INDEPENDENT TRANSVERSAL DOMINATING SETS OF $P_n{}^c$ AND $D(vP_n{}^c)$

Stephen John. B[1], Brainy. F[2]

[1]Associate Professor, P. G. and Research Department of Mathematics, Annai Velankanni College, Tholayavattam, Kanyakumari District - 629157,

[2]Research Scholar Full Time (Reg.No:19113012092004) P. G. and Research Department of Mathematics, Annai Velankanni College, Tholayavattam, Kanyakumari District - 629157,

Affiliated by Manonmaniam Sundaranar University, Abishekapatti, Tirunelveli District - 627012, TamilNadu, India

Email: [1]stephenjohn1963@gmail.com, [2]brainybrainy5@gmail.com

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Abstract

Domination is one of the most application oriented research area in Graph Theory. In this paper, we have found the independent domination number, independent transversal domination number and the corresponding sets to the complement of a path graph P_n and its duplication. Also we have established a general formula to find all sets.

Keywords: Domination number, Independent transversal domination number, Duplication.

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1. Introduction:

In Graph Theory, domination and dominating sets are used in several fields such as wireless networking and efficient routes with in ad-hoc mobile networks. [5] In 2012, the independent transversal dominating set was introduced by Hamid.

Definition: 1.1

A path is a simple graph whose vertices can be arranged in a linear sequence in such a way that two vertices are adjacent if they are consecutive in the sequence and are non-adjacent otherwise.

Definition: 1.2

Let G be a simple graph. The complement G^c of G is the simple graph whose vertex set is $V(G)$ and whose edges are the pairs of nonadjacent vertices of G.

Definition: 1.3

The sum of two graphs G_1 and G_2, denoted by $G_1 + G_2$ is the graph obtained by disjoint copies of G_1 and G_2 and then adding every edge xy whose $x \in V(G_1)$ and $y \in (G_2)$.

Definition: 1.4

Duplication of a vertex v of a graph G produces a new graph G' by adding a new vertex v' such that $N(v') = N(v)$. For a graph G, the graph obtained by duplication of all the vertices of G is denoted by $D(vG)$.

Definition: 1.5

The subset $D \subseteq V(G)$ in a graph G is called a dominating set if every vertex $v \in V(G)$ is either an element of D or is adjacent to at least one element in D. The cardinality of a minimum dominating set in G is called the domination number of G and is denoted by $\gamma(G)$.

Definition: 1.6

A set $I \subseteq V(G)$ is said to an independent set of G if no two elements of I are adjacent in G. A subset I of $V(G)$ is called an independent dominating set if I is both an independent and a dominating set of G. The minimum cardinality of an independent dominating set is called an independent domination number of G, denoted as $\gamma_i(G)$.

Definition: 1.7

A dominating set $D \subseteq V(G)$ is said to be an independent transversal dominating set if D intersect every maximum independent set of G. The minimum cardinality of an independent transversal dominating set of G is called the independent transversal domination number and is denoted by $\gamma_{it}(G)$.

2. Main Results:

Theorem: 2.1

Let $G = P_n{}^c$ be the complement of a path graph of order $n, n > 3$ then

(i) $\gamma(G) = 2$ and $|D(G,\gamma)| = \frac{n^2-3n+4}{2}$

(ii) $\gamma_i(G) = 2$ and $|D_i(G,\gamma)| = n - 1$

(iii) $\gamma_{it}(G) = \left[\frac{n}{2}\right]$ and

$$|D_{it}(G,\gamma)| = \begin{cases} 1 & \text{if } n \text{ is odd}, n > 5 \\ \frac{n}{2} + 1 & \text{if } n \text{ is even}, n \geq 4 \end{cases}.$$

Proof:

Given, $G = P_n{}^c$ is a complement of a path with n vertices and $\frac{(n-1)(n-2)}{2}$ edges.

$V(G) = \{u_i / 1 \leq i \leq n\}$ is the vertex set of G.

$N(u_i) = V(G) - \{u_{i-1}, u_{i+1}, u_i\}$, $1 < i < n$.

$N(u_{i+1}) = V(G) - \{u_i, u_{i+1}, u_{i+2}\}$, $1 \leq i < n-1$.

