$\hat{g}^{**}$s - CONTINUOUS FUNCTIONS IN TOPOLOGICAL SPACES

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Abstract

The aim of this paper is to introduce six classes of $\hat{g}^{**}$s-continuous functions and $\hat{g}^{**}$s-irresolute functions. We investigate the relationship of these functions and its properties. Also, we study the composition and restrictions of these functions.

Keywords: perfectly $\hat{g}^{**}$s-continuous function, contra $\hat{g}^{**}$s-continuous function, slightly $\hat{g}^{**}$s-continuous function, strongly $\hat{g}^{**}$s-continuous function, totally $\hat{g}^{**}$s-continuous function.

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I. Introduction

The word topology was derived from the word topo's meaning discovers or study the surface. Modern topology depends strongly on the ideas of topology developed by Cantor in the later part of the 19th century, during the 1960's research work in general topology has moved into several new areas including set theoretic methods. Continuous function in topology found a valuable place in the applications of Mathematics as it is used in digital signal processing and neural networks. Topologists studied weaker and stronger forms of continuous functions in topology using the sets, stronger and weaker than open and closed sets. Levine introduced the concept of semi-continuous in 1963. In the 1980, Jain introduced totally continuous functions and in the year 1996, Dontchev introduced contra continuous functions. The notion of perfectly continuous function was introduced and studied by Noiris. In 1991, K. Balachandran, H.Maki and P.Sundaram defined a new class of mappings called generalised continuous mappings. The concept of slightly continuous function was introduced by Jain in the year 1997. Recently the authors have introduced the concept of $\hat{g}^{**}$s-closed sets and studied into various properties. Continuing this work, here we introduce a new function slightly $\hat{g}^{**}$s-continuous functions, perfectly $\hat{g}^{**}$s-continuous functions, totally $\hat{g}^{**}$s- continuous functions, contra $\hat{g}^{**}$s-continuous functions, strongly $\hat{g}^{**}$s-continuous functions and weakly $\hat{g}^{**}$s-continuous functions and investigate its properties in terms of comparison, composition and restriction. Also, we have established the relationship of all these continuous functions.

II. Preliminaries

Throughout this paper (X, τ) represent the non-empty topological spaces on which no separation axioms are assumed unless otherwise mentioned. For a subset A of a space (X, τ), $cl(A)$ and $int(A)$ denote the closure and interior of A respectively.

Definition: 2.1: Let (X, τ) be a topological space.

1) A function $f: (X, \tau) \to (Y, \sigma)$ is said to be a continuous function if $f^{-1}(V)$ is a closed set in (X, τ) for every closed set V in (Y, σ).
2) A subset of a topological space is called $\hat{g}^{**}$s-closed **[10]** if $scl(A) \subseteq U$, whenever $A \subseteq U$ and U is $\hat{g}^{*}$-open.
3) A topological space (X, τ) is said to be clopen-T_1 **[17]** if for each pair of distinct points x and y of X, there exist clopen sets U and V containing x and y respectively such that $y \notin U$ and $x \notin V$.
4) A topological space (X, τ) is said to be clopen-T_1 **[17]** if for each pair of distinct points x and y of X, there exist disjoint clopen sets U and V containing x and y respectively such that $x \in U$ and $y \in V$.
5) A topological space (X, τ) is said to be $\hat{g}^{**}$s -T_1 if for each pair of distinct points x and y of X, there exist disjoint $\hat{g}^{**}$s - open sets U and V containing x and y respectively such that $x \in U$ and $y \in V$.
6) A topological space (X, τ) is said to be $\hat{g}^{**}$s -T_2 if for each pair of disjoint points x and y of X, there exist disjoint $\hat{g}^{**}$s - open sets U and V containing x and y respectively such that $x \in U$, $x \notin V$ and $y \in V, y \notin U$.
7) A topological space (X, τ) is said to be $\hat{g}^{**}$s –clopen T_2 if for each pair of disjoint points x and y of X, there exist disjoint $\hat{g}^{**}$s -open sets U and V containing x and y respectively such that $x \in U$, $x \notin V$ and $y \in V, y \notin U$.
8) A topological space (X, τ) is said to be clopen normal **[17]** if for every pair of disjoint clopen subsets F_1 and F_2 of X, there exist disjoint open sets U and V such that $F_1 \subseteq U$ And $F_2 \subseteq V$.
9) A topological space (X, τ) is said to be $\hat{g}^{**}$s -normal if for every pair of disjoint closed subsets F_1 and F_2 of X, there exist disjoint $\hat{g}^{**}$s -open sets U and V such that $F_1 \subseteq U$ And $F_2 \subseteq V$.
10) A topological space (X, τ) is said to be $\hat{g}^{**}$s -clopen normal if for every pair of disjoint clopen subsets F_1 and F_2 of X, there exist disjoint $\hat{g}^{**}$s -open sets U and V such that $F_1 \subseteq U$ And $F_2 \subseteq V$.

Definition: 2.2 [11]: A topological space (X, τ) is called

1) T_b-space if every gs-closed is closed.
2) T_c-space if every gs-closed is g^*-closed.
3) Strongly semi-$T_{1/2}$ if every gs-closed is g^*s-closed.
4) $T^*_{1/2}$ if every g^*-closed is closed.
5) T^*_b-space if every g^*s-closed is closed.
6) T^*_s-space if every $(sg)^*$-closed is closed.
7) $T^{**}_{\hat{b}}$-space if every $\hat{g}^{**}$s-closed is closed.

Lemma: 2.3 [19]: If $f: X \to Y$ is a semi-homeomorphism, then $scl(f^{-1}(A)) = f^{-1}(scl\,A)$ for all $B \subseteq Y$.

Lemma: 2.4 [19]: If $f: X \to Y$ is a semi-homeomorphism, then $scl(f(A)) = f(scl\,A)$ for all $B \subseteq X$.

Lemma: 2.5: $f(f^{-1}(A)) = f^{-1}(f(A))$

Lemma: 2.6: [21] $f^{-1}(V^c) = [f^{-1}(V)]^c$

III. $\hat{g}^{**}$s – CONTINUOUS FUNCTIONS

Definition: 3.1

A function $f: (X, \tau) \to (Y, \sigma)$ is said to be a $\hat{g}^{**}s$ - continuous function if $f^{-1}(V)$ is a $\hat{g}^{**}s$ - closed set in (X, τ), for every closed set V in (Y, σ).

Remark: 3.1.1

Relationship of $\hat{g}^{**}s$-continuous function with other continuous function is shown in the below diagram and the converse of these are not true.

