STRONGLY PRIME LABELING FOR SOME SPECIAL GRAPHS

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Abstract

A graph $G = (V(G), E(G))$ is said to have a prime labeling if its vertices can be labeled with distinct positive integers, such that the label of each pair of adjacent vertices are relatively prime. A graph which admits prime labeling is called a prime graph and a graph G is said to be strongly prime graph if for any vertex, v of G there exists a prime labeling, f satisfying $f(v) = 1$. In this paper we investigate strongly prime labeling of some graphs.

Keywords: Prime Labeling, Prime graph, Strongly Prime Graph.

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1. Introduction

We begin with simple, finite, undirected graph $G = (V(G), E(G))$ with $V(G)$ the vertex set and $E(G)$ the edge set. The set of adjacent vertices of a vertex u of G is denoted by $N(u)$. For all other terminology and notations we refer to Bondy and Murthy[3]. We provide brief summary of definitions which are useful for the present work.

Definition 1.1 If the vertices of the graph are assigned values subject to certain conditions then it is known as graph labeling.

Graph labeling is one of the interesting areas of graph theory with wide range of applications. A systematic study on various applications of graph labeling is carried out in Bloom and Golomb[2]. Graph labeling serves as frontier between number theory and structure of graphs according to Beineke and Hedge[1]. A detailed survey of graph labeling is explained in [4]. A Dynamic Survey of Graph Labeling by Gallian.

Definition 1.2 Let $G = (V(G), E(G))$ be a graph with p vertices. A bijection$f: V(G) \rightarrow \{12, \ldots, p\}$ is called a prime labeling if for each edge $e = uv$, $gcd\{f(u), f(v)\} = 1$. A graph which admits prime labeling is called a prime graph.

The prime labeling notion was originated by Entringer and was discussed in a paper by Tout et al[10]. Many researchers have studied prime labeling of some families of graphs. Entringer proved that all trees have prime labeling. Prime labeling of some classes of graph were discussed by S.K. Vaidya and Udayan M Prajapati in [11]. Some graph operations on prime labeling was discussed by S. Meena and K. Vaithiligam[6]. S. Meena and P. Kavitha have proved strongly prime labeling for some classes of graphs[7].

Definition 1.3 A graph G is said to be a strongly prime graph if for any vertex, v of G there exists a prime labeling, f satisfying, $f(v) = 1$.

Samir K Vaidya and Udayan M Prajapati[12] have introduced the concept of strongly prime graph and proved that the graphs $C_n, P_n, K_{1,n}$ and W_n for every even integer $n \geq 4$ are strongly prime graphs. S. Meena and P. Kavitha[8] have proved the graphs corona of triangular snake, corona of ladder graph, corona of quadrilateral snake and a graph obtained by attaching P_2 at each vertex of outer cycle of Prism D_n by $(D_n; P_2)$, Helm, Gear graph are strongly prime graphs.J. Suresh Kumar and Sarika M. Nair[9] proved the umbrella and flower pot graphs are strongly prime.

Definition 1.4 The sunflower graph, SFl_n, is defined as a graph obtained by starting with an n-cycle C_n with a consecutive vertices $v_1, v_2, \dots, v_n$ and creating new vertices $u_1, u_2, \dots u_n$ with u_i connected to v_i and v_{i+1}. The graph SFl_n has number of vertices $p = 2n$ and a number of edges $q = 3n$.

Definition 1.5 The helm H_n is a graph obtained from a wheel by attaching a pendant edge at each vertex of the n-cycle.

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Theorem 2.1
The graphG $\odot\overline{K}_2$ is a strongly prime graph where G = C_n for all integer n ≥ 2, where n is even.

Proof
Let $\{v_1, v_{2,\dots,} v_n\}$ be a cycle of length n. Let x_i, y_i be the vertices joined to v_i,$1 \leq i \leq n$. The resulting graph is G_1 (i,e) $G \odot K_2$ where $G = C_n$ graph.

