The total vertex strong geodetic number of a graph

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Abstract

A vertex strong geodetic set S of x of G is called a total vertex strong geodetic set of G if $G[S]$ has no isolated vertex. The minimum cardinality of a total vertex strong geodetic set of G is called the total vertex strong geodetic number of G and is denoted by $tsg_x(G)$. Any total vertex strong geodetic set of cardinality $tsg_x(G)$ is called a tsg_x-set of G. Some of the standard graphs are determined. Necessary conditions for $tsg_x(G)$ to be n or $n-1$ are given for some vertex x in G. It is shown for every pair of integers a and b with $2 \leq a \leq b$, there exists a connected graph G such that $sg_x(G) = a$ and $tsg_x(G) = b$ for some vertex x in G, where $sg_x(G)$ is the vertex strong geodetic number of x of G.

Keywords: strong geodetic number, vertex strong geodetic number, total vertex strong geodetic number geodetic number.

AMS Subject Classification: 05C12.

1. Introduction

By a graph $G = (V, E)$, we mean a finite, undirected connected graph without loops or multiple edges. The *order* and *size* of G are denoted by n and m respectively. For basic graph theoretic terminology, we refer to [4]. Two vertices u and v are said to be *adjacent* if uv is an edge of G. Two edges of G are said to be adjacent if they have a common vertex. The *distance* $d(u, v)$ between two vertices u and v in a connected graph G is the length of a shortest u-v path in G.

An $u-v$ path of length $d(u, v)$ is called an $u-v$ *geodesic*. An $x-y$ path of length $d(x, y)$ is called geodesic. A vertex v is said to lie on a geodesic P if v is an internal vertex of P. The closed interval $I[x, y]$ consists of x, y and all vertices lying on some $x - y$ geodesic of G and for a non-empty set $S \subseteq V(G)$, $I[S] = \cup_{x,y \in S} I[x, y]$. A set $S \subseteq V(G)$ in a connected graph G is a geodetic set of G if $I[S] = V(G)$. The geodetic number of G, denoted by $g(G)$, is the minimum cardinality of a geodetic set of G. The geodetic concepts were studied in [1, 2, 5, 10-14, 17-24].

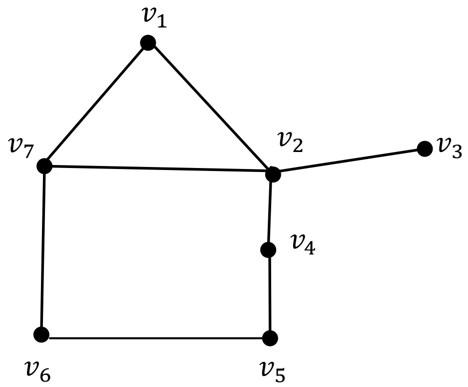
Let $S \subset V(G)$ and $x \in V$ such that $x \notin S$. Let $I_x[y]$ be the set of all vertices that lies in x-y geodesic including x and y, where `$y \in S$ and $I_x[S] = \bigcup_{y \in S} I_x[y]$. Then S is said to be an x-geodetic set of G, if $I_x[S] = V$. The x-geodetic concept were studied in [10]. Let x be a vertex of G and $S \subseteq V - \{x\}$. Then for each vertex $y \in S$, $x \neq y$. Let $\tilde{g}_x[y]$ be a selected fixed shortest x-y path. Then we set $\tilde{I}_x[S] = \{\tilde{g}_x(y): y \in S\}$ and let $V(\tilde{I}_x[S]) = \bigcup_{p \in \tilde{I}_x[S]} V(P)$. If $V(\tilde{I}_x[S]) = V$ for some $\tilde{I}_x[S]$ then the set S is called a vertex strong geodetic set of G. The minimum cardinality of a vertex strong geodetic set of G is called the vertex strong geodetic number of G and is denoted by $sg_x(G)$. These concepts were studied in [3, 5, 6-9, 16, 25]. The following theorem is used in sequel.

