On the doubling of intervals in the pseudo-0-distributive property in the lattice of weak congruences of chains

Gladys Mano Amirtha V [1] and D. Premalatha [2]

[1]*Research Scholar (Register No. 20111172092013),* [2]*Head and Associate Professor,*
[1,2] *PG and Research Department of Mathematics,Rani Anna Government College for Women, Tirunelveli - 627 008,*
Affiliated to Manonmaniam Sundaranar University, Abishekapatti, Tirunelveli - 627 012, Tamilnadu, India.
[1]gladyspeter3@gmail.com,[2]lathaaedward@gmail.com

Abstract

In this paper, we examine how the doubling construction of Alan Day alters the pseudo-0-distributive property after doubling of lower, upper and intermediateintervals in $C_W(L_n)$, that is, the lattice of weak congruences of a chain of length n.
Keywords: lattice theory, doubling construction in lattices, lattices of weak congruences, weak-congruence relation, pseudo-0-distributive.
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1 Introduction

The weak congruence relation in lattices given by Vojvodič G. and Šešelja B. [6] has seen many advances in lattice theory. Many properties of the lattice of weak congruences have been studied by various lattice theories. One of those is the property of pseudo-0-distributivity in the lattice of weak congruences of chains and it was defined by J. C. Varlet in 1968 as a generalisation of the notion of pseudo-complementedness [4]. From G. Gratzer's [3] construction of a lattice L^U from a lattice L by doubling anelement $a = 0$ or 1 in L, Alan Day in his paper [2] introduced a similar construction popularly been referred to as 'Day's definition of doubling of intervals'. In this research paper, we analyse how Day's definition of doubling affects the pseudo-0-distributive property in $C_W(L_n)$. In this chapter, we discuss some preliminary results for the devel- opment of our paper.

Definition 1.1. *[1] A lattice L is said to be pesudo-0-distributive if for all $x, y, z \in L, x \wedge y = 0, x \wedge z = 0$ imply that $(x \vee y) \wedge z = y \wedge z$.*

Definition 1.2. *[6] A weak congruence relation on an algebra $A = (A, F)$ is a symmetric and transitive sub-universe of A^2.*

Theorem 1.3. *[5] $C_W(L_n)$ is pseudo-0-distributive.*

Example 1.4. *The weak congruence lattice $C_W(L_4)$ of which L_4 is* $\{0 \prec a \prec b \prec 1\}$ *is pseudo-0-distributive.*

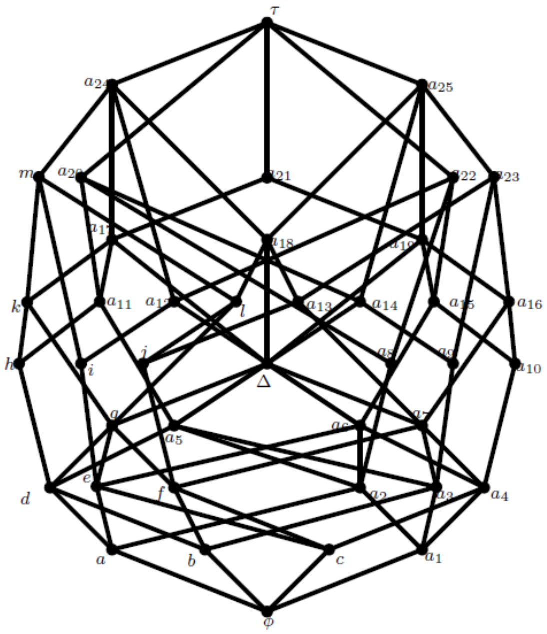

Figure 1: $C_W(L_4)$

Definition 1.5. *[3] The lattice $L[I] = (L \setminus I) \cup (I \times C_2)$ where $C_2 = \{0, 1\}$ is formed by Day's definition of doubling of intervals given by the following ordering: for $x, y \in L[I]$ and $i, j \in C_2$;*

$x \leq y$ *if* $x \leq y$ *in* L;

$(x, i) \leq y$ *if* $x \leq y$ *in* L;

$x \leq (y, j)$ *if* $x \leq y$ *in* L;

$(x, i) \leq (y, j)$ *if* $x \leq y$ *in* L *and* $i \leq j$ *in* C_2.