Now the vertex set
$V(G_1) = \{v_1, v_{2,\dots,} v_n, x_1, x_{2,\dots,} x_n, y_1, y_2, \dots, y_n\}$
and the edge set
$E(G_1) = \{v_i v_{i+1}/1 \leq i \leq n-1\} \cup \{v_i x_i, v_i y_i \leq i \leq n\} \cup \{v_1 v_n\}$.
Let v be the vertex for which we assign label 1 in our labeling method. Then we have the following cases:

Case(i): Let $v = x_j$ for some $j \in$ {1,2,3,...n}, then the function $f: V(G) \to \{1,2,\dots,3n\}$ defined by

$$f(x_i) = \begin{cases} 3n + 3i - 3j + 1 & if\ i = 1,2,\dots,j-1; \\ 3i - 3j + 1 & if\ i = j, j+1,\dots,n\,; \end{cases}$$

$$f(v_i) = \begin{cases} 3n + 3i - 3j + 2 & if\ i = 1,2,\dots,j-1; \\ 3i - 3j + 2 & if\ i = j, j+1,\dots,n\,; \end{cases}$$

$$f(u_i) = \begin{cases} 3n + 3i - 3j + 3 & if i = 1,2,\dots,j-1; \\ 3i - 3j + 3 & if\ i = j, j+1,\dots,n\,; \end{cases}$$

Case(ii): Let $v = v_j$ for some $j \in$ {1,2,3,...n},then define a labeling f_2 using the labeling f defined in case (i) as follows: $f_2(x_j) = f(v_j), f_2(v_j) = f(x_j)$ and $f_2(v) = f(v)$ for all the remaining vertices. Then the resulting labeling f_2 is a prime labeling and it is possible to assign label 1 to any arbitrary vertex of $v = v_j$ in G_1.

Case(iii): Let $v = y_j$ for some $j \in$ {1,2,3,...n},then define a labeling f_3 using the labeling f defined in case (i) as follows: $f_3(x_j) = f(v_j), f_3(v_j) = f(x_j)$ and $f_3(v) = f(v)$ for all the remaining vertices. Then the resulting labeling f_3 is a prime labeling and it is possible to assign label 1 to any arbitrary vertex of $v = y_j$ in G_1.Thus in all the possibilities described above f admits prime labeling and it is possible to assign label 1 to any arbitrary vertex of G_1. Thus G_1 is a strongly prime graph.

Theorem 2.2
The graph $G_1 = (P_n \odot K_1) \cup (P_m \odot K_1)$ is strongly prime.

Proof
Let $\{u_1, u_2, \dots, u_n\}$ be a path of length n and $\{v_1, v_2, \dots, v_m\}$ be a path of length m. Let u_i', $1 \leq i \leq n$ be the vertex joined to u_i and v_i' be the vertex joined to v_i.The resulting graph is $G_1 = (P_n \odot K_1) \cup (P_m \odot K_1)$.
Now the vertex set of
V$(G) = \{u_1, u_2, \dots, u_n, v_1, v_2, \dots, v_m, u_1', u_2', \dots, u_n', v_1', v_2', \dots, v_m'\}$
and the edge set $E(G) = \{u_i u_{i+1}/1 \leq i \leq n-1\} \cup \{v_i v_{i+1}/1 \leq i \leq m-1\} \cup \{u_i u_i'/1 \leq i \leq n\} \cup \{v_i v_i'/1 \leq i \leq m\}$.
Here $|V(G)| = 2n + 2m$. Let v be the vertex for which we assign label 1 in our labeling method. . Then we have the following cases:

Case(i): Let $v = u_j$ for some $j \in$ {1,2,3,...n}, then the function $f: (G) \to \{1,2,3,\dots,2m+2n\}$ defined by

$$f(u_i) = \begin{cases} 2n + 2i - 2j + 1 & if\ \ i = 1,2,\dots,j-1; \\ 2i - 2j + 1 & if\ i = j.j+1,\dots,n\,; \end{cases}$$

$$f(u_i') = \begin{cases} 2n+2i-2j+2 & if \quad i=1,2,\dots,j-1; \\ 2i-2j+2 & if\ i=j.j+1,\dots,n\ ; \end{cases}$$

$$f(v_i) = 2n+2i-1 \quad i=1,2,3,\dots,m;$$

$$f(v_i') = 2n+2i \qquad i=1,2,3,\dots,m;$$

is a prime labeling for G_1 with $f(v) = f(u_j) = 1$. Thus f is a prime labeling and it is also possible to assign label 1 to any arbitrary vertex of $v = u_j$ in G_1.

Case(ii): Let $v = u_j'$ for some $j \in \{1,2,3,\dots n\}$ then define a labeling f_2 using the labeling f defined in case(i) as follows: $f_2(u_j) = f(u_j'), f_2(u_j') = f(u_j)$ and $f_2(v) = f(v)$ for all the remaining vertices. Then the resulting labeling f_2 is a prime labeling and it is possible to assign label 1 to any arbitrary vertex of $v = u_j'$ in G_1.