Theorem 1.1[25] Every extreme vertex of G other than the vertex x (whether x is extreme or not) belongs to every x-geodetic set for any vertex x in G.

2. The Total Vertex Strong Geodetic Number of a Graph

Definition 2.1. A vertex strong geodetic set S of x of G is called a total vertex strong geodetic set of G if $G[S]$ has no isolated vertex. The minimum cardinality of a total vertex strong geodetic set of G is called the total vertex strong geodetic number of G and is denoted by $tsg_x(G)$. Any total vertex strong geodetic set of cardinality $tsg_x(G)$ is called a tsg_x-set of G.

Example 2.2. For the graph G given in Figure 2.1, tsg_x-sets and $tsg_x(G)$ for each vertex x is given in the following Table 2.1.



G
Figure 2.1

Table 2.1

Vertex	tsg_x-sets	$tsg_x(G)$
v_1	$\{v_2, v_3\}, \{v_5, v_6\}$	2
v_2	$\{v_1, v_2, v_3, v_5, v_6\}$	5
v_3	$\{v_1, v_2, v_5, v_6\}$	4
v_4	$\{v_1, v_2, v_3, v_6, v_7\}$	5
v_5	$\{v_1, v_2, v_3, v_7\}$	4
v_6	$\{v_1, v_2, v_3, v_4\}$	4
v_7	$\{v_1, v_2, v_3, v_4, v_5\}$	5

Observation 2.3. Let x be any vertex of a total graph G.

(i) If $y \neq x$ be a simplicial vertex of G, then y belongs to every total x- vertex strong geodetic set of G.

(ii) The eccentric vertices of x belong to every total x- vertex strong geodetic set of G.

(iii) For a total graph of order n, $2 \leq tsg_x(G) \leq n$.

Theorem 2.4. For the path $G = P_n$ $(n \geq 3)$,

$$tsg_x(G) = \begin{cases} 2, & if\ x\ is\ an\ end\ vertex\ of\ G \\ 4, & if\ x\ is\ a\ cut\ vertex\ of\ G \end{cases}$$

Proof. Let P_n be $v_1, v_2, \ldots, v_n$. If $x = v_1$, then $S = \{v_n\}$ is a sg_x-set of G. Since $G[S]$ has isolated vertices, S is not a tsg_x-set of G and so $tsg_x(G) \geq 2$. Let $S_1 = \{v_{n-1}, v_n\}$. Then S_1 is a tsg_x-set of G so that $tsg_x(G) = 2$. By the similar way, if $x = v_n$, then $tsg_x(G) = 2$. Let x be a cut vertex of G. Let $S = \{v_1, v_2, \ldots v_{n-1}, v_n\}$ be the set of all end vertices and support vertices of G. Then by Theorem S is a subset of every total vertex strong geodetic set of G and so $tsg_x(G) \geq 4$. Now S is a tsg_x-set of G so that $tsg_x(G) = 4$. ■

Theorem 2.5. For the cycle $G = C_n$ $(n \geq 4)$, $tsg_x(G) = 2$ for every $x \in G$.

Proof. Let $V(C_n) = \{v_1, v_2, \ldots, v_n\}$. Without loss of generality, let us assume $x = v_1$.

Case (i): Let n be even. Let $n = 2k$ $(k \geq 2)$. Then v_{k+1} is the eccentric vertex of v_1. $G[S]$ has isolated vertices, since $\{v_{k+1}\}$ is not a tsg_x-set of G so that $tsg_x(G) \geq 2$.

Let $S = \{v_{k+1}, v_{k+2}\}$. Then S is a tsg_x-set of G so that $tsg_x(G) = 2$.