2 On pseduo-0-distributive property

Theorem 2.1. *$[C_W(L_n)](I)$ is pseudo-0-distributive where I is a lower interval in $C_W(L_n)$.*

Proof. Let I be a lower interval say $I = [0, \theta_1]$ where $\theta_1 = \{0, x_1, x_2, ..., x_k\}^2$, $k < n$.
Let $A, B, C \in [C_W(L_n)](I)$ such that $A \wedge B = (0, 0)$ and $A \wedge C = (0, 0)$.
Claim: $(A \vee B) \wedge C = B \wedge C$.
We observe that $A \in [(0, 0), 1]$ or $A \in ((0, 0), (\theta, 0)]$.
(I) Assume that $A \notin [(0, 0), 1]$.
(Ii) $B, C \in [(0, 1), 1]$.
(Iia) If $B, C \in [(0, 1), (\theta, 1)]$ then $A_1 \wedge B_1 = 0$ and $A_1 \wedge C_1 = 0$ where $A = (A_1, 0), B = (B_1, 0), C = (C_1, 0)$.

Let $((x_i, x_j), 1) \in (A \vee B) \wedge C = [(A_1, 0) \vee (B_1, 1)] \wedge (C_1, 1)$
$= [(A_1 \vee B_1), 1] \wedge (C_1, 1)$
$= [(A_1 \vee B_1) \wedge (C_1, 1)]$
$\Rightarrow (x_i, x_j) \in (A_1 \vee B_1) \wedge C_1$
$\Rightarrow (x_i, x_j) \in A_1 \vee B_1$ and $(x_i, x_j) \in C_1$
Then, $(A_1 \vee B_1) \wedge C_1 = B_1 \wedge C_1$ since $A_1, B_1, C_1 \in C_W(L_n)$
So, $((x_i, x_j), 1) \in (B_1 \wedge C_1, 1) = (B_1, 1) \wedge (C_1, 1) = B \wedge C$
Hence, $(A \vee B) \wedge C = B \wedge C$ is true in this case.
(Iib) Let $B \in [(0,1), (\theta, 1)]$ and $C \notin [(0,1), (\theta, 1)]$ and $C \in C_w(L_n)$
$B_1 \wedge C \leq B_1$ and $C \notin [(0,0), (\theta, 0)]$
Therefore, $B_1 \wedge C = (\theta_1, 1)$
$((x_i, x_j), 1) \in (A \vee B) \wedge C = [(A_1, 0) \vee (B_1, 1)] \wedge C$
$= ((A_1, B_1) \wedge C, 1)$
$(x_i, x_j) \in (A_1 \vee B_1) \wedge C$
$\Rightarrow (x_i, x_j) \in B_1 \wedge C$
$\Rightarrow ((x_i, x_j), 1) \in (B_1 \wedge C_1, 1) = ((B_1, 1) \wedge C = B \wedge C$

So, in this case, $(A \vee B) \wedge C = B \wedge C$ holds good.
(Iic) $B, C \in [(0,0), (\theta, 0)]$ where $B = (B_1, 0), C = (C_1, 0)$.
$(A \vee B) \wedge C = B \vee C$ is true by a similar argument as in Case(Iia).
(II) Assume that $A \in [(0,1), 1]$
Therefore, $B, C \in [(0,0), (\theta, 0)]$
(IIa) $A \in [(0,1), (\theta, 1)]$
$(A \vee B) \wedge C = [(A_1, 1) \vee (B_1, 0)] \wedge (C_1, 0)$
$= ((A_1 \vee B_1), 1) \wedge (C_1, 0)$
$= ((A_1 \vee B_1) \wedge C_1, 0)$
$= (B_1 \wedge C_1, 0)$
$= B \wedge C$
(IIb) Let $A \notin [(0,1), (\theta, 1)]$
$(A \vee B) \wedge C = [A \vee (B_1, 0)] \wedge (C_1, 0)$
$= (A \vee B_1, 0) \wedge (C_1, 0)$
$= ((A \vee B_1) \wedge C_1, 0)$
$= (B_1 \wedge C_1, 0)$
$= B \wedge C$
In this case, we get $[C_W(L_n)](I)$ is pseudo-0-distributive.
Hence, $[C_W(L_n)](I)$ is pseudo-0-distributive for a lower interval I. □