(i) Let $D_{1,l} = \{\{u_l, u_{l+1}\}\}$, $1 \leq l \leq n-1$. …… (1)

Hence, every pair of vertices $(u_l, u_{l+1}) \in V(G)$, $N[u_l] \cup N[u_{l+1}] = V(G)$ for all $1 \leq l \leq n-1$.

$\Longrightarrow \gamma(G) = 2$. The dominating sets of G with $\gamma = 2$ are,

$$D(G,\gamma) = \begin{cases} D_{1,l} & = \{\{u_l, u_{l+1}\}\} \\ {}_{1 \leq l \leq n-1} & \\ D_{i,j} & = \{\{u_i, u_{j+3}\}\} \\ {}_{1 \leq i \leq n-3,\ i \leq j \leq n-3} & \end{cases}$$

$|D(G,\gamma)| = n - 1 + [1 + 2 + \cdots + (n-3)]$
$= \frac{1}{2}[n^2 - 3n + 4]$.

(ii) Since, $u_l u_{l+1} \notin E(G)$ for all l, hence every set contained in $D_{1,l}$ / $1 \leq l \leq n-1$ is the required independent dominating sets of G.

Therefore, $\gamma_i(G) = 2$ and $|D_i(G,\gamma)| = n - 1$.

(iii) Let $I(G,\alpha)$ be the collection of maximum independent sets of G.

$I(G,\alpha) = I_{1,i} = \{\{u_i, u_{i+1}\}\}$, $1 \leq i < n$ …… (2)

If n is odd, $n > 3$,

Let $D_1 = \{u_{2i} / 1 \leq i \leq \left[\frac{n}{2}\right]\}$ …… (3)

$N[D_1] = V(G)$ and $|D_1| = \left[\frac{n}{2}\right]$.

From equations (2) and (3), every maximum independent set belongs to $I(G,\alpha)$ contains a vertex of D_1 . Therefore, D_1 is the required independent transversal dominating set of G with cardinality $\left[\frac{n}{2}\right]$. Hence, $\gamma_{it}(G) = \left[\frac{n}{2}\right]$ and $|D_{it}(G,\gamma)| = 1$.

If n is even, $n \geq 4$, let $D_2 = \left\{u_{2i-1} / 1 \leq i \leq \frac{n}{2}\right\}$; $N[D_2] = V(G)$. D_2 intersect all the maximum independent sets of G and $|D_2| = \left[\frac{n}{2}\right]$.

The independent transversal dominating sets of G with cardinality $\left[\frac{n}{2}\right]$ are,

$$D_{it}(G,\gamma) = \begin{cases} D_1 & = \left\{u_{2i} / 1 \leq i \leq \frac{n}{2}\right\} \\ D_2 & = \left\{u_{2i-1} / 1 \leq i \leq \frac{n}{2}\right\} \\ D_{1,j} & = \{u_{2t} / 1 \leq i \leq j\} \\ {}_{1 \leq j \leq \frac{n}{2}-1} & \cup \left\{u_{2r+1} / j \leq r \leq \frac{n}{2} - 1\right\} \end{cases}$$

$|D_{it}(G,\gamma)| = \frac{n}{2} + 1$.

Hence, the theorem proved.

Result: 2.2

(i) $\gamma(P_n) \geq \gamma(P_n^c)$, $n > 3$

(ii) $\gamma_i(P_n) \geq \gamma_i(P_n^c)$, $n > 3$

(iii) $\gamma_{it}(P_n) \leq \gamma_{it}(P_n^c)$, $n > 3$.

Theorem: 2.3

Let $G = D(vP_n^c)$ where P_n^c is the complement of a path graph P_n then,

(i) $\gamma(G) = 2$ and $|D(G,\gamma)| = \frac{n^2-5n+6}{2}$, $n > 3$

(ii) $\gamma_i(G) = 4$ and $|D_i(G,\gamma)| = n - 1$, $n > 4$

(iii) $\gamma_{it}(G) = 3$ and

$|D_{it}(G,\gamma)| = \frac{n(n^2-5n+6)}{2}$, $n > 4$.

Proof:

Let $V(P_n) = \{v_i / 1 \leq i \leq n\}$ and

$V(P_n^c) = \{u_i / 1 \leq i \leq n\}$.

Let $G = D(vP_n^c)$ is a duplication of all the vertices of P_n^c .The vertex set of G is, $V(G) = \{u_i, u_i' / 1 \leq i \leq n\}$ with $d(u_1) = d(u_n) = 2n - 4$, $d(u_1') = d(u_n') = n - 2$; $d(u_i) = 2n - 6$ and $d(u_i') = n - 3$, for all $1 < i < n$ and $|V(G)| = 2n$.