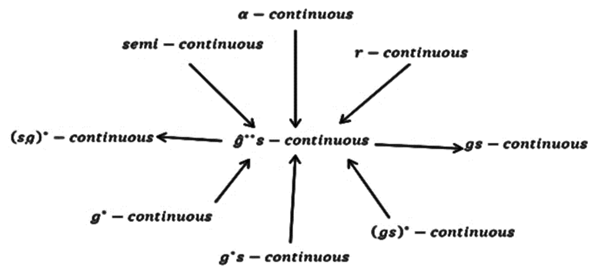

Figure:1

Theorem: 3.1.2: Let $f: X \to Y$ be a function. Then the following are equivalent.

(i) f is $\hat{g}^{**}s$ -continuous.
(ii) the inverse image of each open set in Y is $\hat{g}^{**}s$ -open in X.

Proof: ***i***) ⇒ ***ii***) Assume that $f: X \to Y$ is a $\hat{g}^{**}s$ - continuous map and V is open in Y. This implies V^c is a closed set in Y. As f is $\hat{g}^{**}s$ -continuous, $f^{-1}(V^c)$ is a $\hat{g}^{**}s$ -closed set in X. Also, $f^{-1}(V)$ is a $\hat{g}^{**}s$-open set in X, since $f^{-1}(V^c) = X - f^{-1}(V)$ is $\hat{g}^{**}s$ -closed in X. Therefore, the inverse image of each open set in Y is $\hat{g}^{**}s$ -open in X.

ii) ⇒ ***i***) Assume that the inverse image of each open set in Y is $\hat{g}^{**}s$ -open in X. Let V_1 be any closed set in Y ⇒ V_1^c is open in Y ⇒ $f^{-1}(V_1^c)$ is $\hat{g}^{**}s$ -open in X ⇒ $X - f^{-1}(V_1)$ is $\hat{g}^{**}s$ -open in X⇒ $f^{-1}(V_1)$ is $\hat{g}^{**}s$ -closed in X ⇒ f is $\hat{g}^{**}s$ - continuous.

Remark: 3.1.3: Composition of two $\hat{g}^{**}s$-continuous functions need not be $\hat{g}^{**}s$-comntinuous.

Example: 3.1.4: Let $X = Y = Z = \{a, b, c, d\}$, $\tau = \{\emptyset,\ X, \{a\}, \{a, b, c\}\}$, $\sigma = \{\emptyset,\ X,\ \{a\}, \{d\},\ \{a, d\}, \{a, b, c\}\}$,
$\eta = \{\emptyset,\ X, \{a\}, \{a, d\},\ \{a, b, c\}\}$.
$\hat{g}^{**}s(X, \tau) = \left\{\begin{matrix}\emptyset, X, \{b\}, \{c\}, \{d\}, \{a, d\}, \{c, d\}, \{b, d\}, \{b, c\}, \{a, b, d\}, \\ \{a, c, d\}, \{b, c, d\}\end{matrix}\right\}$,
$\hat{g}^{**}s(Y, \sigma) = \{\emptyset, X, \{b\}, \{c\}, \{d\}, \{c, d\}, \{b, d\}, \{b, c\}, \{a, b, c\}, \{b, c, d\}\}$,
$\hat{g}^{**}s(Z, \eta) = \{\emptyset, X, \{b\}, \{c\},\ \{c, d\}, \{b, c\}, \{b, d\}, \{b, c, d\}\}$. Define $f: (X, \tau) \to (Y, \sigma)$ as $f(a) = b, f(b) = a, f(c) = d,\ f(d) = c$ and define $g: (Y, \sigma) \to (Z, \eta)$ as $g(a) = a,\ g(b) = d, g(c) = c, g(d) = b$ **.** Clearly, f and are $\hat{g}^{**}s$ -continuous. But for a closed set $\{b, c\}$ in (Z, η), $(f \circ g)^{-1}\{b, c\}$ does not belongs to $\hat{g}^{**}s$ –closed set in (X, τ).

IV. $\hat{g}^{**}s$ - IRRESOLUTE FUNCTION:

Definition: 4.1: A function $f: (X, \tau) \to (Y, \sigma)$ is said to be a $\hat{g}^{**}s$ - irresolute function if $f^{-1}(V)$ is a $\hat{g}^{**}s$ -closed set in (X, τ) for every $\hat{g}^{**}s$ -closed set V in (Y, σ).

Theorem: 4.1: Every irresolute function is $\hat{g}^{**}s$ -continuous.

Theorem: 4.2: Every semi-irresolute function is $\hat{g}^{**}s$ -continuous.

Theorem: 4.3: Every g^*s-irresolute function is $\hat{g}^{**}s$ -continuous.

Theorem: 4.4: Every g^*-irresolute function is $\hat{g}^{**}s$ -continuous.

Theorem: 4.5: Every $(gs)^*$-irresolute function is $\hat{g}^{**}s$ -continuous.

Theorem: 4.6: Every $\hat{g}^{**}s$ -irresolute function is gs -continuous.

Theorem: 4.7: Every $\hat{g}^{**}s$ -irresolute function is $(sg)^*$ -continuous.

Theorem: 4.8: Every $\hat{g}^{**}s$ -irresolute function is a $\hat{g}^{**}s$ -continuous function.

Theorem: 4.9: Let $f: (X, \tau) \to (Y, \sigma)$ be a $\hat{g}^*$-irresolute function, semi-homeomorphism and injective function. Then for every $\hat{g}^{**}s$ –closed set A in (X, τ), $f(A)$ is $\hat{g}^{**}s$ -closed in (Y, σ).

Proof: Let A be a $\hat{g}^{**}s$ -closed in (X, τ). Consider U as any $\hat{g}^*$-open subset of (Y, σ) such that $f(A) \subseteq U$ **.** Given that, f is a $\hat{g}^*$-irresolute map, which implies that $f^{-1}(U)$ is an $\hat{g}^*$-open subset of (X, τ) such that $A \subseteq f^{-1}(U)$. Moreover, A is $\hat{g}^{**}s$ -closed set $\Rightarrow scl\ A \subseteq f^{-1}(U) \Rightarrow f(scl\ A) \subseteq U \Rightarrow scl\ (f(A)) \subseteq U$ Therefore, $f(A)$ is a $\hat{g}^{**}s$ -closed in (Y, σ).

Theorem: 4.10: Let $f: (X, \tau) \to (Y, \sigma)$ and $g: (Y, \sigma) \to (Z, \eta)$ be two functions. Then

(i) $g \circ f$ is $\hat{g}^{**}s$ -continuous if g is continuous and f is $\hat{g}^{**}s$ -continuous.

(ii) $g \circ f$ is $\hat{g}^{**}s$ -irresolute if g is $\hat{g}^{**}s$ -irresolute and f is $\hat{g}^{**}s$ - irresolute.

