BIORTHOGONAL POLYNOMIALS FOR THE WEIGHT FUNCTION $\frac{|x|^{2\mu}}{(-x^2q^2;q^2)_\infty}$

N. M. Kavathekar[1], Gajanan A. Dhanorkar[2] , T.B.Jagtap[3]

Article History: Received: 10.06.2023 **Revised:** 20.06.2023 **Accepted:** 28.07.2023

Abstract :

Al Salam and Verma [1] discussed two polynomial sets $\left\{Z_n^{(\alpha)}(x,x/q)\right\}$ and $\left\{Y_n^{(\alpha)}(x,k/q\right\}$ which are bi-orthogonal on $(0,\infty)$ with respect to continuous or discrete distribution function. Present paper attempts to construct a pair of bi-orthogonal polynomial sets $\left\{S_n^{\mu}(x,k/q)\right\}$ and $\left\{T_n^{\mu}(x,k/q)\right\}$. $(\mu > -1/2, n = 0,1,2,\dots)$ which are biorthogonal with respect to weight function $\frac{|x|^{2\mu}}{(-x^2q^2;q^2)_\infty}$. We also obtain some interesting properties of these polynomials

Keywords and phrases: Biorthogonal polynomials, generating functions, q-analogue of gamma function, weight function.
2000 Mathematics Subject Classification: 33C45, 33D05, 33D99, 33E50

[1]Department of Mathematics, Mudhoji College, Phaltan, Dist-Satara (M.S.) India.
[2]Department of Mathematics, Karmaveer Bhaurao Patil College,Vashi, Navi Mumbai,India.
[3]Department of Mathematics, Y. C. Institute of Science, Satara (MS) India.

DOI: 10.31838/ecb/2023.12.s3.768

1 Introduction

For real or complex q, $|q|<1$, let

$$(a;q)_\mu = \prod_{j=0}^{\infty} (1-aq^j)/(1-aq)^{\mu+j} \tag{1.1}$$

For arbitrary a and μ, so that

$$(a;q)_n = \begin{cases} 1, & if n = 0 \\ (1-a)(1-aq)\dots(1-aq^{n-1}), & if n = 1,2,\dots \end{cases} \tag{1.2}$$

and

$$(a;q)_\infty = \prod_{j=0}^{\infty} (1-aq^j) \tag{1.3}$$

Jackson [2] defined a q-analogue of the gamma function as

$$\Gamma_q(x) = \frac{[q]_\infty}{[q^x]_\infty}(1-q)^{1-x},\ 0<q<1 \tag{1.4}$$

which satisfy the functional equation

$$\Gamma_q(x+1) = \frac{q^x-1}{q-1}\Gamma_q(x) \tag{1.5}$$

Using (1.5), it is easy to verify that

$$[q^\alpha]_n = \frac{\Gamma_q(\alpha+n)}{\Gamma_q(\alpha)}(1-q)^n \tag{1.6}$$

Let $\delta x, q$ be the q-derivative defined by

$$\delta x, qf(x) = \frac{f(x)-f(xq)}{x} \tag{1.7}$$

For simplicity we shall write δ for $\delta x, q$. It is easy to see that

$$\delta\{f(x)g(x)\} = f(xq)\delta g(x) + g(x)\delta f(x) \tag{1.8}$$

The q-binomial coefficient for arbitary λ is defined by

$$\begin{bmatrix}\lambda\\ n\end{bmatrix}_q = (-1)^n q^{n(2\lambda-n+1)/2}[q^{-\lambda}]_n/[q]_n \tag{1.9}$$

q-konhauser bi-orthogonal polynomials are given by Al-salam and Verma [1].

$$\begin{aligned} Z_n^\alpha(x,k/q) &= \frac{[q^{1+\alpha}]_{nk}}{(q^k;q^k)_n}\sum_{j=0}^{n} \frac{q^{-nk};q^k)_j x^{kj}}{(q^k;q^k)_j[q^{1+\alpha}]_{kj}} q^{\frac{1}{2}kj(kj-1)+kj(n+\alpha+1)} \\ &= \frac{[q^{1+\alpha}]_{nk}}{(q^k;q^k)_n}\sum_{j=0}^{n} (-1)^j \begin{bmatrix}n\\ j\end{bmatrix}_{q^k} q^{kj(kj+j+2\alpha)/2}\frac{x^{kj}}{[q^{1+\alpha}]_{kj}} \end{aligned} \tag{1.10}$$

$$Y_n^\alpha(x,k/q) = \frac{1}{[q]_n}\sum_{r=0}^{n} \frac{x^r}{[q]_r} q^{r(r-1)/2}\sum_{j=0}^{r} \frac{[q^{-r}]_j}{[q]_j} q^j (q^{1+\alpha+j};q^k)_n \tag{1.11}$$

which are bi-orthogonal on $(0,\infty)$ with respect to continuous or discrete distribution function.