Example 2.2. *Now, consider the interval $I = [0, a_{32}]$ in the lattice $C_W(L_4)$ (figure: 1). We can form the new lattice $[C_W(L_4)](I) = \{C_W(L_4) \setminus I\} \cup (I \times C_2)$ as in figure: 2 by doubling the interval I. The resultant lattice is also pseudo-0-distributive.*

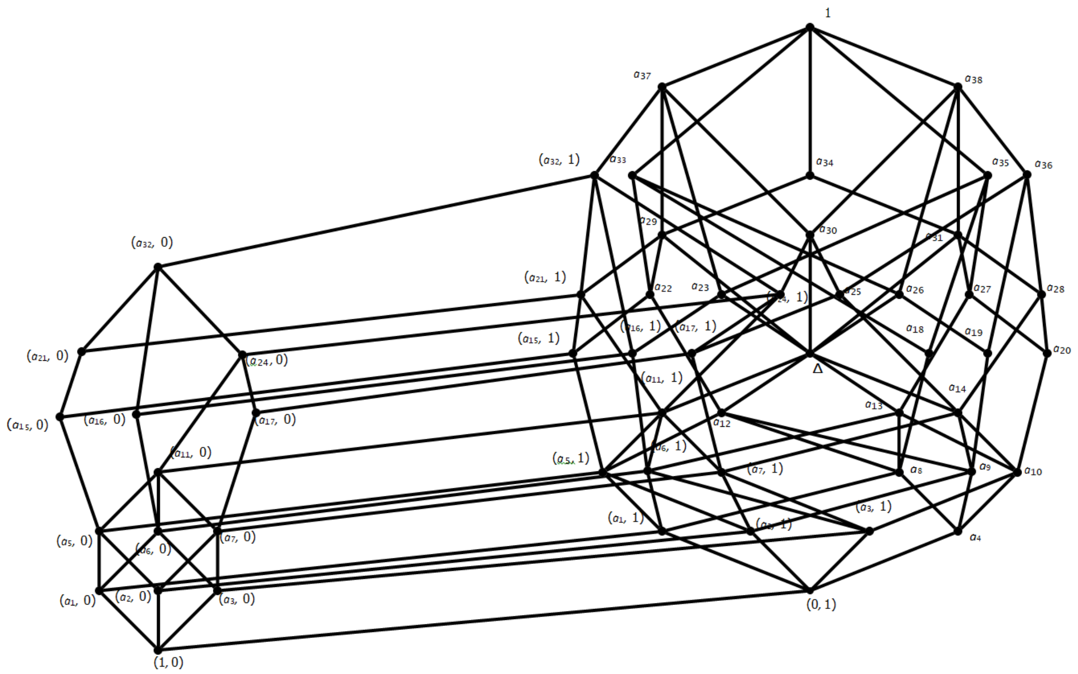

Figure 2: $[C_W(L_4)](I)$ where $I = [0, a_{32}]$

Theorem 2.3. *$[C_W(L_n)](I)$ is pseudo-0-distributive where I is an upper interval in $C_W(L_n)$.*