(iii) $g \circ f$ is $\hat{g}^{**}s$ -continuous if g is $\hat{g}^{**}s$ -continuous and f is $\hat{g}^{**}s$ - irresolute.

Proof:

i) Consider V as a closed set in (Z, η). Given that, g is continuous and so $g^{-1}(V)$ is closed in (Y, σ). Moreover, f is $\hat{g}^{**}s$ -continuous, $f^{-1}(g^{-1}(V))$ is $\hat{g}^{**}s$ -closed in $(X, \tau) \Rightarrow (g \circ f)^{-1}(V)$ is $\hat{g}^{**}s$ -closed in (X, τ). Therefore, $g \circ f$ is $\hat{g}^{**}s$ -continuous.

ii) Let V be $\hat{g}^{**}s$ -closed in (Z, η). Given that g is $\hat{g}^{**}s$ -irresolute, so $g^{-1}(V)$ is a $\hat{g}^{**}s$ -closed set in (Y, σ). Also, f is a $\hat{g}^{**}s$ -irresolute function which implies that $f^{-1}(g^{-1}(V))$ is $\hat{g}^{**}s$ -closed in (X, τ). So, $(g \circ f)^{-1}(V)$ is a $\hat{g}^{**}s$ -closed set in (X, τ). Therefore, $g \circ f$ is $\hat{g}^{**}s$ -irresolute.

iii) Consider V as a closed set in (Z, η). Assume that g is $\hat{g}^{**}s$ -continuous, which shows $g^{-1}(V)$ is $\hat{g}^{**}s$ -closed in (Y, σ). Also, given that f is $\hat{g}^{**}s$ -irresolute $\Rightarrow f^{-1}(g^{-1}(V))$ is $\hat{g}^{**}s$ -closed in $(X, \tau) \Rightarrow (g \circ f)^{-1}(V)$ is $\hat{g}^{**}s$ -closed in (X, τ). Therefore, $g \circ f$ is $\hat{g}^{**}s$ -continuous.

V. SOME MORE CONTINUOUS FUNCTIONS:

Definition: 5.1:

1) A function $f: (X, \tau) \to (Y, \sigma)$ is said to be a contra $\hat{g}^{**}s$ -continuous function if $f^{-1}(V)$ is a $\hat{g}^{**}s$ -closed set in (X, τ) for every open set V in (Y, σ).
2) A function $f: (X, \tau) \to (Y, \sigma)$ is said to be a perfectly $\hat{g}^{**}s$ -continuous function if $f^{-1}(V)$ is a clopen set in (X, τ) for every $\hat{g}^{**}s$ -open set V in (Y, σ).
3) A function $f: (X, \tau) \to (Y, \sigma)$ is said to be a slightly $\hat{g}^{**}s$ -continuous function if $f^{-1}(V)$ is a $\hat{g}^{**}s$ -open set in (X, τ) for every clopen set V in (Y, σ).
4) A function $f: (X, \tau) \to (Y, \sigma)$ is said to be a totally $\hat{g}^{**}s$ -continuous function if $f^{-1}(V)$ is a $\hat{g}^{**}s$ -clopen set in (X, τ) for every open set V in (Y, σ).
5) A function $f: (X, \tau) \to (Y, \sigma)$ is said to be a strongly $\hat{g}^{**}s$ -continuous function if $f^{-1}(V)$ is a open set in (X, τ) for every $\hat{g}^{**}s$ -open set V in (Y, σ).
6) A function $f: (X, \tau) \to (Y, \sigma)$ is said to be a weakly $\hat{g}^{**}s$ -continuous function if

$f^{-1}(V)$ is a $\hat{g}^{**}s$ -open set in (X, τ) for every open set V in (Y, σ).

7) A function $f: (X, \tau) \rightarrow (Y, \sigma)$ is said to be $\hat{g}^{**}s$-open map if f(U) is $\hat{g}^{**}s$-open in (Y, σ) for every open set U in (X, τ).
8) A function $f: (X, \tau) \rightarrow (Y, \sigma)$ is said to be $\hat{g}^{**}s$-closed map if f(U) is $\hat{g}^{**}s$-closed in (Y, σ) for every closed set U in (X, τ).

RESULTS:

Theorem: 5.1: Let f be $\hat{g}^{*}$-open map, semi-homeomorphism and bijective. Then for every $\hat{g}^{**}s$-closed set A of Y, $f^{-1}(A)$ is gs-closed in X.

Proof: Let A be $\hat{g}^{**}s$-closed set in Y. To prove that $f^{-1}(A)$ is s –closed in X. Assume $f^{-1}(A) \subseteq U$ and U is open in X. By hypothesis, f(U) is $\hat{g}^{*}$-open in Y. Therefore, $A \subseteq f(U)$ and f(U) is $\hat{g}^{*}$-open in Y. Since, A is $\hat{g}^{**}s$-closed in Y $\Rightarrow scl(A) \subseteq f(U) \Rightarrow f^{-1}\big(scl(A)\big) \subseteq U \Rightarrow scl(f^{-1}(A)) \subseteq U$. Hence $scl\big(f^{-1}(A)\big) \subseteq U$, whenever $f^{-1}(A) \subseteq U$ and U is open in X. Therefore, $f^{-1}(A)$ is gs-closed in X.

Theorem: 5.2: Every $\hat{g}^{**}s$-irresolute is weakly $\hat{g}^{**}s$- continuous.

Proof: Follows from te definition.

Theorem: 5.3: If X is $T_{\hat{b}}^{**}$, then every $\hat{g}^{**}s$-irresolute is strongly $\hat{g}^{**}s$- continuous.

Proof: Let V be $\hat{g}^{**}s$-open in Y, then V^c is $\hat{g}^{**}s$-closed in Y which implies that $f^{-1}(V^c)$ is $\hat{g}^{**}s$-closed in X. By hypothesis, $f^{-1}(V^c)$ is closed in X and so, $f^{-1}(V)$ is open in X.

Theorem: 5.4: If Y is $T_{\hat{b}}^{**}$, then every weakly $\hat{g}^{**}s$-continuous is $\hat{g}^{**}s$-irresolute.

Proof: Let V be $\hat{g}^{**}s$-closed in Y. By assumption, V is closed in Y which implies V^c is open in Y. $f^{-1}(V^c)$ is $\hat{g}^{**}s$-open in X, so $f^{-1}(V)$ is $\hat{g}^{**}s$-closed in X.

Theorem: 5.5: If f is a weakly $\hat{g}^{**}s$- continuous, closed and injective function and Y is normal, then X is $\hat{g}^{**}s$-normal.