In present paper we have constructed following pair of polynomials

$$\begin{aligned} S_n^\mu(x,k/q) = \ & \Gamma_{q^2}(\epsilon+\frac{k}{2}(n-\epsilon+1)+\mu) \\ & .\sum_{j=0}^{N} (-1)^j \begin{bmatrix}N\\ j\end{bmatrix}_{q^{2k}} (1-q^2)^{kj}\frac{x^{nk-2kj}q^u}{\Gamma_{q^2}(\frac{1}{2}(kn+1+\epsilon)-kj+\mu)} \end{aligned} \tag{1.12}$$

$$\begin{aligned} T_n^\mu(x,k/q) = \ & (-1)^N\sum_{r=0}^{N} \frac{x^{n-2r}}{(q^2;q^2)_{N-r}}. \\ & .\sum_{j=0}^{N-r} (-1)^j \begin{bmatrix}N-r\\ j\end{bmatrix}_{q^2} q^V (q^{1+2j+2\mu+k\epsilon+\epsilon};q^{2k})_N \end{aligned} \tag{1.13}$$

where

$$U = \frac{K}{4}\{n(kn+n+2)-(k+3)\epsilon) + \mu k(n-\epsilon) + kj(j+kj-n-1-kn-2\mu),$$

$$V = \frac{1}{4}(n+\epsilon)(n+2-3\epsilon) + r(r-1-n+\epsilon) + j(j+1-n+2r+\epsilon)$$

$N=\left[\frac{n}{2}\right]$ is the greatest integer less than or equal to n/2 and value of ϵ is zero or one according to even or odd nature of n.
By reverting the order of summation for even and odd integers in (1.12) and (1.13) we have the following relations

$$S_{2n}^{\mu}(x,k/q)=(-1)^n(q^{2k};q^{2k})_n\frac{\Gamma_{q^2}(kn+k/2+\mu)}{\Gamma_{q^2}(kn+\mu+1/2)}.Z_n^{\mu-1/2}(x^2q^2;k/q^2) \qquad (1.14)$$

$$S_{2n+1}^{\mu}(x,k/q)=(-1)^n(q^{2k};q^{2k})_nx^kZ_n^{\mu+k/2}(x^2q^2,k/q^2) \qquad (1.15)$$

$$T_n^{\mu}(x,k/q)=(-1)^n(q^2;q^2)_nY_n^{\mu}(x^2q^2,k/q^2) \qquad (1.16)$$

$$T_{2n+1}^{\mu}(x,k/q)=(-1)^n(q^2;q^2)_nxY_n^{\mu+k/2}(x^2q^2,k/q^2) \qquad (1.17)$$

where $Z_n^{\alpha}(x;k/q)$ and $Y_n^{\alpha}(x;k/q)$ are q-konhauser biorthogonaal polynomials as given in (1.10) and (1.11)
It is interesting to note that

$$\lim_{q\to 1^-}2^n(1-q^2)^{-nk/2}S_n^{\mu}(x(1-q^2)^{1/2};k/q)=S_n^{\mu}(x,k) \qquad (1.18)$$

$$\lim_{q\to 1^-}2^n(1-q^2)^{-n/2}k^{-N}T_n^{\mu}(x(1-q^2)^{1/2};k/q)=T_n^{\mu}(x,k) \qquad (1.19)$$

where $S_n^{\mu}(x;k)$ and $T_n^{\mu}(x;k)$ are biorthogonal polynomials defined and studied by Thakare and Madhekar [9].

2 Biorthogonality

For $\alpha>-1$, q-konhauser bi-orthogonal polynomials (1.10) and (1.11) satisfy the orthogonality relation

$$\int_0^\infty Z_n^{\alpha}(x,k/q)Y_m^{\alpha}(x,k/q)\frac{x^\alpha}{[-x]_\infty}\,dx = \frac{(1-q)^{1+\alpha}\Gamma_{(-\alpha)}\Gamma_{(1+\alpha)}[q^{1+\alpha}]_{kn}q^{-kn}}{\Gamma_q(-\alpha)[q]_n}.\delta_{mn} \qquad (2.1)$$

where δ_{mn} is the familiar kronecker delta and $\alpha>-1$
Considering (1.14), (1.16) and using (2.1) we obtain

$$\begin{aligned}&\int_{-\infty}^{\infty}S_{2n}^{\mu}(x,k/q)T_{2m}^{\mu}(x,k/q)\frac{|x|^{2\mu}}{(-x^2q^2;q^2)_\infty}dx\\&=(q^{2k};q^{2k})_n\frac{\Gamma_{q^2}(kn+k/2+\mu)\Gamma(1-\frac{1}{2})\Gamma(\mu+1/2)}{\Gamma_{q^2}(\mu+1/2)\Gamma_{q^2}(1/2-\mu)}\\&.(1-q^2)^{kn+\mu+1/2}q^{2k-2\mu-1}\delta_{mn}\end{aligned} \qquad (2.2)$$