The neighbours of $V(G)$ is as follows,

$N(u_1) = V(G) - \{u_1, u_2, u_1', u_2'\}$.

$N(u_n) = V(G) - \{u_n, u_{n-1}, u_n', u_{n-1}'\}$

$N(u_i) = V(G)$
$-\{u_{i-1}, u_i, u_{i+1}, u_{i-1}', u_i', u_{i+1}'\}$, $1 < i < n$.

$N(u_1') = V(G) - \{u_1, u_2, u_i' / 1 \leq i \leq n\}$;

$N(u_n') = \{u_i' / 1 \leq i \leq n-2\}$.

$N(u_i') = \{u_i / 1 \leq i \leq n\}$
$-\{u_{i-1}, u_i, u_{i+1}\}$, $1 < i < n$

(i) Let $D_{1,i} = \{\{u_i, u_{i+3}\} / 1 \leq i \leq n - 3\}$

$N[u_i] \cup N[u_{i+3}] = V(G)$, $1 \le i \le n-3$.
Therefore, every set in $D_{1,i}/_{1 \le i \le n-3}$ is the required dominating set of G. Hence, $\gamma(G) = 2$.
The dominating sets of G with cardinality 2 are,

$$D(G,\gamma) = \underset{\substack{1 \le t \le n-3 \\ t \le j \le n-3}}{D_{t,j}} = \left\{\{u_t, u_{j+3}\}\right\} \quad \ldots\ldots (4)$$

$|D(G,\gamma)| = \frac{n^2-5n+6}{2}, n > 3$.

(ii) Let $D_{2,l} = \{u_l, u_{l+1}, u_l{}', u_{l+1}{}'\}, {}_{1 \le l \le n-1}$.
$N[D_{2,l}] = V(G)$, $1 \le l \le n-1$. Therefore, every set contained in $D_{2,l}$ / $1 \le l \le n-1$ is the dominating set of G. Since, $u_l u_{l+1}$, $u_l{}' u_{l+1}{}'$, $u_l u_l{}' \notin E(G)$ for all l. Hence any set containing four elements of the form $\{u_l, u_{l+1}, u_l{}', u_{l+1}{}'\}$ is an independent dominating set of G. $\Longrightarrow \gamma_i(G) = 4$.
The independent dominating sets with $\gamma = 4$ are,

$$D_i(G,\gamma) = \begin{cases} \underset{1 \le l \le n-1}{D_{2,l}} = \left\{\{u_l, u_{l+1}, u_l{}', u_{l+1}{}'\}\right\} \\ \underset{1 \le l \le n-1}{D_{3,l}} = \left\{\{u_l, u_{l-1}{}', u_l{}', u_{l+1}{}'\}\right\} \end{cases}$$

$|D_i(G,\gamma)| = 2n-3, n > 4$.

(iii) Let us take $I(G,\alpha) = I_1 = \{u_i'/1 \le i \le n\}$.
Clearly, I_1 is the maximum independent set for G.
Any arbitrary set S belongs to
$D_{t,j}/_{1 \le t \le n-3,\, t \le j \le n-3}$ in equation (4) does not intersect the maximum independent set I_1 of G.
Now we choose $D_{s,j}$ such that

$$\underset{\substack{1 \le s \le n-3 \\ s \le j \le n-3}}{D_{s,j}} = \left\{\{u_s, u_{j+3}\} \cup \{u_l{}'\}\right\}, {}_{1 \le l \le n}.$$

Clearly, every set in $D_{s,j}$ / $1 \le s \le n-3$, $s \le j \le n-3$ intersect all the maximum independent sets of G.
Hence, all the sets of $D_{s,j}/_{1 \le s \le n-3,\, s \le j \le n-3}$ is the independent transversal dominating sets of G.
Hence, $\gamma_{it}(G) = 3$ and $|D_{it}(G,\gamma)| = \frac{n(n^2-5n+6)}{2}$.
Hence Proved.

Result: 2.4

(i) $\gamma(P_n) \ge \gamma(D(vP_n^c))$, $n > 3$
(ii) $\gamma_i(P_n) \ge \gamma_i(D(vP_n^c))$, $n > 3$
(iii) $\gamma_{it}(P_n) \ge \gamma_{it}(D(v(P_n^c))$, $n > 5$.