Case (ii): Let n be odd. Let $n = 2k + 1$ $(k \geq 2)$. Then $S = \{v_{k+1}, v_{k+2}\}$ is the eccentric vertices of v_1. S is a subset of every tsg_x-set of G and so $tsg_x(G) \geq 2$. Since S is a sg_x-set of G and $G[S]$ has no isolated vertices, S is a tsg_x-set of G so that $tsg_x(G) = 2$.

■

Theorem 2.6. For the complete graph $G = K_n$ $(n \geq 4)$, $tsg_x(G) = n - 1$ for every $x \in G$.

Proof. Let x be a vertex of G. Let $S = V(G) - \{x\}$. Since every vertex of G is an extreme vertex of G, it follows from Observation S is the unique tsg_x-set of G so that $tsg_x(G) \geq n - 1$ for every x in G.

■

Theorem 2.7. For the fan graph $G = K_1 + P_{n-1} (n \geq 5)$,

$$tsg_x(G) = \begin{cases} n-1, & \text{if } x \in V(K_1) \\ n-3, & \text{if } x \text{ is extreme vertex of } P_{n-1} \\ n-2, & \text{if } x \text{ is internal vertex of } P_{n-1} \end{cases}$$

Proof. Let $V(K_1) = y$ and $V(P_{n-1}) = \{v_1, v_2, \ldots, v_{n-1}\}$.

Case (i): If $x = y$. Then $S = \{v_1, v_2, \ldots, v_n\}$ is a set of all eccentric vertices for x. By Observation. S is a subset of every tsg_x-set of G and so $tsg_x(G) \geq n - 1$. Since $G[S]$ has no isolated vertices, S is a tsg_x-set of G so that $tsg_x(G) = n - 1$.

Case (ii): If $x \in V(P_{n-1})$. Let $x = v_1$. Then $S = \{v_3, v_4, \ldots, v_{n-1}\}$ are eccentric vertices for v_1. By Observation.2.3 (i) S is a subset of every tsg_x-set of G and so $tsg_x(G) \geq n - 3$. Now S is a sg_x-set of G and $G[S]$ has no isolated vertices. Therefore S is a tsg_x-set of G so that $tsg_x(G) = n - 3$. If $x = v_{n-1}$, by the similar way we can prove that $tsg_x(G) = n - 3$.

Case (iii): Let $x \in \{v_2, v_3, \dots, v_{n-2}\}$. Without loss of generality, let us assume that $x = v_2$. Then $\{v_1, v_{n-1}\}$ is set of extreme vertices of G. By Observation 2.3 (i) $\{v_4, v_5, \dots, v_{n-2}\}$ is the set of eccentric vertices of v_2. Then $\{v_4, v_5, \dots, v_{n-2}\}$ is a subset of every tsg_x-set of G. Let $S' = \{v_1, v_2, v_4, v_5, \dots, v_{n-2}, v_{n-1}\}$. Then S' is a sg_x-set of G but $G[S']$ has isolated vertex. Therefore $S' \cup \{y\}$ is a tsg_x-set of G so that $tsg_x(G) = n - 2$. ■

Theorem 2.8. For the wheel graph $G = K_1 + C_{n-1}$ $(n \geq 5)$,

$$tsg_x(G) = \begin{cases} n-1, & if\ x \in v_1 \\ n-3, & if\ x \in V(C_{n-1}) \end{cases}$$

Proof. Let $V(K_1) = y$ and $V(C_{n-1}) = \{v_1, v_2, \dots, v_{n-1}\}$.

Case (i): Let $x = y$. Then $S = \{v_1, v_2, \dots, v_{n-1}\}$ is a set of all eccentric vertices for x. By Observation 2.3 (i) S is a subset of every tsg_x-set of G and so $tsg_x(G) \geq n - 1$. Now, S is a sg_x-set of G and $G[S]$ has no isolated vertices. Therefore S is a tsg_x-set of G so that $tsg_x(G) = n - 1$.