Proof: Let F_1 and F_2 be disjoint closed sets in X. Since f is closed and injective, $f(F_1)$ and $f(F_2)$ are disjoint closed sets in Y. Since Y is normal, there exist disjoint open sets U and V such that $f(F_1) \subset U$ and $f(F_2) \subset V$. (ie)$F_1 \subset f^{-1}(U)$ and $F_2 \subset f^{-1}(V)$. Since f is weakly $\hat{g}^{**}s$- continuous, $f^{-1}(U)$ and $f^{-1}(V)$ are $\hat{g}^{**}s$-open sets containing the closed sets F_1 and F_2 in X. Also, $f^{-1}(U) \cap f^{-1}(V) = \emptyset$. Hence, X is $\hat{g}^{**}s$-normal.

Theorem: 5.6: If f is a totally $\hat{g}^{**}s$ - continuous, closed and injective function and Y is normal, then X is $\hat{g}^{**}s$-clopen normal.

Proof: Let F_1 and F_2 be disjoint clopen sets in X $\Rightarrow F_1$ and F_2 are disjoint closed sets in X. Since f is closed and injective, $f(F_1) \cap f(F_2) = \emptyset$ in Y. Since Y is normal, there exist disjoint open sets U and V such that $f(F_1) \subset U$ and $f(F_2) \subset V$. (ie)$F_1 \subset f^{-1}(U)$ and $F_2 \subset f^{-1}(V)$. Since f is totally $\hat{g}^{**}s$-continuous, $f^{-1}(U)$ and $f^{-1}(V)$ are $\hat{g}^{**}s$-clopen sets containing the closed sets F_1 and F_2 in X. Also, $f^{-1}(U) \cap f^{-1}(V) = \emptyset$. Hence, X is $\hat{g}^{**}s$-clopen normal.

Theorem: 5.7: If f is a perfectly $\hat{g}^{**}s$- continuous, closed and injective function and Y is $\hat{g}^{**}s$ -normal, then X is normal.

Proof: Let F_1 and F_2 be disjoint closed sets in X. Since f is closed and injective, $f(F_1)$ and $f(F_2)$ are disjoint closed sets in Y. Since Y is normal, there exist disjoint $\hat{g}^{**}s$-open sets U and V such that $f(F_1) \subset U$ and $f(F_2) \subset V$. (ie)$F_1 \subset f^{-1}(U)$ and $F_2 \subset f^{-1}(V)$. Since f is perfectly $\hat{g}^{**}s$- continuous, $f^{-1}(U)$ and $f^{-1}(V)$ are clopen sets containing the closed sets F_1 and F_2 in X. Also, $f^{-1}(U) \cap f^{-1}(V) = \emptyset$. Hence, X is normal.

Theorem: 5.8: If f is a strongly $\hat{g}^{**}s$- continuous, closed and injective function and Y is normal, then X is clopen normal.

Proof: Let F_1 and F_2 be disjoint closed sets in X. Since f is closed and injective, $f(F_1)$ and $f(F_2)$ are disjoint closed sets in Y. Since Y is normal, there exist disjoint $\hat{g}^{**}s$-open sets U and V such that $f(F_1) \subset U$ and $f(F_2) \subset V$. (ie)$F_1 \subset f^{-1}(U)$ and $F_2 \subset f^{-1}(V)$. Since f is strongly $\hat{g}^{**}s$- continuous, $f^{-1}(U)$ and $f^{-1}(V)$ are open sets containing the closed sets F_1 and F_2 in X. Also, $f^{-1}(U) \cap f^{-1}(V) = \emptyset$. Hence, X is clopen normal.

Theorem: 5.9: Let f be pre semi closed and $\hat{g}^{*}$-irresolute function then for every $\hat{g}^{**}s$-closed set A in X, f(A) is $\hat{g}^{**}s$-closed in Y.

Proof: Let A be $\hat{g}^{**}s$-closed in X. to prove f(A) is $\hat{g}^{**}s$-closed in Y it is enough to prove that $scl(f(A)) \subseteq U$. Assume$f(A) \subseteq U$, where U is $\hat{g}^{*}$-open in Y implies $A \subseteq f^{-1}(U)$, where $f^{-1}(U)$ is $\hat{g}^{*}$-open in X. Since, A is $\hat{g}^{**}s$-closed in X implies $scl(A) \subseteq f^{-1}(U) \Rightarrow f\big(scl(A)\big) \subseteq U \Rightarrow scl\big(f(A)\big) \subseteq U$. Therefore, f(A) is $\hat{g}^{**}s$-closed in Y.

Theorem: 5.10: Let f be a perfectly $\hat{g}^{**}s$-continuous and injective function and Y is $\hat{g}^{**}s - T_1$ space, then X is clopen T_1.

Proof: Let x and y be two distinct points of X implies $f(x) \neq f(y)$ in Y. Since Y is $\hat{g}^{**}s - T_1$, there exist $\hat{g}^{**}s$-open sets U and V such that $f(x) \in U$ and $f(y) \in V \Rightarrow x \in f^{-1}(U)$ and $y \in f^{-1}(V)$. Since f is perfectly continuous, $f^{-1}(U)$ and $f^{-1}(V)$ are clopen in X containing x and y, then X is clopen T_1.

Theorem: 5.11: Let f be a strongly $\hat{g}^{**}s$-continuous and injective function and Y is $\hat{g}^{**}s - T_1$ space, then X is T_1.

Proof: Let x and y be two distinct points of X implies $f(x) \neq f(y)$ in Y. Since Y is $\hat{g}^{**}s - T_1$, there exist $\hat{g}^{**}s$-open sets U and V such that $f(x) \in U$ and $f(y) \in V \Rightarrow x \in f^{-1}(U)$ and $y \in f^{-1}(V)$. Since f is strongly $\hat{g}^{**}s$- continuous, $f^{-1}(U)$ and $f^{-1}(V)$ are open in X containing x and y, then X is T_1.

Theorem: 5.12: If f is a contra $\hat{g}^{**}$s- continuous and injective function and Y is T_2, then X is $\hat{g}^{**}s - T_2$.

Proof: Let x and y be two distinct points of X implies f(x) ≠ f(y) in Y. Since Y is T_2, there exist disjoint open sets U and V such that f(x) ∈ U and f(y) ∈ V ⇒ x ∈ $f^{-1}(U)$ and y ∈ $f^{-1}(V)$. Since f is contra $\hat{g}^{**}$s- continuous, $f^{-1}(U)$ and $f^{-1}(V)$ are $\hat{g}^{**}$s-closed in X containing x and y, then X is $\hat{g}^{**}s - T_2$.

Theorem: 5.13: If f is a totally $\hat{g}^{**}$s- continuous and injective function and Y is T_2, then X is $\hat{g}^{**}s -$ clopen T_2.