Result: 2.5

(i) $\gamma(P_n{}^c) = \gamma(D(vP_n{}^c))$, $n > 2$
(ii) $\gamma_i(P_n{}^c) = \gamma_i(D(vP_n{}^c))$, $n > 3$
(iii) $\gamma_{it}(P_n{}^c) > \gamma_{it}(D(vP_n{}^c))$, $n > 4$.

Theorem: 2.6

Let $G = P_m{}^c + D(vP_n{}^c)$ be a graph of order $2n + m$ where $m, n > 2$. Then

(i) $|D(G,\gamma)| = \frac{1}{2}[n^2 - 3n + 4] + 2(mn)$, ${}_{m>2}$
(ii) $\gamma_i(G) = 2$ and $|D_i(G,\gamma)| = m - 1$
(iii) $\gamma_{it}(G) = 2$ and $|D_{it}(G,\gamma)| = mn$.

Proof:
Let $V(P_m{}^c) = \{u_i/_{1 \le i \le m}\}$ and
$V(D(vP_n{}^c)) = \{v_j/_{1 \le j \le n}\}$.
Given, $G = P_m{}^c + D(vP_n{}^c)$, $|V(G)| = 2n + m$.
$V(G) = \{u_i, v_j, v_j{}' /_{1 \le i \le m,\ 1 \le j \le n}\}$.
$N(u_i) = V(G) - \{u_{i-1}, u_i, u_{i+1}\}, 1 < i < m$
$N(v_n{}') = V(G) - \{v_n, v_{n-1}\} - \{v_i{}'/1 \le j \le n\}$
$N(v_j) = V(G) - \{v_{j-1}, u_j, u_{j+1}, v'_{j-1}, v'_j, v_{j+1}{}'\}, {}_{1<j<n}$

(i) Let $D_{1,i} = \{u_i, u_{i+1}\}, 1 \le i \le m-1$.
$N[u_i] \cup N[u_{i+1}] = V(G), 1 \le i \le m-1$.
Hence, any set containing two elements of the form $\{u_i, u_{i+1}\}, 1 \le i \le m-1$ is a dominating set of G. Hence, $\gamma(G) = 2$.
The dominating sets of G with cardinality 2 are,

$$D(G,\gamma) = \begin{cases} \underset{1 \le i \le m-1}{D_{1,i}} = \left\{\{u_i, u_{i+1}\}\right\} \\ \underset{\substack{1 \le t \le m-3 \\ t \le i \le m-3}}{D_{t,i}} = \left\{\{u_t\} \cup \{u_{i+3}\}\right\} \\ \underset{\substack{1 \le p \le m \\ 1 \le j \le n}}{D_{p,j}} = \left\{\{u_p\} \cup \{v_j\}\right\} \\ \underset{\substack{1 \le r \le m \\ 1 \le j \le n}}{D_{r,j}} = \left\{\{u_r\} \cup \{v_j{}'\}\right\} \end{cases} \quad \ldots\ldots (5)$$

$|D(G,\gamma)| = \frac{1}{2}[m^2 - 3m + 4] + 2(mn)$.

(ii) Clearly, the pair of vertices $\{u_i, u_{i+1}\}/_{1 \le i \le m-1}$ are not adjacent in G . Therefore, all the sets contained in $D_{1,i}/_{1 \le i \le m-1}$ are also an independent dominating sets of G.
$\Rightarrow \gamma_i(\mathrm{G}) = 2$ and $|D_i(G,\gamma)| = m - 1$.

(iii) Let $I(G,\alpha) = \{v_i{}'/1 \le i \le n\}$ is the maximum independent set for G.
From equation (5), every set of $D_{r,j}$ for all $_{1 \le r \le m, 1 \le j \le n}$ intersect the maximum independent set of G .Therefore, every set belong to $D_{r,j}, {}_{1 \le r \le m, 1 \le j \le n}$ is the independent transversal dominating sets of G.
$\Longrightarrow \gamma_{it}(G) = 2$ and $|D_{it}(G,\gamma)| = mn$.
Hence proved.

Conclusion:

In this paper, we find the domination, independent domination and independent transversal domination number and the corresponding sets to the complement of a path graph P_n and its duplication. In addition, we have established a general formula to find all sets and discussed some of its result.

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