Case (ii): Let $x \in V(C_{n-1})$. Without loss of generality, let us assume that $x = v_1$. Then $S = \{v_3, v_4, \dots, v_{n-1}\}$ are eccentric vertices of G. By Observation 2.3 (i) S is a subset of every tsg_x-set of G and so $tsg_x(G) \geq n - 3$. Now S is a sg_x-set of G and $G[S]$ has no isolated vertices. Therefore S is a tsg_x-set of G so that $tsg_x(G) = n - 3$. ■

Theorem 2.9. For the star graph $G = K_{1,n-1}$ $(n \geq 3)$,

$$tsg_x(G) = \begin{cases} n-1, & if\ x\ is\ end\ vertex\ of\ G \\ n, & if\ x\ is\ the\ cut\ vertex\ of\ G \end{cases}$$ for every $x \in G$.

Proof. Let y be the cut vertex of G and $S = \{v_1, v_2, \dots, v_{n-1}\}$ is a set of all eccentric vertices of G. Let $x = y$ is a set of all eccentric vertices for x. By Observation S is a subset of every tsg_x-set of G and so $tsg_x(G) \geq n - 1$. Since $G[S]$ has isolated vertices, S is not a sg_x-set of G and so $tsg_x(G) = n$. Then $S = V(G)$ is the unique tsg_x-set of G so that $tsg_x(G) = n$ for every vertex x in G.

Let $x \in \{v_1, v_2, \dots, v_{n-1}\}$ without loss of generality, let us assume that $x = v_1$. Then $S = S - \{x\}$ is a subset of every tsg_x-set of G and so $tsg_x(G) \geq n - 2$. Since $G[S_1]$ has isolated vertices, S_1 is not a sg_x-set of G and so $tsg_x(G) \geq n - 1$. Let $S_2 = S_1 \cup \{y\}$. Then S_2 is a tsg_x-set of G so that $tsg_x(G) = n - 1$. ■

Theorem 2.10. For the double star graph $G = B_{r,s}$ $(r, s \geq 2)$,

$$tsg_x(G) = \begin{cases} n, if\ x\ is\ a\ cut\ vertex\ of\ G \\ n-1, if\ x\ is\ an\ end\ vertex\ of\ G \end{cases}.$$

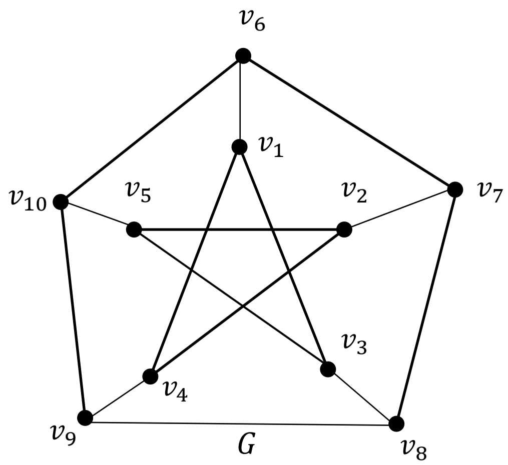
Proof. Let $X = \{x, y\}$ be the set of cut vertices of G and $Z = \{y_1, y_2, \dots, y_r\} \cup = \{z_1, z_2, \dots, z_s\}$ be the end vertices of G. Let $x \in X$. Then Z is a subset of every tsg_x-set of G and so $tsg_x(G) \geq n-1$. Since $G[S]$ has isolated vertices, S is not a sg_x-set of G and so $tsg_x(G) \geq n$. Then $S = V(G)$ is the unique tsg_x-set of G so that $tsg_x(G) = n$. Let $x \in Z$ without loss of generality, let us assume that $x = x_1$. Then $S_1 = S - \{x\}$ is a subset of every tsg_x-set of G and so $tsg_x(G) \geq n-1$. Let $S_2 = S_1 \cup X$. Then S_2 is a sg_x-set of G. Since $G[S_2]$ has no isolated vertices, S_2 is a tsg_x-set of G so that $tsg_x(G) = n-1$.