Similarly in view of (1.15), (1.17) and using (2.1) we readily obtain

$$\begin{aligned}&\int_{-\infty}^{\infty}S_{2n+1}^{\mu}(x,k/q)T_{2n+1}^{\mu}(x,k/q)\frac{|x|^{2\mu}}{(-x^2q^2;q^2)_\infty}dx\\&=(q^{2k};q^{2k})_n\Gamma(-\mu-k/2)\Gamma(1+\mu+k/2)\\&.(1-q^2)^{2kn+k+2\mu+2)/2}\frac{\Gamma_{q^2}(\left(\frac{2kn+k+2\mu+2}{2}\right)}{\Gamma_{q^2}\left(-\mu-\frac{k}{2}\right)\Gamma_{q^2}(1+\mu+k/2)}\delta_{mn}\end{aligned} \qquad (2.3)$$

Combining (2.2) and (2.3) we obtain the biorthogonality for the polynomials $S_n^{\mu}(x,k/q)$ and

$T_n^\mu(x,k/q)$.

$$\int_{-\infty}^{\infty} S_n^\mu(x,k/q)T_m^\mu(x,k/q)\frac{|x|^{2\mu}}{(-x^2q^2;q^2)_\infty}dx$$
$$= (q^{2k};q^{2k})\frac{\Gamma_{q^2}\left(\mu+\frac{nk+k-k\epsilon+2\epsilon}{2}\right)}{\Gamma_{q^2}\left(\mu+\frac{1+\epsilon+k\epsilon}{2}\right)\Gamma_{q^2}\left(-\mu+\frac{1-k\epsilon-\epsilon}{2}\right)}$$
$$.\Gamma\left(\mu+\frac{1+\epsilon+k\epsilon}{2}\right)\Gamma\left(-\mu+\frac{1-k\epsilon-\epsilon}{2}\right).$$
$$(1-q^2)^{\frac{1}{2}(1+kn+\epsilon)+\mu}q^{-kn-1-\epsilon-2\mu}\delta_{mn} \quad (2.4)$$

which is integral form of biorthogonality polynomials of $S_n^\mu(x,k/q)$ and $T_n^\mu(x,k/q)$ in terms of gamma function.

3 Generating Function

Consider the explicit representation of $Z_n^\alpha(x,k/q)$ given in (1.10) and using q-binomial theorem [[7], p-92]

$$\sum_{n=0}^{\infty}\frac{[a]_n}{[q]_n}x^n = \frac{[ax]_\infty}{[x]_\infty} \quad (3.1)$$

and elementary identity

$$[a]_{kn} = \prod_{j=1}^{k}(aq^{j-1};q^k)_n \quad (3.2)$$

we obtain the following generating function

$$\sum_{n=0}^{\infty}\frac{(c;q^k)_n}{[q^{1+\alpha}]_{kn}}Z_n^\alpha(x,k/q)t^n$$
$$= \frac{(ct;q^k)_\infty}{(t;q^k)_\infty}{}_1\phi_{k+1}\begin{bmatrix}c;q^k,-tx^kq^{k(1+k+2\alpha)/2}\\ ct;\Delta(q^{1+\alpha},k/q)\end{bmatrix} \quad (3.3)$$

where $\Delta(\beta,k/q)$ abbreviates the array of k parameters $\beta,\beta q,\dots,\beta q^{k-1}$ $(k\geq 1)$
For $c=0$ equation (3.3) get reduced to the generating function

$$\sum_{n=0}^{\infty}Z_n^\infty(x,k/q)\frac{t^n}{[q^{1+\alpha}]_{kn}} = \frac{1}{(t;q^k)_\infty}{}_0\phi_k\begin{bmatrix}-;q^k,-tx^kq^{k(1+k+2\alpha)/2}\\ \Delta(q^{1+\alpha};k/q)\end{bmatrix} \quad (3.4)$$

The generating function (3.3) is obtained by Al-salam and Verma.
From (3.3) and (1.14) we obtain the following generating function

$$\sum_{n=0}^{\infty}\frac{(c;q^{2k})_n}{(q^{k+2\mu};q^2)_{kn}}S_{2n}^\mu(x,k/q)\frac{t^{2n}}{(q^{2k};q^{2k})_n} = \frac{\Gamma_{q^2}(\mu+k/2)}{\Gamma_{q^2}(\mu+1/2)}\frac{(-ct^2;q^{2k})_\infty}{(-t^2;q^{2k})_\infty}\ 1\phi_{k+1}\begin{bmatrix}c,q^{2k},t^2x^{2k}q^{k(k+2\mu+2)}\\ -ct^2,\Delta(q^{1+2\mu};k/q^2)\end{bmatrix} \quad (3.5)$$