Proof: Let x and y be two distinct points of X implies f(x) ≠ f(y) in Y. Since Y is T_2, there exist disjoint open sets U and V such that f(x) ∈ U and f(y) ∈ V ⇒ x ∈ $f^{-1}(U)$ and y ∈ $f^{-1}(V)$. Since f is totally $\hat{g}^{**}$s- continuous, $f^{-1}(U)$ and $f^{-1}(V)$ are $\hat{g}^{**}$s-clopen in X containing x and y, then X is $\hat{g}^{**}s -$ clopen T_2.

Proposition: 5.14: A function $f: X \to Y$ is totally $\hat{g}^{**}$s- continuous iff inverse image of every closed set in(Y, σ) is $\hat{g}^{**}$s-clopen in (X, τ).

Proof: Assume f is totally $\hat{g}^{**}$s- continuous. Let A be closed in $(Y, \sigma) \Rightarrow A^c$ is open in $(Y, \sigma) \Rightarrow f^{-1}(A^c)$ is $\hat{g}^{**}$s-clopen in $(X, \tau) \Rightarrow (f^{-1}(A)^c$ is $\hat{g}^{**}$s-clopen in $(X, \tau) \Rightarrow f^{-1}(A)$ is $\hat{g}^{**}$s-clopen in X. Conversely, let V be open in $(Y, \sigma) \Rightarrow V^c$ is closed in $(Y, \sigma) \Rightarrow f^{-1}(V^c)$ is $\hat{g}^{**}$s-clopen in $(X, \tau) \Rightarrow f^{-1}(V)$ is $\hat{g}^{**}$s-clopen in (X, τ). Hence, f is totally $\hat{g}^{**}$s- continuous.

Proposition: 5.15: A function $f: X \to Y$ is strongly $\hat{g}^{**}$s- continuous iff inverse image of every $\hat{g}^{**}$s-closed set in(Y, σ) is closed in (X, τ).

Proof: Assume f is strongly $\hat{g}^{**}$s- continuous. Let A be $\hat{g}^{**}$s-closed in $(Y, \sigma) \Rightarrow A^c$ is $\hat{g}^{**}$s-open in $(Y, \sigma) \Rightarrow f^{-1}(A^c)$ is open in $(X, \tau) \Rightarrow (f^{-1}(A)^c$ is open in $(X, \tau) \Rightarrow f^{-1}(A)$ is closed in X. Conversely, let V be $\hat{g}^{**}$s-open in $(Y, \sigma) \Rightarrow V^c$ is $\hat{g}^{**}$s-closed in $(Y, \sigma) \Rightarrow f^{-1}(V^c)$ is closed in $(X, \tau) \Rightarrow f^{-1}(V)$ is open in (X, τ). Hence, f is strongly $\hat{g}^{**}$s-continuous.

Proposition: 5.16: If the topological space (X, τ) is $\hat{g}^{**}$s –clopen and (Y, σ) is clopen, then every totally $\hat{g}^{**}$s-continuous is $\hat{g}^{**}$s-continuous and vise-versa.

Proof: Assume that f is $\hat{g}^{**}$s-continuous. Let V be open in $(Y, \sigma) \Rightarrow$ V is closed in Y $\Rightarrow f^{-1}(V)$ is $\hat{g}^{**}$s-closed in $(X, \tau) \Rightarrow f^{-1}(V)$ is $\hat{g}^{**}$s-open in $(X, \tau) \Rightarrow f^{-1}(V)$ is $\hat{g}^{**}$s-clopen in (X, τ).

Conversely, assume that f is totally $\hat{g}^{**}$s-continuous. Let V be closed in $(Y, \sigma) \Rightarrow$ V is open in $(Y, \sigma) \Rightarrow f^{-1}(V)$ is $\hat{g}^{**}$s-clopen in $(X, \tau) \Rightarrow f^{-1}(V)$ is $\hat{g}^{**}$s-closed in (X, τ).

Proposition: 5.17: A function $f: X \to Y$ is contra $\hat{g}^{**}$s- continuous iff inverse image of every closed set in(Y, σ) is $\hat{g}^{**}$s-open in (X, τ).

Proof: Assume f is contra $\hat{g}^{**}$s- continuous. Let A be closed in $(Y, \sigma) \Rightarrow A^c$ is open in $(Y, \sigma) \Rightarrow f^{-1}(A^c)$ is $\hat{g}^{**}$s-closed in $(X, \tau) \Rightarrow (f^{-1}(A)^c$ is $\hat{g}^{**}$s-closed in $(X, \tau) \Rightarrow f^{-1}(A)$ is $\hat{g}^{**}$s-open in X. Conversely, let V be open in $(Y, \sigma) \Rightarrow V^c$ is closed in $(Y, \sigma) \Rightarrow f^{-1}(V^c)$ is $\hat{g}^{**}$s-open in $(X, \tau) \Rightarrow f^{-1}(V)$ is $\hat{g}^{**}$s-closed in (X, τ). Hence, f is contra $\hat{g}^{**}$s- continuous.

Proposition: 5.18: A function $f: X \to Y$ is perfectly $\hat{g}^{**}$s- continuous iff inverse image of every $\hat{g}^{**}$s-closed set in(Y, σ) is clopen in (X, τ).

Proof: Assume f is perfectly $\hat{g}^{**}$s- continuous. Let A be $\hat{g}^{**}$s-closed in $(Y, \sigma) \Rightarrow A^c$ is $\hat{g}^{**}$s-open in $(Y, \sigma) \Rightarrow f^{-1}(A^c)$ is clopen in $(X, \tau) \Rightarrow f^{-1}(A)$ is clopen in X. Conversely, let V be $\hat{g}^{**}$s-open in $(Y, \sigma) \Rightarrow V^c$ is $\hat{g}^{**}$s-closed in $(Y, \sigma) \Rightarrow f^{-1}(V^c)$ is clopen in $(X, \tau) \Rightarrow f^{-1}(V)$ is clopen in (X, τ). Hence, f is perfectly $\hat{g}^{**}$s- continuous.

Theorem: 5.19: If X is T_{s^*}, then every $\hat{g}^{**}$s-continuous function is irresolute function.

Proof: Let V be closed in Y⇒ $f^{-1}(V)$ is $\hat{g}^{**}$s-closed in X⇒ $f^{-1}(V)$ is $(sg)^*$-closed in X⇒ $f^{-1}(V)$ is closed in X.

Theorem: 5.20: If X is strongly semi-$T_{1/2}$ and Y is T_b^*, then every $\hat{g}^{**}$s-continuous function is g^*s-irresolute function.