■

Theorem 2.11. For the Peterson graph G, $tsg_x(G) = 6$ for every $x \in G$.

Proof. Case (i) Let $x \in \{v_1, v_2, v_3, v_4, v_5\}$. Without loss of generality, let us assume that $x = v_1$. Then $S = \{v_2, v_5, v_7, v_8, v_9, v_{10}\}$ is a set of all eccentric vertices of x. By Observation (ii) S is a subset of every tsg_x-set of G and so $tsg_x(G) \geq 6$. Since S is a sg_x-set of G and $G[S]$ has no isolated vertices. Then S is a tsg_x-set of G so that $tsg_x(G) = 6$.

Case (ii) Let $x \in \{v_6, v_7, v_8, v_9, v_{10}\}$. Without loss of generality, let us assume $x = v_6$. Then $S = \{v_2, v_3, v_4, v_5, v_8, v_9\}$ is the set of all eccentric vertices of x. By Observation (ii) S is a subset of every tsg_x-set of G and so $tsg_x(G) \geq 6$. Since S is a sg_x-set of G and $G[S]$ has no isolated vertices. Then S is a tsg_x-set of G so that $tsg_x(G) = 6$. ■



G

Figure 2.2

Theorem 2.12. For every pair of integers a and b with $2 \leq a \leq b$, there exists a connected graph G such that $sg_x(G) = a$ and $tsg_x(G) = b$ for some x in G.

Proof. For $a = b$, let $G = K_{a+1}$. Then by Theorems $sg_x(G) = tsg_x(G) = a$ for all $x \in V(G)$. So let $2 \leq a < b$. Let $V(\overline{K}_2) = \{y, z\}$ and $V(\overline{K}_{a-1}) = \{z_1, z_2, \dots, z_{a-1}\}$. Let $P_i: u_i, v_i$ $(1 \leq i \leq b - a + 1)$ be a copy of path on two vertices. Let G be the graph obtained from $\overline{K}_2, \overline{K}_{a-1}$ and P_i $(1 \leq i \leq b - a + 1)$, by introducing the edges yz_i $(1 \leq i \leq a - 1)$, yu_i and zv_i $(1 \leq i \leq b - a + 1)$. The graph G is shown in Figure 2.3. Let $x = z$. Let $Z = \{z_1, z_2, \dots, z_{a-1}\}$.

First we prove that $sg_x(G) = a$. By Theorem 1.1, Z is a subset of sg_x-set of G and so $sg_x(G) \geq a - 1$. Since Z is a sg_x-set of G, $sg_x(G) \geq a$. Let $S = Z \cup \{y\}$. Then Z is a sg_x-set of G so that $sg_x(G) = a$.

Next we prove that $tsg_x(G) = b$. Now $Z \cup \{y\}$ is a subset of every tsg_x-set of G. We fix $x - v_1 - u_1 - y - z_1$ geodesic. Hence it follows that every tsg_x-set of G contains each u_i $(2 \leq i \leq b - a + 1)$ and so $tsg_x(G) \geq a - b - a = b$. Let $S_1 = S \cup \{u_2, u_3, \dots, u_{b-a+1}\}$. Then S_1 is a tsg_x-set of G so that $tsg_x(G) = b$. ■

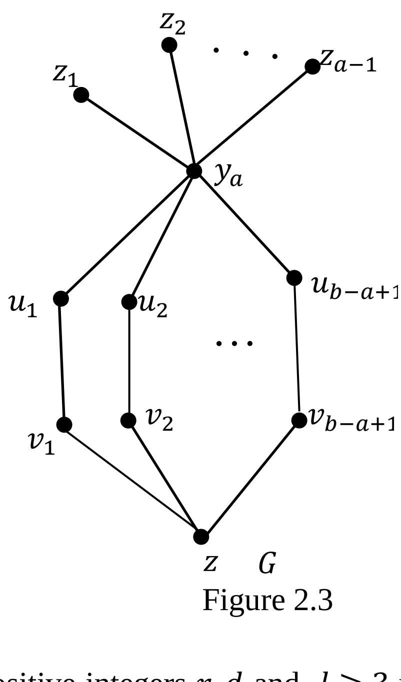


Figure 2.3

Theorem 2.13. For a positive integers r, d and $l \geq 3$ with $r \leq d \leq 2r$, there exists a connected graph G with $radG = r$, $diamG = d$ and $tsg_x(G) = l$ for some vertex x in G.