Similarly from (3.3) and (1.15) we obtain the following generating function

$$\sum_{n=0}^{\infty}\frac{(c;q^{2k})_n}{(q^{2+2\mu+k};q^2)_{kn}(q^{2k};q^{2k})_n}S_{2n+1}^\mu(x,k/q)t^{2n}$$
$$= x^k\frac{(-ct^2;q^{2k})_\infty}{(-t^2;q^{2k})_\infty}{}_1\phi_{k+1}\begin{bmatrix}c,q^{2k},zq^{k(k+1)}\\ -ct^2,\Delta(q^{2+2\mu+k};k/q^2)\end{bmatrix} \quad (3.6)$$

where $z=t^2x^{2k}q^{k(k+2\mu+2)}$
Now using (1.7) we can verify that

$$\sum_{n=0}^{\infty}\frac{(c,q^{2k})}{(q^{2+k+2\mu};q^2)_{kn}}S_{2n+1}^\mu(x,k/q)\frac{1}{(q^{2k};q^{2k})_n}\left[q^{2kn}+\frac{t\delta_t,q^k}{1-q^{k+2\mu}}\right]t^{2n}$$
$$= \sum_{n=0}^{\infty}\frac{(c,q^{2k})_n(1-q^{2nk+k+2\mu})S_{2n+1}^\mu(x,k/q)t^{2n}}{(q^{2+2\mu+k};q^2)_{kn}(q^{2k};q^{2k})_n(1-q^{k+2\mu})} \quad (3.7)$$
$$= \sum_{n=0}^{\infty}\frac{(c,q^{2k})_n}{(q^{k+2\mu};q^2)_{kn}(q^{2k};q^{2k})_n}S_{2n+1}^\mu(x,k/q)t^{2n}$$

Using (3.6) and (1.8) we get

$$\sum_{n=0}^{\infty} \frac{(c,q^{2k})}{(q^{2\mu+k};q^2)_{kn}(q^{2k};q^{2k})_n} S_{2n+1}^{\mu}(x,k/q)t^{2n}$$
$$= x^k \frac{(-ct^2q^{2k},q^{2k})_\infty}{(-t^2q^{2k};q^{2k})_\infty}\, {}_1\phi_{k+1}\begin{bmatrix} c;q^{2k},zq^{k(k+3)} \\ -ct^2q^{2k};\Delta(q^{2+2\mu+k},k/q^2) \end{bmatrix}$$
$$+\frac{x^k t}{1-q^{k+2\mu}}\delta_{t,q^k}\left\{\frac{(-ct^2;q^{2k})_\infty}{(-t^2;q^{2k})_\infty}\cdot {}_1\phi_{k+1}\begin{bmatrix} c;q^{2k},zq^{k(k+1)} \\ -ct^2,\Delta(q^{2+2\mu+k},k/q^2 \end{bmatrix}\right\}$$
$$= \frac{x^k(-ct^2q^{2k},q^{2k})_\infty}{(-t^2q^{2k};q^{2k})_\infty}\, {}_1\phi_{k+1}\begin{bmatrix} c;q^{2k},zq^{k(k+3)} \\ -ct^2q^{2k};\Delta(q^{2+2\mu+k},k/q^2) \end{bmatrix}$$
$$+\frac{x^k(c-1)t^2}{1-q^{k+2\mu}}\, {}_1\phi_{k+1}\begin{bmatrix} c;q^{2k},zq^{k(k+3)} \\ -ct^2q^{2k};\Delta(q^{2+2\mu+k},k/q^2) \end{bmatrix}\frac{(-ct^2q^{2k};q^{2k})_\infty}{(-t^2;q^{2k})_\infty} \tag{3.8}$$
$$+\frac{x^k(1-c)zq^{k(k+1)}}{(1-q^{k+2\mu})(1+ct^2)(1+ct^2q^{2k})(q^{2+2\mu+k},q^2)_k}\cdot {}_1\phi_{k+1}\begin{bmatrix} cq^{2k};q^{2k},zq^{3k(k+1)} \\ -ct^2q^{4k},\Delta(q^{2+2\mu+3k},k/q^2 \end{bmatrix}$$
$$= \frac{x^k(-ct^2q^{2k},q^{2k})_\infty}{(-t^2;q^{2k})_\infty}\left\{1+t^2+\frac{(c-1)t^2}{1-q^{k+2\mu}}\right\}{}_1\phi_{k+1}\begin{bmatrix} c;q^{2k},zq^{k(k+3)} \\ -ct^2q^{2k};\Delta(q^{2+2\mu+k},k/q^2) \end{bmatrix}$$
$$+\frac{x^k(1-c)zq^{k(k+1)}(-ct^2q^{4k};q^{2k})_\infty}{(1-q^{k+2\mu})(q^{2+2\mu+k};q^2)_k(-t^2;q^{2k})_\infty}\, {}_1\phi_{k+1}\begin{bmatrix} cq^{2k};q^{2k},zq^{3k(k+1)} \\ -ct^2q^{4k};\Delta(q^{2+2\mu+3k},k/q^2) \end{bmatrix}$$