Case(iii): Let $v = v_j$ for some $j \in \{1,2,3,\dots n\}$ then the function $f:(G) \to \{1,2,3,\dots,2m+2n\}$ defined by

$$f_3(v_i) = \begin{cases} 2m+2i-2j+1 & if \quad i=1,2,\dots,j-1; \\ 2i-2j+1 & if\ i=j.j+1,\dots,m\ ; \end{cases}$$

$$f_3(v_i') = \begin{cases} 2m+2i-2j+2 & if\ i=1,2,\dots,j-1; \\ 2i-2j+2 & if\ i=j.j+1,\dots,m\ ; \end{cases}$$

$$f_3(u_i) = 2m+2i-1 \quad i=1,2,3,\dots,n;$$

$$f_3(u_i') = 2m+2i \qquad i=1,2,3,\dots,n;$$

Case(iv): Let $v = v_j'$ for some $j \in \{1,2,3,\dots m\}$ then define a labeling f_4 using the labeling f_3 defined in case(i) as follows: $f_4(v_j) = f_3(v_j')$, $f_4(v_j') = f_3(v_j)$ and $f_4(v) = f_3(v)$ for all the remaining vertices. Then the resulting labeling f_4 is a prime labeling and it is possible to assign label 1 to any arbitrary vertex of $v = v_j'$ in G_1. Thus from all the cases described above G_1 is a strongly prime graph.

Theorem 2.3
The graph $G \odot K_1$ is a strongly prime graph where $G = SFl_n$ for all integer $n \geq 3$.

Proof
Let $\{u_1, u_2, \dots, u_n\}$ be a cycle of length n. Let $v_i, 1 \leq i \leq n$ be the new vertex added to u_i and u_{i+1}. Let u_i' be the vertex joined to u_i and v_i' be the vertex joined to $v_i 1 \leq i \leq n$. The resulting graph we denote by G_1 (i.e) $G \odot K_1$ where $G = SFl(n)$ graph. Now the vertex set of $V(G_1) = \{u_1, u_{21} \dots, u_n, u_1', u_{2,\dots,}' u_n', v_1, v_2, \dots, v_n, v_1', v_{2,\dots,}' v_n'\}$ and the edge set $E(G_1) = \{u_i u_{i+1}, v_i u_{i+1} / 1 \leq i \leq n-1\} \cup \{u_i v_i, u_i u_i', v_i v_i' / 1 \leq i \leq n$. Let v be the vertex for which we assign label 1 in our labeling method. Then we have the following cases:

$$f(u_i) = \begin{cases} 4n+4i-4j+1 & if\ i=1,2,\dots,j-1; \\ 4i-4j+1 & if\ i=j.j+1,\dots,n\ ; \end{cases}$$

$$f(v_i) = \begin{cases} 4n+4i-4j+3 & if \quad i=1,2,\dots,j-1; \\ 4i-4j+3 & if\ i=j.j+1,\dots,n\ ; \end{cases}$$

$$f(u_i') = \begin{cases} 4n+4i-4j+2 & if \quad i=1,2,\dots,j-1; \\ 4i-4j+2 & if\ i=j.j+1,\dots,n\ ; \end{cases}$$

$$f(v_i') = \begin{cases} 4n+4i-4j+4 & if i=1,2,\dots,j-1; \\ 4i-4j+4 & if\ i=j.j+1,\dots,n\ ; \end{cases}$$

Case(ii): Let $v = u_j'$ for some $j \in \{1,2,3,\dots n\}$ then define a labeling f_2 using the labeling f defined in case(i) as follows: $f_2(u_j) = f(u_j'), f_2(u_j') = f(u_j)$ and $f_2(v) = f(v)$ for all the remaining vertices. Then the resulting labeling f_2 is a prime labeling and it is possible to assign label 1 to any arbitrary vertex of $v = u_j'$ in G_1.

Case(iii): Let $v = v_j$ for some $j \in \{1,2,3,\dots n\}$ then define a labeling f_3 using the labeling f defined in case(i) as follows: $f_3(v_i) = f(u_i), f_3(v_i') = f(u_i'), f_3(u_i) = f(v_i), f_3(u_i) = f(v_i')$ and $f_2(v) = f(v)$ for all the remaining vertices. Then the resulting labeling f_3 is a prime labeling and it is possible to assign label 1 to any arbitrary vertex of $v = v_j$ in G_1.