Proof. Let $r = 1$. Then $d = 1$ or 2. If $d = 1$, let $G = K_{l+1}$ satisfies the given condition for any vertex. If $d = 2$, let $G = K_{1,l-1}$. Let x be the cut vertices of G. By Theorem 2.9

$tsg_x(G) = l$ for the cut vertex x in G. Let $r \geq 2$. Let $C_{2r}: v_1, v_2, \dots, v_{2r}, v_1$ be a cycle of order $2r$ and let $P_{d-r+1}: u_0, u_1, u_2, \dots, u_{d-r}$ be a path of length $d - r + 1$. Let H be the graph obtained from C_{2r} and P_{d-r+1} by identifying v_1 in C_{2r} and u_0 in P_{d-r+1}. Now add $l - 2$ vertices $w_1, w_2, \dots, w_{l-2}$ to H and join each w_i $(1 \leq i \leq l-2)$ to the vertex u_{d-r+1} and obtain the graph G is shown in Figure 2.4. Then G has radius r and diameter d. Let $x = v_{r+1}$. Let $M = \{u_{d-r}, w_1, w_2, \dots, w_{l-2}\}$ be the cut vertices of G. Then M is a subset of every tsg_x-set of G and so $tsg_x(G) \geq l - 1$. Since $G[M]$ has isolated vertices, M is not a tsg_x-set of G. Let $M_1 = M \cup \{u_{d-r-1}\}$. Then M is a tsg_x-set of G so that $tsg_x(G) = l$.

■

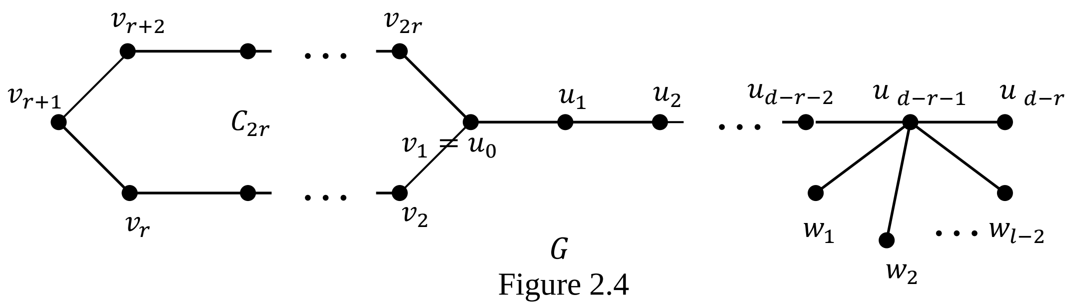


Figure 2.4

References

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[1] D. Anusha, J. John and S. Joseph Robin, The geodetic hop domination number of complementary prisms, Discrete Mathematics, Algorithms and Applications , 13(6), (2021),2150077

[2] S. Beulah Samli, J. John and S. Robinson Chellathurai, The double geo chromatic number of a graph, Bulletin of the International Mathematical virtual Institute, 11(1), 2021, 25 - 38.

[3] L. G. Bino Infanta and D. Antony Xavier, Strong upper geodetic number of graphs, *Communications in Mathematics and Applications* 12(3), (2021)737–748.

[4] F. Buckley and F. Harary, Distance *in Graphs*, Addison-Wesley, Redwood City, CA, 1990

[5] G. Chartrand, F. Harary and P. Zhang, On the geodetic number of a graph, *Networks*, 39, (2002), 1-6.