Infact, one obtains after combining (3.5) and (3.8) the following generating function

$$\sum_{n=0}^{\infty} \frac{(c;q^{2k})_N}{(q^{k+2\mu};q^2)_{kN}(q^{2k};q^{2k})_N} S_n^{\mu}\left(x,\frac{k}{q}\right)t^n$$
$$= \frac{\Gamma_{q^2}\left(\mu+\frac{k}{2}\right)}{\Gamma_{q^2}\left(\mu+\frac{1}{2}\right)}\frac{(-ct^2;q^{2k})_\infty}{(-t^2;q^{2k})_\infty}\, {}_1\phi_{k+1}\begin{bmatrix} c;q^{2k}z \\ -ct^2,\Delta\left(q^{1+2\mu},\frac{k}{q^2}\right) \end{bmatrix}$$
$$+\frac{x^kt(-ct^2q^{2k};q^{2k})_\infty}{(-t^2;q^{2k})_\infty}\left\{1+t^2+\frac{(c-1)t^2}{1-q^{k+2\mu}}\right\}{}_1\phi_{k+1}\begin{bmatrix} c;q^{2k},zq^{k(k+3)} \\ -ct^2q^{2k},\Delta\left(q^{2+2\mu+k},\frac{k}{q^2}\right) \end{bmatrix}$$
$$+\frac{(1-c)tx^kzq^{k(k+1)}}{(1-q^{k+2\mu})(q^{2+2\mu+k};q^2)_k}\frac{(-ct^2q^{4k};q^{2k})_\infty}{(-t^2;q^{2k})_\infty}\, {}_1\phi_{k+1}\begin{bmatrix} cq^{2k};q^{2k},zq^{3k(k+1)} \\ -ct^2q^{4k},\Delta(q^{2+2\mu+3k},k/q^2) \end{bmatrix} \tag{3.9}$$

where $z=t^2x^{2k}q^{k(k+2\mu+2)}$

Particular Cases:

(i) For $k=1$ in (1.14) and (1.15) we have

$$S_{2n}^{\mu}(x/q)=q^{n(1+2n+2\mu)}H_{2n}^{\mu}(x/q) \tag{3.10}$$

and

$$S_{2n+1}^{\mu}(x/q)=q^{n(3+2\mu+2n)}H_{2n+1}^{\mu}(x/q) \tag{3.11}$$

where $H_n^{\mu}(x/q)$ is a q-analogue of Szego-Hermite polynomials defined by Madhekar [3]. Then one readily obtained following generating function for $H_n^{\mu}(x/q)$ for (3.7) with $k=1$

$$\begin{aligned}&\sum_{n=0}^{\infty} q^{n(1+n+2\mu)/2}\frac{(c;q^2)_N H_n^{\mu}(x/q)}{(q^{1+2\mu};q^2)_N(q^2;q^2)_N}t^n\\&=\frac{(-ct^2;q^2)}{(-t^2;q^2)}\,{}_1\phi_2\begin{bmatrix}c;q^2x^2t^2q^{3+2\mu}\\-ct^2,q^{1+2\mu}\end{bmatrix}\\&+xtq^{1+\mu}\frac{(-ct^2q^2;q^2)_\infty}{(-t^2;q^2)_\infty}\left\{1+t^2+\frac{(c-1)}{1-q^{1+2\mu}}t^2\right\}\\&\cdot{}_1\phi_2\begin{bmatrix}c;q^2,x^2t^2q^{7+2\mu}\\-ct^2q^2,q^{3+2\mu}\end{bmatrix}\\&+\frac{x^3t^3(1-c)q^{6+3\mu}(-ct^2q^4;q^2)_\infty}{(1-q^{1+2\mu})(1-q^{3+2\mu})(-t^2;q^2)_\infty}\\&\cdot{}_1\phi_2\begin{bmatrix}cq^2;q^2,x^2t^2q^{9+2\mu}\\-ct^2q^4,q^{5+2\mu}\end{bmatrix}\end{aligned}\tag{3.12}$$

which is due to Madhekar [[3], p-132] (ii) For $k=1,\ \mu=0$ we have

$$S_{2n}(x/q)=q^{n(1+2n)}H_{2n}(x/q)$$

and

$$S_{2n+1}(x/q)=q^{n(3+2n)}H_{2n+1}(x/q)$$

where $H_n(x/q)$ is q-Hermite polynomials. Hence putting $k=1$ and $\mu=0$ in (3.9) we get generating function for q-Hermite polynomials. The same generating function is also obtained by putting $\mu=0$ in (3.12)