Proof. Let I be an upper interval say $I = [\{(x_k, x_k)\}, 1]$ of $C_W(L_n)$.
To prove that $[C_W(L_n)](I)$ is pseudo-0-distributive i.e., to prove that for all
$A, B, C \in [C_W(L_n)](I)$ whenever $A \wedge B = 0$ and $A \wedge C = 0$,
$(A \vee B) \wedge C = B \wedge C$.
We observe that when $A \in I \times C_2$, we have $B, C \in C_W(L_n) \setminus I$.
(I) $A \in I \times C_2$, then $A = (A_1, j), j = 0$ or 1 and $B, C \in C_W(L_n) \setminus I$.
We claim that, $(A \vee B) \wedge C = B \wedge C$.
$(A \vee B) \wedge C = [(A_1, j) \vee B] \wedge C$
$= (A_1 \vee B, j) \wedge C$
$= (A_1 \vee B) \wedge C$
$= B \wedge C$ (As $C_W(L_n)$ is pseudo-0-distributive)
(II) $A \notin I \times C_2$
There are three possibilities:
(a) $B, C \in I \times C_2$
(b) $B, C \in C_W(L_n) \setminus I$
(c) $B \in C_W(L_n) \setminus I$ and $C \in I \times C_2$
(IIa) $B, C \in I \times C_2$.
Let $B = (B_1, j), C = (C_1, j)$ where $j = 0$ or 1.
Now, $(A \vee B) \wedge C = [A \vee (B_1, j)] \wedge (C_1, j)$
$= (A \vee B_1, j) \wedge (C_1, j)$
$= ((A \vee B_1) \wedge C_1, j)$
$= (B_1 \wedge C_1, j)$
$= (B_1, j) \wedge (C_1, j)$
$= B \wedge C$ (As $C_W(L_n)$ is pseudo-0-distributive)

(IIb) If $B, C \in C_W(L_n) \setminus I$, then $(A \vee B) \wedge C = B \wedge C$, as $C_W(L_n)$ is pseudo-0-distributive.
(IIc) If $B \in C_W(L_n) \setminus I$ and $C \in I \times C_2$, then
$(A \vee B) \wedge C = (A \vee B) \wedge (C_1, j)$
$= (A \vee B) \wedge C_1$
$= B \wedge C_1$, as $C_W(L_n)$ is pseudo-0-distributive.
Therefore, we conclude that $[C_W(L_n)](I)$ is pseudo-0-distributive. □

Example 2.4. *Consider the interval $[a_1, 1]$ in figure:1. The lattice formed by the doubling of the interval is given in (figure: 3). The resultant lattice also preserves the pseudo-0-distributive property.*

