ell, t \in \mathbb{Z}^+$. Since $\mathrm{M}(\Gamma)$ is planar, either $\ell \leq 2$ or $t \leq 2$. If $\ell \leq 2$, then $t \leq 2$ and so $|\Gamma| \leq 5$. If $t \leq 2$ then $\ell \leq 2$ and so $|\Gamma| \leq 5$. Hence Γ is isomorphic to one of the following groups: $\mathbb{Z}_2$, $\mathbb{Z}_3$, $\mathbb{Z}_4$, $\mathbb{Z}_5$ or $\mathbb{Z}_2 \times \mathbb{Z}_2$.

Conversely if Γ is isomorphic to one of the following groups: $\mathbb{Z}_2$, $\mathbb{Z}_3$, $\mathbb{Z}_4$, $\mathbb{Z}_5$ or $\mathbb{Z}_2 \times \mathbb{Z}_2$, then $\mathrm{M}(\Gamma)$ is subgraph of K_5 and so $\mathrm{M}(\Gamma)$ is planar.

Theorem 2.18. *Let Γ be a finite group. If there is at least two distinct primes $p \geq 3$ and $q \geq 3$ divides $|\Gamma|$ and $p < q$, then Γ is non-planar.*

Proof. Let $\Omega_1 = \{x \in \Gamma : |x| = p\}$ and $\Omega_2 = \{y \in \Gamma : |x| = q\}$. Then $|\Omega_1| \geq 2$, $|\Omega_2| \geq 4$, every vertices of Ω_1 are adjacent to every vertices of Ω_2 and so $K_{2,4}$is a subgraph of $\mathrm{M}(\Gamma)$. Since e is adjacent to all other vertices of $\mathrm{M}(\Gamma)$, $K_{3,4}$ is a subgraph of $\mathrm{M}(\Gamma)$ and hence $\mathrm{M}(\Gamma)$ is non-planar.

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