(iii) For $x\to x\sqrt{1-q^2},\ t\to 2t,\ c\to q^{2c}$ in (3.12) and using the relation

$$\lim_{q\to 1}2^n(1-q^2)^{-n/2}H_n^{\mu}(x\sqrt{1-q^2}/q)=H_n^{\mu}(x)\tag{3.13}$$

where $H_n^{\mu}(x)$, $(\mu>-1/2)$ are Szegö-Hermite polynomials which are orthogonal with respect to the Szegö-Hermite weight function $|x|^{2\mu}\exp(-x^2)$ over the interval $(-\infty,\infty)$ (see G. Szegö [[8], p-380]) we obtain

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{(c)_N}{N!(\mu+1/2)_N}H_n^{\mu}(x)t^n&=(1+4t^2)^{-c}\,{}_1F_1\begin{bmatrix}c;4x^2t^2/1+4t^2\\\mu+1/2;\end{bmatrix}\\&+\frac{2xt(1+4t^2-8ct^2)}{(1+4t^2)^{c+1}(1+2\mu)}\,{}_1F_1\begin{bmatrix}c;\frac{4x^2t^2}{1+4t^2}\\3/2+\mu;\end{bmatrix}\\&+\frac{32cx^3t^3}{(1+2\mu)(3+2\mu)(1+4t^2)^{c+2}}\,{}_1F_1\begin{bmatrix}c+1;\frac{4x^2t^2}{1+4t^2}\\5/2+\mu;\end{bmatrix}\end{aligned}\tag{3.14}$$

For $\mu=0$ the generating function (3.14) reduces to the generating function of classical Hermite polynomials $H_n(x)$ see [6].

4 Recurrence relations and Some Properties

Using known results for q-konhauser polynomials we easily obtain the recurrence relations for the polynomials $S_n^{\mu}(x,k/q)$ and $T_n^{\mu}(x;x/q)$

Madhekar and Chamle [[5], equation (2.3), p-145] obtained following recurrence relation for $Y_n^{\alpha}(x;k/q)$

$$(1-q^{n+1})Y_{n+1}^{\alpha}(x,k/q)=Y_n^{\alpha}(x,k/q)-q^{1+\alpha+nk}Y_n^{\alpha}(xq,k/q)\tag{4.1}$$

For $\alpha=\mu-1/2,\ q\to q^2,\ x\to x^2q^2$ in (4.1) we get

$$\begin{aligned}(1-q^{2n+2})Y_n^{\mu-1/2}(x^2q^2,k/q^2)&=Y_n^{\mu-1/2}(x^2q^2;k/q^2)\\-q^{2+2nk+2\mu-1}(1+x^2q^2)&Y_n^{\mu-1/2}(x^2q^4,k/q^2)\end{aligned}\tag{4.2}$$

From (4.2), in view of (1.16) we have

$$-T_{2n+2}^{\mu}(x,k/q)=T_{2n}^{\mu}(x,k/q)-(1+x^2q^2).q^{1+2nk+2\mu}T_{2n}^{\mu}(xq,k/q)\tag{4.3}$$

Similarly for $\alpha = \mu + k/2,\ q \to q^2,\ x \to x^2q^2$ in (4.1) and using (1.17) we obtain

$$-T_{2n+3}^{\mu}(x,k/q) = T_{2n+1}^{\mu}(x,k/q) - q^{1+2\mu+(2n+1)k}.(1+x^2q^2)T_{2n+1}^{\mu}(xq;k/q) \quad (4.4)$$

combining (4.3) and (4.4) we get the following recurrence relation for the polynomial set $T_n^{\mu}(x,k/q)$

$$-T_{n+2}^{\mu}(x,k/q) = q^{1+2\mu+nk}(1+x^2q^2)T_n^{\mu}(xq,k/q) - T_n^{\mu}(x,k/q) \quad (4.5)$$

using the definition (1.7) of q-derivative equation (4.5) can be written as

$$q^{1+2\mu+k}(1+x^2q^2)x\delta T_n^{\mu}(x,k/q) = [1 - q^{1+2\mu+nk}(1+x^2q^2)]T_n^{\mu}(x,k/q) \\ +T_{n+2}^{\mu}(x,k/q) \quad (4.6)$$