Proof: Let V be g^*s-closed in Y. Since Y is T_b^*, V is closed in Y⇒ $f^{-1}(V)$ is $\hat{g}^{**}$s-closed in $X \Rightarrow f^{-1}(V)$ is gs-closed in X. Since X is strongly semi-$T_{1/2}$, $f^{-1}(V)$ is g^*s-closed in X.

Theorem: 5.21: If X is T_c and Y is $T_{1/2}^*$, then every $\hat{g}^{**}$s-continuous function is g^*-irresolute function.

Proof: Let V be g^*-closed in Y. Since Y is $T_{1/2}^*$, V is closed in Y⇒ $f^{-1}(V)$ is $\hat{g}^{**}$s-closed in $X \Rightarrow f^{-1}(V)$ is gs-closed in X. Since X is T_c, $f^{-1}(V)$ is g^*-closed in X.

Theorem: 5.22: If X is T_s^* and Y is T_b, then every $(sg)^*$-continuous function is $\hat{g}^{**}$s-irresolute function.

Proof: Let V be $\hat{g}^{**}$s-closed in Y ⇒ V is gs-closed in Y. Since Y is T_b, V is closed in Y ⇒ $f^{-1}(V)$ is $(sg)^*$-closed in X. Since X is T_s^*, $f^{-1}(V)$ is closed in X. T_c, $f^{-1}(V)$ is $\hat{g}^{**}$s-closed in X.

Theorem: 5.23: If X and Y are both T_b-space, then every gs-continuous function is $\hat{g}^{**}$s-irresolute function.

Proof: Let V be $\hat{g}^{**}$s-closed in Y⇒ V is gs-closed in Y. Since Y is T_b, V is closed in Y⇒ $f^{-1}(V)$ is gs-closed in X. Since X is T_b, $f^{-1}(V)$ is closed inX ⇒ $f^{-1}(V)$ is $\hat{g}^{**}$s-closed in X.

Theorem: 5.24: If Y is T_b, every $\hat{g}^{**}$s-continuous function is $\hat{g}^{**}$*s*-irresolute function.

Proof: Let V be $\hat{g}^{**}$s-closed in Y⇒ V is gs-closed in Y. Since Y is T_b, V is closed in Y⇒ $f^{-1}(V)$ is $\hat{g}^{**}$s-closed in (X, τ).

Theorem: 5.25: A function f is $\hat{g}^{**}$s-continuous iff it is $\hat{g}^{**}$s-weakly continuous.

Proof: Let V be open in Y⇒ V^c is closed in Y⇒ $f^{-1}(V^c)$ is $\hat{g}^{**}$s-closed in X⇒ $f^{-1}(V)$ is $\hat{g}^{**}$s-open in X. Conversely, Let V be closed in Y⇒ V^c is open in Y⇒ $f^{-1}(V^c)$ is $\hat{g}^{**}$s-open in X⇒ $f^{-1}(V)$ is $\hat{g}^{**}$s-closed in X.

Theorem: 5.26: Every strongly $\hat{g}^{**}$s- continuous function is weakly $\hat{g}^{**}$s- continuous.

Proof: Let V be open in Y⇒ V is $\hat{g}^{**}$s-open in Y⇒ $f^{-1}(V)$ is open in X⇒ $f^{-1}(V)$ is $\hat{g}^{**}$s-open in X.

Theorem: 5.27: If X is T_b and Y is $T_{\hat{b}}^{**}$, then every weakly $\hat{g}^{**}$s- continuous function is strongly $\hat{g}^{**}$s-continuous function.

Proof: Let V be $\hat{g}^{**}$s-open in Y⇒ V^c is $\hat{g}^{**}$s-closed in Y. Since Y is $T_{\hat{b}}^{**}$, V^c is closed in Y and so V is open in Y⇒ $f^{-1}(V)$ is $\hat{g}^{**}$s-open in X⇒ $f^{-1}(V)$ is gs-open in X. Since X is a T_b-space, $f^{-1}(V)$ is open in X.

Proposition: 5.28: For any bijection $f: X \longrightarrow Y$, the following are equivalent:

i) $f^{-1}: Y \longrightarrow X$ is $\hat{g}^{**}$s-continuous
ii) f is $\hat{g}^{**}$s-open
iii) f is $\hat{g}^{**}$s-closed.

Proof: i) ⇒ ii) Let F be open in X⇒ $X - F$ is closed in X ⇒ $(f^{-1})^{-1}(X - F)$ is $\hat{g}^{**}$s-closed in Y ⇒ $f(X - F)$ is $\hat{g}^{**}$s-closed in Y ⇒ $Y - f(F)$ is $\hat{g}^{**}$s-closed ⇒ f(F) is $\hat{g}^{**}$s-open ⇒ f is $\hat{g}^{**}$s-open.

ii) ⇒ iii) Let F be closed in X⇒ $X - F$ is open in X ⇒ $f(X - F)$ is $\hat{g}^{**}$s-open in Y ⇒ $Y - f(F)$ is $\hat{g}^{**}$s-open ⇒ f(F) is $\hat{g}^{**}$s-closed.

iii) ⇒ i) Let V be closed in X ⇒ f(V) is $\hat{g}^{**}$s-closed in Y ⇒ $(f^{-1})^{-1}(V)$ is $\hat{g}^{**}$s-closed in Y ⇒ f^{-1} is $\hat{g}^{**}$s-continuous.

Theorem: 5.29: Every totally $\hat{g}^{**}$s-continuous function is slightly $\hat{g}^{**}$s-continuous.

Proof: Let $f: X \rightarrow Y$ be a totally $\hat{g}^{**}$s-continuous function. Let V be clopen in Y, then V is open in Y. By assumption, $f^{-1}(V)$ is $\hat{g}^{**}$s-clopen in X. Hence, $f^{-1}(V)$ is $\hat{g}^{**}$s-open in X.

Theorem: 5.30: Every perfectly $\hat{g}^{**}$s-continuous is strongly $\hat{g}^{**}$s-continuous.

Proof: Let $f: X \rightarrow Y$ be a perfectly $\hat{g}^{**}$s-continuous function. Let V be $\hat{g}^{**}$s-open in Y, then $f^{-1}(V)$ is clopen in X. Hence $f^{-1}(V)$ is open in X.

Theorem: 5.31: Every strongly $\hat{g}^{**}$s-continuous is $\hat{g}^{**}$s-continuous.

Proof: Let $f: X \rightarrow Y$ be a strongly $\hat{g}^{**}$s-continuous function. Let V be closed in Y, then V is $\hat{g}^{**}$s-closed in Y, which implies V^c is $\hat{g}^{**}$s-open in Y. By hypothesis, $f^{-1}(V^c)$ is open in X which implies that $(f^{-1}(V))^c$ is open in X and so, $f^{-1}(V)$ is closed in X. Hence, $f^{-1}(V)$ is $\hat{g}^{**}$s-closed in X.