Case(iv): Let $v = v_j'$ for some $j \in \{1,2,3,\dots n\}$ then define a labeling f_4 using the labeling f_3 defined in case(i) as follows: $f_4(v_j') = f_3(v_j), f_4(v_j) = f_2(v_j')$ and $f_4(v) = f(v)$ for all the remaining vertices. Then the resulting labeling f_2 is a prime labeling and it is possible to assign label 1 to any arbitrary vertex of $v = v_j'$ in G_1. Thus from all the cases described above G_1 is a strongly prime graph.

Theorem 2.4
The graph obtained by fusing any two consecutive vertices other than the apex vertex in a helm graph H_n is a strongly prime graph when $(n+1) \not\equiv 0 (mod\, 3)$.

Proof
Let v_0 be the apex vertex $v_1, v_{2,\dots,} v_{n-1}$ be the consecutive rim vertices of H_n and $v_1', v_2', \dots, v_n'$ be the pendent vertices of H_n. Let v be the vertex for which we assign label 1 in our labeling method. Then we have the following cases:
Case(i): If v is the apex vertex $v = v_0$ then the function $f: V(H_n) \to \{1,2,\dots,2n\}$ defined as

$$f(v_0) = 1, f(v_1') = 2,$$

$$f(v_1 = v_2) = 3, f(v_2') = 4$$

$$f(v_i) = 2i - 1 \qquad if \quad 3 \le i \le n$$

$$f(v_i') = 2i \qquad if \quad 3 \le i \le n$$

then clearly f is an injection. For an arbitrary edge $e = xy$ of H_n we claim that $gcd(f(x), f(y)) = 1$.
Subcase(i): If $e = v_0 v_i$ for some $i \in \{1,2,\dots,n\}$ then $gcd(f(v_0), f(v_i)) = gcd(1, f(v_i)) = 1$.
Subcase(ii): If $e = v_i v_{i+1}$ for some $i \in \{1,2,\dots,n-1\}$ then $gcd(f(v_i), f(v_{i+1})) = gcd(2i-1, 2i+1) = 1$ as $2i-1$ and $2i+1$ are consecutive odd positive integers. If $e = v_1 v_{n-1}$ then $gcd(f(v_1), f(v_{n-1})) = gcd(3, 2n-3) = 1$.
Subcase(iii): If $e = v_i v_i'$ for some $i \in \{1,2,\dots,n\}$, then $gcd(f(v_i), f(v_i')) = gcd(2i-1, 2i) = 1$ as $2i-1$ and $2i$ are consecutive positive integers.

Case(ii): If $v = v_j$ for some $j \in \{1,2,\dots,n\}$, then define a labeling f_2 using the labeling f defined in case(i) as follows:$f_2(v_0) = f(v_1')$, $f_2(v_1 = v_2) = f(v_0), f_2(v_1') = f(v_1 = v_2)$ and $f_2(v) = f(v)$ for all the remaining vertices. Clearly f is an injection.
Subcase(i): If $e = v_0 v_i$ for some $i \in \{1,2,\dots,n\}$ then $gcd(f(v_0), f(v_i)) = gcd(2, 2i-1) = 1$ as $2i-1$ is an odd positive integer and it is not divisible by 2. When $e = v_i v_{i+1}$ for some $i \in \{2,3,\dots,n-1\}$ are as same as subcase(ii) in case(i). If $e = v_n v_1$ then $gcd(f(v_n), f(v_1)) = gcd(2n-1, 1) = 1$.
Subcase(ii): If $e = v_i v_i'$ for some $i \in \{2,3,\dots,n\}$ are as same as subcase(iii) in case(i). If $e = v_1 v_1'$ then $gcd(1,3) = 1$ and if $e = v_2 v_2'$ then $gcd(1,4) = 1$.

Case(iii): If $v = v_j$ for some $j \in \{1,2,\dots,n\}$, then define a labeling f_3 using the labeling f_2 defined in case(ii) as follows:$f_3(v_0) = f_2(v_0), f_3(v_1 = v_2) = f_2(v_1')$ and $f_3(v) = f_2(v)$ for all the remaining vertices. If $v = v_2'$ then $f_3(v_1') = f_2(v_2')$. Clearly f is an injection.
Subcase(i): If $e = v_0 v_i$ for some $i \in \{1,2,\dots,n\}$ are as same as subcase(i) in case(ii).
Subcase(ii): If $e = v_i v_{i+1}$ for some $i \in \{1,2,\dots,n-1\}$ are as same as subcase(ii) in case(i).
Subcase(iii): If $e = v_i v_i'$ for some $i \in \{1,2,\dots,n\}$, then $gcd(f(v_i), f(v_i')) = gcd(2i+1, 1) = 1$.
Thus in all the cases described above f is a prime labeling and it is possible to assign label 1 to any arbitrary vertex of H_n. Hence G is strongly prime graph.