From Madhekar and Chamle [[4], equation (3.3), p-362] we have

$$xq^{2nk}\delta z_n^{\alpha}(x^2q^2,k/q^2) = (1-q^{2nk})z_n^{\alpha}(x^2q^4,k/q^2) \\ 2cm - (q^{2(1+\alpha+nk-k)};q^2)_k z_{n-1}^{\alpha}(x^2q^4,k/q^2) \quad (4.7)$$

In view of (1.14) and (1.15), from (4.7) we obtain

$$xq^{2nk}\delta S_{2n}^{\mu}(x,k/q) = (1-q^{2nk})S_{2n}^{\mu}(xq,k/q) \\ 2cm + (1-q^{2nk})(q^{2nk+2\mu-k};q^2)_k S_{2n-2}^{\mu}(xq,k/q) \quad (4.8)$$

and

$$xq^{(2n+1)k}\delta S_{2n+1}^{\mu}(x,k/q) = (1-q^{(2n+1)k})S_{2n+1}^{\mu}(xq,k/q) \\ 2cm + (1-q^{2nk})(q^{2+2nk-k+2\mu};q^2)_k S_{2n-1}^{\mu}(xq,k/q) \quad (4.9)$$

Results (4.8) and (4.9) can be combined fruitfully in the form

$$xq^{2nk}\delta S_{2n}^{\mu}(x,k/q) = (1-q^{nk})S_n^{\mu}(xq,k/q) \\ 2cm + (1-q^{nk-k\epsilon})(q^{(n-1-\epsilon)k+2\epsilon+2\mu};q^2)_k S_{2n-2}^{\mu}(xq,k/q) \quad (4.10)$$

From generating function (3.2) by routine method we have expansion of x^{kn} in the form

$$x^{kn} = q^{-kn(1+kn+2\alpha)/2}[q^{1+\alpha}]_{kn}\sum_{r=0}^{n}\frac{(q^{-nk};q^k)_r}{[q^{1+\alpha}]_{rk}}q^{rk}z_r^{\alpha}(x,k/q) \quad (4.11)$$

which is corrected form of earlier result due to Al-salam and Verma [[1], equation (4.6), p-6]. From (4.11) and in view of (1.14) we have

$$\begin{aligned} x^{2kn} = \; & q^{-2kn-kn(2\mu+kn)} \frac{\Gamma_{q^2}(\mu+1/2+kn)}{\Gamma_{q^2}(\mu+k/2+kn)} \\ & (q^{k+2\mu};q^2)_{kn} \sum_{n=0}^{\infty} \frac{1}{(q^{k+2\mu};q^2)_{km}} \begin{bmatrix} n \\ m \end{bmatrix}_{q^{2k}} \\ & q^{2km-km(2n-m+1)} S_{2m}^{\mu}(x;k/q) \\ x^{2kn} = \; & q^{-kn(1+n+kn+2\mu)} \frac{\Gamma_{q^2}(\mu+1/2+kn)}{\Gamma_{q^2}(\mu+k/2+kn)} \\ & \sum_{m=0}^{n} \begin{bmatrix} n \\ m \end{bmatrix}_{q^{2k}} \frac{(q^{k+2\mu},q^2)_{kn}}{(q^{k+2\mu};q^2)_{km-kn}} q^{km(m-1)} \\ & S_{2n-2m}^{\mu}(x,k/q) \end{aligned} \tag{4.12}$$

Similarly from (4.11) and an account of (1.15) we have

$$\begin{aligned} x^{(2n+1)k} = q^{-kn(2+n+kn+2\mu+k)} \sum_{m=0}^{n} \begin{bmatrix} n \\ m \end{bmatrix}_{q^{2k}} \frac{(q^{k+2\mu};q^2)_{kn+1}}{(q^{k+2\mu};q^2)_{kn-km+1}} \\ q^{km(m-1)} S_{2n-2m+1}^{\mu}(x,k/q) \end{aligned} \tag{4.13}$$

After combining (4.12) and (4.13) we obtain the expansion formula of x^{kn} in the form

$$\begin{aligned} x^{kn} = \frac{\Gamma_{q^2}((1+nk+2\mu)/2)}{\Gamma_{q^2}((k+nk+2\mu+\epsilon-k\epsilon)/2)} \sum_{m=0}^{N} q^p \begin{bmatrix} N \\ m \end{bmatrix}_{q^{2k}} \frac{(q^{k+2\mu};q^2)_{kN+\epsilon}}{(q^{k+2\mu};q^2)_{kN+\epsilon-km}} \\ S_{n-2m}^{\mu}(x,k/q) \end{aligned} \tag{4.14}$$

where $N = [n/2]$

$$p = -1/2kN(2+n+kn+k\epsilon+\epsilon) + km(m-1) + k\mu(\epsilon-n)$$

From multiplication formula for polynomials $Z_n^{\alpha}(x,k/q)$ given by Mahekar and Chamle [[4], 3.12, p-364] we have

$$\begin{aligned} Z_n^{\alpha}(x^2y^2q^2,k/q^2) = \sum_{j=0}^{n} \frac{(q^{2+2\alpha};q^2)_{kn}(y^{2k};q^{2k})_j}{q^{2+2\alpha};q^2)_{kn-kj}(q^{2k};q^{2k})_j} y^{2kn-2kj} \\ Z_{n-j}^{\alpha}(x^2q^2,k/q^2) \end{aligned} \tag{4.15}$$