Theorem: 5.32: Every perfectly $\hat{g}^{**}$s-continuous function is totally $\hat{g}^{**}$s-continuous.

Proof: Let $f: X \rightarrow Y$ be a perfectly $\hat{g}^{**}$s-continuous function. Let V be open in Y, which implies that V is $\hat{g}^{**}$s-open in Y. By given condition, $f^{-1}(V)$ is clopen in X. Hence, $f^{-1}(V)$ is $\hat{g}^{**}$s-clopen in X.

Theorem: 5.33: Every strongly $\hat{g}^{**}$s-continuous function is slightly $\hat{g}^{**}$s-continuous.

Proof: Let $f: X \rightarrow Y$ be a strongly $\hat{g}^{**}$s-continuous function. Let V be clopen in Y, and so, V is $\hat{g}^{**}$s-open in Y. By hypothesis, $f^{-1}(V)$ is open in X. Hence, $f^{-1}(V)$ is $\hat{g}^{**}$s-open in X.

Theorem: 5.34: Every totally $\hat{g}^{**}$s-continuous function is contra $\hat{g}^{**}$s- continuous.

Proof: Let $f: X \rightarrow Y$ be a totally $\hat{g}^{**}$s-continuous function. Let V be open in Y. By hypothesis, $f^{-1}(V)$ is $\hat{g}^{**}$s-clopen in X, so $f^{-1}(V)$ is $\hat{g}^{**}$s-closed in X.

Theorem: 5.35: Every perfectly $\hat{g}^{**}$s-continuous function is perfectly continuous.

Proof: Let $f: X \rightarrow Y$ be a perfectly $\hat{g}^{**}$s-continuous function. Let V be open in Y, clearly V is $\hat{g}^{**}$s-open in Y, so by hypothesis, $f^{-1}(V)$ is clopen in X.

Theorem: 5.36: Every totally continuous function is totally $\hat{g}^{**}$s-continuous.

Proof: Let $f: X \rightarrow Y$ be totally continuous. Let V be open in Y, then by given condition, $f^{-1}(V)$ is clopen in X, which implies that $f^{-1}(V)$ is $\hat{g}^{**}$s-clopen in X.

Theorem: 5.37: Every strongly continuous function is strongly $\hat{g}^{**}$s-continuous.

Proof: Let $f: X \rightarrow Y$ be strongly continuous. Let V be clopen in Y, which implies V is $\hat{g}^{**}$s-open in Y, therefore, $f^{-1}(V)$ is open in X.

Theorem: 5.38: Every contra continuous is contra $\hat{g}^{**}$s-continuous.

Proof: Let $f: X \rightarrow Y$ be contra continuous. Let V be open in Y, so $f^{-1}(V)$ is closed in X, therefore, $f^{-1}(V)$ is $\hat{g}^{**}$s-closed in X.

Theorem: 5.39: Every strongly $\hat{g}^{**}$s-continuous function is weakly $\hat{g}^{**}$s-continuous.

Proof: Let $f: X \rightarrow Y$ be strongly $\hat{g}^{**}$s-continuous. Let V be open in Y, so V is $\hat{g}^{**}$s-open in Y.
By hypothesis, $f^{-1}(V)$ is open in X, which shows $f^{-1}(V)$ is $\hat{g}^{**}$s-open in X.

Theorem: 5.40: Every perfectly $\hat{g}^{**}$s-continuous is weakly $\hat{g}^{**}$s-continuous.

Proof: Let $f: X \rightarrow Y$ be perfectly $\hat{g}^{**}$s-continuous and V be open in Y, then V is $\hat{g}^{**}$s-open in Y. By given condition, $f^{-1}(V)$ is clopen in X, so $f^{-1}(V)$ is $\hat{g}^{**}$s-open in X.

Theorem: 5.41: Every weakly $\hat{g}^{**}$s-continuous function is slightly $\hat{g}^{**}$s-continuous.

Proof: Let $f: X \rightarrow Y$ be weakly $\hat{g}^{**}$s-continuous and V be clopen in Y, so V is open in Y. Then, by hypothesis, $f^{-1}(V)$ is $\hat{g}^{**}$s-open in X.

Theorem: 5.42: Every totally $\hat{g}^{**}$s-continuous function is weakly $\hat{g}^{**}$s-continuous.

Proof: Let $f: X \rightarrow Y$ be totally $\hat{g}^{**}$s-continuous and V is open in Y. By hypothesis, $f^{-1}(V)$ is $\hat{g}^{**}$s-clopen in X, so $f^{-1}(V)$ is $\hat{g}^{**}$s-open in X.

Theorem: 5.43: If (Y, σ) is clopen in a function $f: X \rightarrow Y$, then every contra $\hat{g}^{**}$s-continuous function is $\hat{g}^{**}$s-continuous and vice versa.

Proof: Assume f is contra $\hat{g}^{**}$s-continuous and V is closed in Y. Since, Y is clopen V is open in Y, then $f^{-1}(V)$ is $\hat{g}^{**}$s-closed in X. Conversely, assume f is

$\hat{g}^{**}$s-continuous and V is open in Y, Since, Y is clopen V is closed in Y. By hypothesis, $f^{-1}(V)$ is $\hat{g}^{**}$s-closed in X.

Theorem: 5.44: If (X,τ) is clopen, then strongly $\hat{g}^{**}s$-continuous is totally $\hat{g}^{**}s$-continuous.

Proof: Let V be open in $(Y,\sigma) \Rightarrow$ V is $\hat{g}^{**}$s-open in (Y,σ). By hypothesis, $f^{-1}(V)$ is open in $(X,\tau) \Rightarrow f^{-1}(V)$ is closed in $(X,\tau) \Rightarrow f^{-1}(V)$ is clopen in $(X,\tau) \Rightarrow f^{-1}(V)$ is $\hat{g}^{**}$s-clopen in (X,τ).

Theorem: 5.45: If (X,τ) is clopen, then slightly $\hat{g}^{**}$s-continuous is $\hat{g}^{**}$s-continuous.

Proof: Let V be clopen in $(Y,\sigma) \Rightarrow$ V is closed in (Y,σ). By hypothesis, $f^{-1}(V)$ is $\hat{g}^{**}$s -closed in $(X,\tau) \Rightarrow f^{-1}(V)$ is $\hat{g}^{**}$s-open in (X,τ).

Theorem: 5.46: If (X,τ) is clopen, then strongly $\hat{g}^{**}$s-continuous is perfectly $\hat{g}^{**}$s-continuous.