Theorem 2.5
The graph $G \odot K_1$ is a strongly prime graph where $G = C_n(C_n)$ when $(n+1) \not\equiv 0 (mod\, 2)$ and for all integer $n \ge 2$.
Proof
Let u_i and v_i, $1 \le i \le n$ be vertices of the inner and outer cycle respectively and u_i' and v_i', $1 \le i \le n$ be their corresponding pendant vertices. The resulting graph is G_1. (i.e.) $G \odot K_1$ where $G = C_n(C_n)$ graph.
The vertex set
$V(G_1) = \{u_1, u_{2,\dots,} u_n, v_1, v_2, \dots, v_n, u_1', u_2', \dots, u_n', v_1', v_{2,\dots,}' v_n'\}$
and the edge set $E(G_1) = \{u_i v_i, u_i u_i', v_i v_i' / 1 \le i \le n\} \cup \{u_n u_1, v_n u_1\} \cup \{v_i u_{i+1}, u_i u_{i+1} / 1 \le i \le n-1\}$.Here $|V(G_1)| = 4n$. Let v be the vertex for which we assign label 1 in our labling method.
Then we have the following cases:

Case(i): Let $v = u_j$ for some $j \in \{1,2,\dots,n\}$ then the function $f: V(G_1) \to \{1,2,\cdots,4n\}$ defined by

$$f(u_i) = \begin{cases} 4n + 4i - 4j + 1 & if \ i = 1,2,\dots,j-1; \\ 4i - 4j + 1 & if \ i = j.j+1,\dots,n; \end{cases}$$

$$f(v_i) = \begin{cases} 4n + 4i - 4j + 3 & if \ i = 1,2,\dots,j-1; \\ 4i - 4j + 3 & if \ i = j.j+1,\dots,n; \end{cases}$$

$$f(u_i') = \begin{cases} 4n + 4i - 4j + 2 & if \ i = 1,2,\dots,j-1; \\ 4i - 4j + 2 & if \ i = j.j+1,\dots,n; \end{cases}$$

$$f(v_i') = \begin{cases} 4n + 4i - 4j + 4 & if \ i = 1,2,\dots,j-1; \\ 4i - 4j + 4 & if \ i = j.j+1,\dots,n; \end{cases}$$

Case(ii): Let $v = u_j'$ for some $j \in \{1,2,\dots,n\}$, then define a labeling f_2 using the labeling f

defined in case(i) as follows: $f_2(u_j') = f(u_j), f_2(u_j) = f(u_j')$ for $j \in \{1,2,\dots,n\}$ and $f_2(v) = f(v)$ for all the remaining vertices. Then the resulting labeling f_2 is a prime labeling and it is possible to assign label 1 to any arbitrary vertex of $v = u_j'$ in G_1.

Case(iii): Let $v = v_j$ for some $j \in \{1,2,\dots,n\}$, then define a labeling f_3 using the labeling f defined in case(i) as follows: $f_3(v_i) = f(u_i), f_3(u_i) = f(v_i), f_3(u_i') = f(v_i')$ and $f_3(v_i') = f(u_i')$ for $1 \le i \le n$ in G_1. Then the resulting labeling f_3 is a prime labeling and it is possible to assign label 1 to any arbitrary vertex of $v = v_j$ in G_1.

Case(iii): Let $v = v_j'$ for some $j \in \{1,2,\dots,n\}$, then define a labeling f_4 using the labeling f_3 defined in case(i) as follows: $f_4(v_j') = f_3(v_j), f_4(v_j) = f_3(v_j')$ for $j \in \{1,2,\dots,n\}$ and $f_4(v) = f_3(v)$ for all the remaining vertices. Then the resulting labeling f_4 is a prime labeling and it is possible to assign label 1 to any arbitrary vertex of $v = v_j'$ in G_1. Thus from all the above cases we can conclude that the graph G_1 is a strongly prime graph.

III. CONCLUSION

In this paper we showed that some families of graph like $C_n \odot \overline{K}_2$, $(P_n \odot K_1) \cup (P_m \odot K_1)$, $SFl_n \odot K_1$ and the graph obtained by fusing any two consecutive vertices other than the apex vertex in a helm graph H_n are all strongly prime.

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