Using the definition of (1.14) and (1.15), from (4.15) we obtain

$$\begin{aligned} S_{2n}^{\mu}(xy,k/q) = (q^{k+2\mu};q^2)_{kn} \sum_{j=0}^{n} (-1)^j \begin{bmatrix} n \\ j \end{bmatrix}_{q^{2k}} \frac{(y^{2k};q^{2k})_j y^{2kn-2kj}}{(q^{k+2\mu};q^2)_{kn-kj}} \\ S_{2n-2j}^{\mu}(x,k/q) \end{aligned} \tag{4.16}$$

and

$$\begin{aligned} S_{2n+1}^{\mu}(xy,k/q) = (q^{k+2\mu};q^2)_{kn+1} \sum_{j=0}^{n} (-1)^j \begin{bmatrix} n \\ j \end{bmatrix}_{q^{2k}} \\ \frac{(y^{2k};q^{2k})_j y^{(2n+1)k-2kj}}{(q^{k+2\mu};q^2)_{kn-kj}} S_{2n+1-2j}^{\mu}(x,k/q) \end{aligned} \tag{4.17}$$

Combining (4.16) and (4.17) we have the multiplication formula for the polynomials $S_n^{\mu}(x,k/q)$ as given below

$$S_n^\mu(xy,k/q) = (q^{k+2\mu};q^2)_{kN+\epsilon} \sum_{j=0}^{N} (-1)^j \begin{bmatrix} N \\ j \end{bmatrix}_{q^{2k}} \frac{(y^{2k};q^{2k})_j y^{kn-2kj}}{(q^{k+2\mu};q^2)_{kN-kj+\epsilon}} S_{n-2j}^\mu(x,k/q) \tag{4.18}$$

For $k = 1$ and in view of (3.10) and (3.11) from (4.18) we have multiplication formula for the polynomials $H_n^\mu(x/q)$

$$H_n^\mu(x/q) = \sum_{j=0}^{N} (-1)^j \begin{bmatrix} N \\ j \end{bmatrix}_{q^2} \frac{(q^{1+2\mu};q^2)_{N+\epsilon}}{(q^{1+2\mu};q^2)_{N-j+\epsilon}} q^{j(2j-1-2n-2\mu)} (y^2;q^2)_j y^{n-2j} H_{n-2j}^\mu(x/q) \tag{4.19}$$

which is due to Madhekar [[3], equation 3.9, p-130].

5 Conclusion

In the above calculation we have find out relation in biorthogonal polynomial integral form into gamma function (2.4). In section we have discussed generating functions with particular cases. Also we have developed recurrence relance for some particular cases. We can use this theory for the development in properties and recurrence relations of biorthogonal polynomial.

6. References

1. W. A. Al-Salam and A. Verma : q-konhauser polynomials, Pacific Jour. Math, 108,(1983), 1–7.
2. F. H. Jackson : On q-definite integrals, Quart. Jour. Pure Appl. Math, 41, (1910), 193–203.
3. H. C. Madhekar : A q-Analogue of Szegö-Hermite polynomials, Soochow J. of Math. (Taipei, Taiwan, Republic of China), Vol. 15, No. 2 (1989), 125–136.
4. H. C. Madhekar and V. T. Chamle : Note on the q-konhauser polynomials, Bulliten of the Institute of Mathematics Academica Sinica, Vol. 15, No. 3 (1987), 357–365.
5. H. C. Madhekar and V. T. Chamle : On the q-konhauser biorthogonal polynomials, Internal J. Math. and Math. Sci., 10(1987), 413–415.
6. E. D. Rainville : Special Functions, Macmillan, New York, (1960).
7. L. J. Slater : Generalized Hypergeometric Functions, Cambridge University Press, (1966).
8. G. Szegö : Orthogonal Polynomials, Amer. Math. Soc. Collog. Publ., Vol-23, third edition, second printing, Amer. Math. Soc. Providence, Rhode Island, (1974).
9. [N. K. Thakare and H. C. Madhekar : A pair of Biorthogonal polynomials for the Szegö-Hermite weight function, Internat. J. Math and Math. Sci., Vol. 11, No. 4, (1988), 736–767.