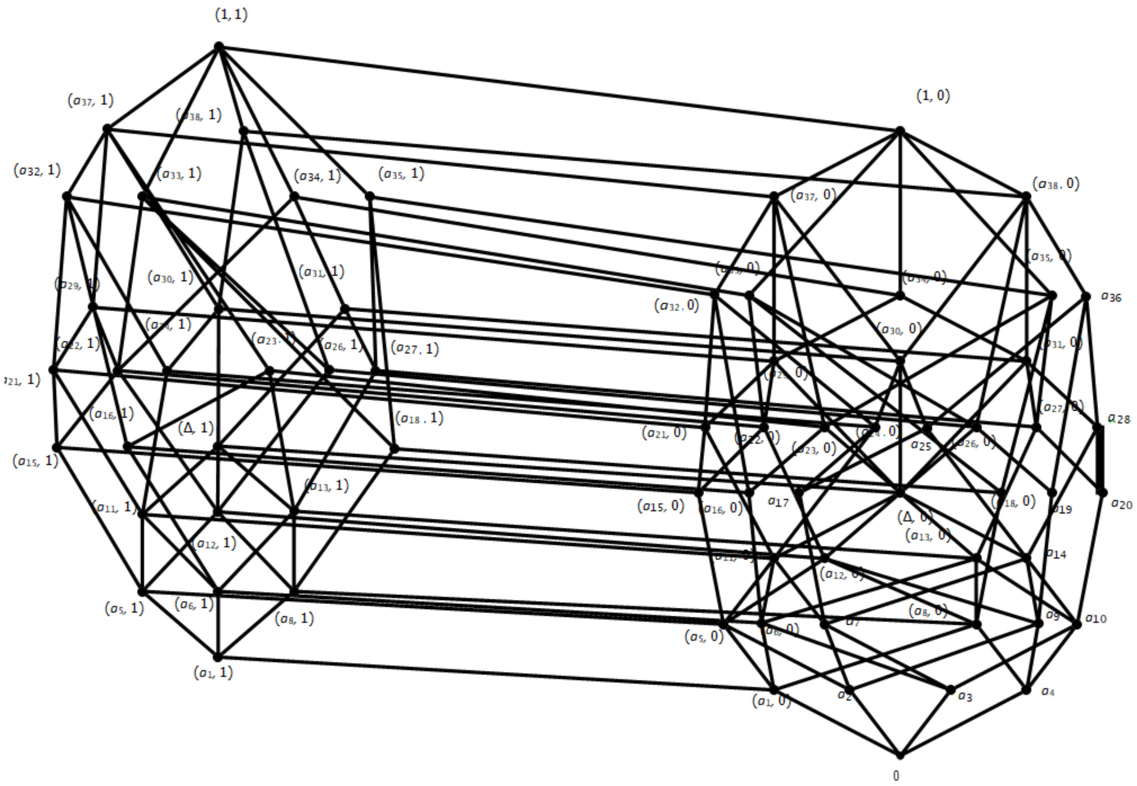

Figure 3: $[C_W(L_4)](I)$ where $I = [a_1, 1]$

Theorem 2.5. *$[C_W(L_n)](I)$ is pseudo-0-distributive where I is an intermediate interval in $C_W(L_n)$.*

Proof. Assume $1 < i < k < n$.
Without loss of generality, let us assume that $I = [\{x_i, x_k\}^2, \{x_{i-1}, x_i, x_{i+1}, ..., x_k\}^2]$.
Claim: $[C_W(L_n)](I)$ is pseudo-0-distributive.
That is, to prove that $A \wedge B = 0, A \wedge C = 0 \Rightarrow (A \vee B) \wedge C = B \wedge C$ is true in $[C_W(L_n)](I)$.
We observe that either $A \in I \times C_2$ or $A \in C_W(L_n) \setminus I$.

(i) $A \in I \times C_2$, then $B, C \in C_W(L_n) \setminus I$

(ii) $A \in C_W(L_n) \setminus I$, then $B, C \in C_W(L_n) \setminus I$ or $B \in I \times C_2$ and $C \in C_W(L_n) \setminus I$ or $B, C \in I \times C_2$

Case (i): Let $A \in I \times C_2$ and $B, C \in C_W(L_n) \setminus I$
Therefore, $A = (A_1, j), j = 0$ or 1.
Definitely, $x_i, x_k \notin B$ and $x_i, x_k \notin C$.
Case (ia): B does not contain any of $x_{i-1}, x_{i+1}, ..., x_{k-1}$ in their generating set. Then $A \vee B \in C_W(L_n) \setminus I$.
$(A \vee B) \wedge C = [(A_1, j) \vee B] \wedge C = (A_1 \vee B) \wedge C = B \wedge C$ (As $C_W(L_n)$ is pseudo-0-distributive)
Case (ib): Suppose the atoms in B are exactly some of the atoms in θ_2 then $A \vee B \in I \times C_2$
Subcase (ib_1): If C does not contain any of the atoms of θ_2, then
$(A \vee B) \wedge C = (\theta_3, j) \wedge C, j = 0$ or 1
$= \theta_3 \wedge C = 0$
$B \wedge C = 0$ definitely.
So, $(A \vee B) \wedge C = B \wedge C$.
Subcase (ib_2): If C also contains exactly some of the atoms of θ_2, then
$(A \vee B) \wedge C = (\theta_4, j) = B \wedge C$.
Case (ic): Suppose the atoms in B are some in common with that of θ_2 and some outside and $A \vee B \in C_W(L_n) \setminus I$
Therefore, $(A \vee B) \wedge C = (A_1 \vee B) \wedge C$ (Since $C_W(L_n)$ is pseudo-0-distributive)
$= B \wedge C$
Case (ii): Let $A \in C_W(L_n) \setminus I$.
Case (iia): Let $B, C \in C_W(L_n) \setminus I$.
$(A \vee B) \wedge C = B \wedge C$ is true as $C_W(L_n)$ is pseudo-0-distributive.
Case (iib): Let $B \in I \times C_2$ and $C \in C_W(L_n) \setminus I$.
Let $B = (B_1, j)$.
$A \vee B = (A \vee B_1, j)$, if A contains exactly atoms inside θ_2.
Now, $(A \vee B) \wedge C = (A \vee B_1, j) \wedge C$
$= (B_1, j) \wedge C$, if C contains exactly some common atoms of θ_2.
$= B \wedge C$ (Since $A \wedge C = 0$)
Case (iic): Let $B, C \in I \times C_2$. Then, $B = (B_1, j), C = (C_1, j)$
Subcase (iic_1): If A contains exactly some atoms of θ_2, then
$A \vee B = A \vee (B_1, j)$
$= (A \vee B_1, j)$
$(A \vee B) \wedge C = (A \vee B_1, j) \wedge (C_1, j)$
$= ((A \vee B_1) \wedge C_1, j)$
$= (B_1 \wedge C_1, j)$
$(B_1, j) \wedge (C_1, j)$
$= B \wedge C$
Subcase (iic_2): If A contains some atoms outside θ_2 that is no atoms inside $B \wedge C$ then $A \vee B \in C_W(L_n) \setminus I$
Therefore, $(A \vee B) \wedge C = (A \vee B) \wedge (C_1, j)$
$= (A \vee B_1) \wedge (C_1, j)$
$(B_1 \wedge C_1, j) = B \wedge C$
Therefore, we conclude that $[C_W(L_n)](I)$ is pseudo-0-distributive. □