[6] V. Gledel, V. Irsic, and S. Klavzar, Strong geodetic cores and cartesian product graphs, arXiv:1803.11423 [math.CO] (30 Mar 2018).

[7] Huifen Ge, Zao Wang-and Jinyu Zou Strong geodetic number in some networks, *Journal of Mathematical Resarch*-11(2), (2019), 20-29.

[8] V. Irsic, Strong geodetic number of complete bipartite graphs and of graphs with specified diameter, *Graphs and Combin.* 34 (2018) 443–456.

[9] V. Irsic, and S. Klavzar, Strong geodetic problem on Cartesian products of graphs, RAIRO Oper. Res. 52 (2018) 205–216.

[10] J.John, The forcing monophonic and the forcing geodetic numbers of a graph, Indonesian Journal of Combinatorics .4(2), (2020) 114-125.

[11] J. John and D.Stalin,The edge geodetic self decomposition number of a graph, RAIRO Operations Research, RAIRO- 55,(2021), S1935-S1947

[12] J. John and D.Stalin, Distinct edge geodetic decomposition in Graphs, Communication in Combinatorics and Optimization, 6 (2),(2021), 185-196

[13] J. John, On the vertex monophonic, vertex geodetic and vertex Steiner numbers of graphs, Asian-European Journal of Mathematics 14 (10), (2021), 2150171

[14] J.John, and V. Sujin Flower, The edge-to-edge geodetic domination number of a graph, Proyecciones journal of Mathematics, 40(3), (2021), 635-658.

[15] P. Manuel, S. Klavzar, A. Xavier, A. Arokiaraj, and E. Thomas, Strong edge geodetic problem in networks, Open Math. 15 (2017) 1225–1235.

[16] A.L. Merlin Sheela, J.John and M. Antony, The edge-to-vertex strong geodetic number of a graph, 12(1), (2023), 1243-1251.

[17] A. P. Santhakumaran and J. John, Edge Geodetic Number of a Graph, Journal of Discrete Mathematical Sciences and Cryptography 10(3), (2007) ,415-432.

[18] A. P. Santhakumaran and J. John, The edge Steiner number of a graph, Journal of Discrete Mathematical Sciences and Cryptography, 10 (2007), 677 - 696.

[19] A. P. Santhakumaran, and J. John, The upper edge geodetic number and the forcing edge geodetic number of a graph, Opuscula Mathematica 29, 4, (2009), 427 - 441.

[20] A. P. Santhakumaran, P. Titus and J. John, The upper connected geodetic number and the forcing connected geodetic number of a graph, Discrete Applied Mathematics, 157 (7), (2009), 1571 - 1580.

[21] A. P. Santhakumaran, P. Titus and J. John, On the connected geodetic number of a graph, Journal of Comb. Math. and Comb. Comp., 69, (2009), 219 - 229.

[22] A.P. Santhakumaran and J. John, The connected edge geodetic number of a graph, SCIENTIA Series A: Mathematical Sciences, 17, (2009), 67 - 82.

[23] A.P. Santhakumaran and J. John, On the forcing geodetic and forcing Steiner numbers of a graph, Discussiones Mathematicae Graph Theory,(2011),31, 611-

624

[24] A. P. Santhakumaran and J. John, The upper connected edge geodetic number of a graph, Filomat, 26(1), (2012), 131 - 141.

[25]C. Saritha and T.Muthu Nesa Beula, The vertex strong geodetic number of a graph (Communicated)

[26] D. Stalin and J. John, The forcing edge geodetic domination number of a graph, Journal of Advanced Research in Dynamical and Control Systems, 10(4),(2018), 172-177.

[27] D. Stalin and J. John, Edge geodetic dominations in graphs, International Journal of Pure and Applied Mathematics, 116,(22),(2017), 31-40.