Proof: Let V be $\hat{g}^{**}$s -open in (Y,σ). By hypothesis, $f^{-1}(V)$ is open in $(X,\tau) \Rightarrow f^{-1}(V)$ is closed in $(X,\tau) \Rightarrow f^{-1}(V)$ is clopen in (X,τ).

Theorem: 5.47: Let $f:(X,\tau) \to (Y,\sigma)$ and $g:(Y,\sigma) \to (Z,\eta)$ be two functions. Then

i) $g \circ f$ is strongly continuous if f is perfectly $\hat{g}^{**}$s-continuous and g is slightly $\hat{g}^{**}$s-continuous.

ii) $g \circ f$ is contra $\hat{g}^{**}$s-continuous if f is $\hat{g}^{**}$s -irresolute and g is contra $\hat{g}^{**}$s-continuous.

iii) $g \circ f$ is strongly $\hat{g}^{**}$s-continuous if f is strongly continuous and perfectly $\hat{g}^{**}$s-continuous.

iv) $g \circ f$ is contra $\hat{g}^{**}$s-continuous if f is $\hat{g}^{**}$s-irresolute and g is totally $\hat{g}^{**}$s-continuous.

v) $g \circ f$ is slightly $\hat{g}^{**}$s-continuous if f is totally $\hat{g}^{**}$s-continuous and g is strongly continuous.

vi) $g \circ f$ is totally $\hat{g}^{**}$s-continuous if f is totally $\hat{g}^{**}$s-continuous and g is totally continuous.

vii) $g \circ f$ is perfectly $\hat{g}^{**}$s-continuous if g is strongly $\hat{g}^{**}$s-continuous and f is totally continuous.

viii) $g \circ f$ is totally $\hat{g}^{**}$s-continuous and contra $\hat{g}^{**}$s-continuous if f is perfectly $\hat{g}^{**}$s-continuous and g is $\hat{g}^{**}$s-continuous.

Proof: i) Let V be clopen in (Z,η). Since g is slightly $\hat{g}^{**}$s-continuous, $g^{-1}(V)$ is $\hat{g}^{**}$s-open in (Y,σ). Also, given that f is perfectly $\hat{g}^{**}$s-continuous, $f^{-1}\left(g^{-1}(V)\right)$ is clopen in (X,τ). Hence $(g \circ f)^{-1}(V)$ is open in (X,τ).

ii) Let V be open in (Z,η). Since g is contra $\hat{g}^{**}$s -continuous, $g^{-1}(V)$ is $\hat{g}^{**}$s-closed in (Y,σ). Also, given that f is $\hat{g}^{**}$s-irresolute, $f^{-1}\left(g^{-1}(V)\right)$ is $\hat{g}^{**}$s-closed in (X,τ). Hence $(g \circ f)^{-1}(V)$ is $\hat{g}^{**}$s-closed in (X,τ).

iii) Let V be $\hat{g}^{**}$s -open in (Z,η). Since g is perfectly $\hat{g}^{**}$s -continuous, $g^{-1}(V)$ is clopen in (Y,σ). Also, given that f is strongly continuous, $f^{-1}\left(g^{-1}(V)\right)$ is open in (X,τ). Hence $(g \circ f)^{-1}(V)$ is open in (X,τ).

iv) Let V be open in (Z,η). Since g is totally $\hat{g}^{**}$s-continuous, $g^{-1}(V)$ is $\hat{g}^{**}$s -clopen in (Y,σ). $g^{-1}(V)$ is $\hat{g}^{**}$s -closed in (Y,σ). Also, given that f is $\hat{g}^{**}$s-irresolute $f^{-1}\left(g^{-1}(V)\right)$ is $\hat{g}^{**}$s -closed in (X,τ). Hence $(g \circ f)^{-1}(V)$ is $\hat{g}^{**}$s -closed in (X,τ).

v) Let V be clopen in (Z,η). Since g is strongly continuous, $g^{-1}(V)$ is open in (Y,σ). Also, given that f is totally $\hat{g}^{**}$s-continuous, $f^{-1}\left(g^{-1}(V)\right)$ is $\hat{g}^{**}$s -clopen in (X,τ). Hence $(g \circ f)^{-1}(V)$ is $\hat{g}^{**}$s -open in (X,τ).

vi) Let V be open in (Z,η). Since g is totally continuous, $g^{-1}(V)$ is clopen in (Y,σ) and so, $g^{-1}(V)$ is clopen in (Y,σ). Also, given that f is totally $\hat{g}^{**}$s -continuous, $f^{-1}\left(g^{-1}(V)\right)$ is $\hat{g}^{**}$s-clopen in (X,τ). Hence $(g \circ f)^{-1}(V)$ is $\hat{g}^{**}$s-clopen in (X,τ).

vii) Let V be $\hat{g}^{**}$s-open in (Z,η). Since g is strongly $\hat{g}^{**}$s -continuous, $g^{-1}(V)$ is open in (Y,σ). Also, given that f is totally continuous, $f^{-1}\left(g^{-1}(V)\right)$ is clopen in (X,τ). Hence $(g \circ f)^{-1}(V)$ is clopen in (X,τ).

viii) Let V be open in $(Z,\eta) \Rightarrow V^c$ is closed in (Z,η). Since g is $\hat{g}^{**}$s-continuous, $g^{-1}(V^c)$ is $\hat{g}^{**}$s -closed in (Y,σ). $g^{-1}(V)$ is $\hat{g}^{**}$s -open in (Y,σ). Also, given that f is perfectly $\hat{g}^{**}$s-continuous, $f^{-1}\left(g^{-1}(V)\right)$ is clopen in (X,τ). Hence $(g \circ f)^{-1}(V)$ is $\hat{g}^{**}$s -clopen in (X,τ). Therefore $g \circ f$ is totally $\hat{g}^{**}$s-continuous. Also, $(g \circ f)^{-1}(V)$ is $\hat{g}^{**}$s -closed in (X,τ) which implies that $g \circ f$ is contra $\hat{g}^{**}$s-continuous. Hence proved.

Remark: 5.49:

The following diagram shows the relationship of various continuous functions.

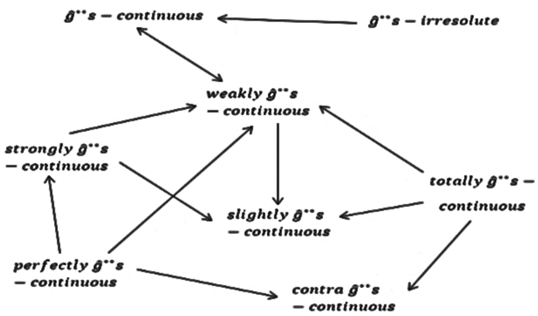

Figure:2

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