Example 2.6. *Consider the intermediate interval $I = [a_4, a_{38}]$ in $C_W(L_4)$. On doubling this interval I, we get the lattice $[C_W(L_4)](I)$ which remains pseudo-0-distributive.*

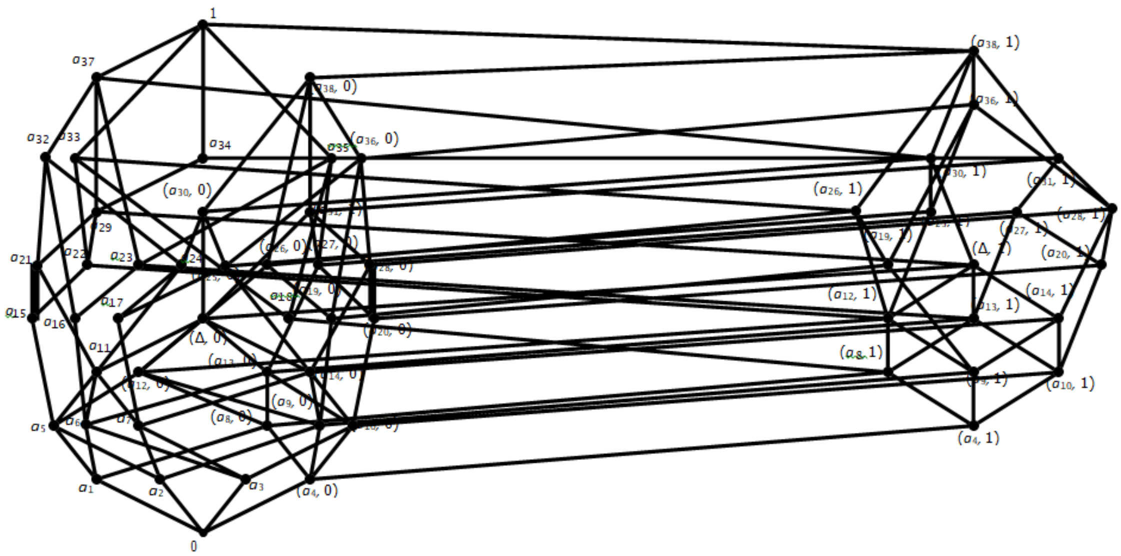

Figure 4: $[C_W(L_4)](I)$ where $I = [a_4, a_{38}]$

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