HUB DOMINATING SETS AND HUB DOMINATION POLYNOMIALS OF THE COMPLETE BIPARTITE GRAPH $K_{2,n}$

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Abstract— Let $G = (V, E)$ be a simple graph. Let $HD(G, i)$ be the family of hub dominating sets in G with cardinality i. Then the polynomial

$$HD(G, x) = \sum_{i=\hbar d(G)}^{|V(G)|} hd(G, i)x^i$$

is called the hub domination polynomial of G where $hd(G, i)$ is the number of hub dominating sets of G with cardinality i and $\hbar d(G)$ is the hub domination number of G. Let $K_{2,n}$ denotes the complete bipartite graph with $n + 2$ vertices and $HD(K_{2,n}, i)$ denotes the family of hub dominating sets of $K_{2,n}$ with cardinality i. Then, the polynomial,

$$HD(K_{2,n}, x) = \sum_{i=\hbar d(k_{2,n})}^{|V(K_{2,n})|} hd(K_{2,n}, i)x^i$$

is called the hub domination polynomial of $K_{2,n}$ where $hd(K_{2,n}, i)$ is the number of hub dominating sets of $K_{2,n}$ with cardinality i and $\hbar d(K_{2,n})$ is the hub domination number of $K_{2,n}$.

In this paper, we obtain a recursive formula for $hd(K_{2,n}, i)$. Using this recursive formula, we construct the hub domination polynomial of $K_{2,n}$ as,

$$HD(K_{2,n}, x) = \sum_{i=2}^{n+2} hd(K_{2,n}, i)x^i$$

where $hd(K_{2,n}, i)$ is the number of hub dominating sets of $K_{2,n}$ with cardinality i and some of the properties of this polynomial also have been studied.

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INTRODUCTION

A graph $G = (V, E)$ is called a bipartite graph if its vertices V can be partitioned into two subsets V_1 and V_2 such that each edge of G connects a vertex of V_1 to a vertex of V_2. If G contains every edge joining a vertex of V_1 and a vertex of V_2 then G is called a complete bipartite graph. It is denoted by $K_{m,n}$, where m and n are the number of vertices in V_1 and V_2 respectively.

A set $D \subseteq V$ is a dominating set of G if $N[D] = V$ or equivalently, every vertex in $V - D$ is adjacent to atleast one vertex in D. The domination number of a graph G is defined as the minimum cardinality taken over all the dominating sets D of vertices in G and is denoted by $\gamma(G)$.

The families of hub-dominating sets of $\boldsymbol{K_{2,n}}$ are built using a recursive method in the following part. Using the findings from section II, we investigate the hub domination equations of the full bipartite graph $\boldsymbol{K_{2,n}}$ in section III. For the typical combination n to i, we use $\binom{n}{i}$. Additionally, we use $[n]$ to indicate the set $\{1,2, \dots \dots . n\}$

I. HUB DOMINATING SETS OF THE COMPLETE BIPARTITE GRAPH $K_{2,n}$

In this section, we list the hub domination number and some of the characteristics of the hub dominating sets of the full bipartite graph $K_{2,n}$. We use $V(K_{2,n}) = \{v_1, v_2, v_3, \dots \dots . v_{n+1}, v_{n+2}\}$ and $E(K_{2,n})\{(v_1, v_3), (v_1, v_4), \dots \dots . (v_1, v_{n+1}), (v_1, v_{n+2}), (v_2, v_3), (v_2, v_4), \dots \dots . (v_2, v_{n+1}), (v_2, v_{n+2})\}$ throughout this paper.

Definition 2.1

Let G be a simple graph of order n with no isolated vertices. A set $D \subseteq V$ is said to be a hub dominating set if every vertex in $V - D$ is adjacent to atleast one vertex in D and every pair of vertices in $V - D$ has a path in G such that all the internal vertices of the path are in D. The hub dominaton number of a graph G is defined as the minimum cardinality taken over all the dominating sets D of vertices in G and is denoted by $\boldsymbol{hd}(G)$.

Lemma 2.2

For all $n \in Z^+, \binom{n}{i} = 0$ if $i > n$ or $i < 0$.

Theorem 2.3

Let $K_{2,n}$ be the complete bipartite graph with $n+2$ vertices. Then,

$$hd(K_{2,n},\ i) = \begin{cases} \binom{n+2}{i} - \binom{n}{i} & when\ 2 \leq i \leq n+2\ and\ i \neq n \\ \binom{n+2}{i} - \binom{n}{i} + 1 & when\ i = n \end{cases}$$

Proof:

Let $K_{2,n}$ be the complete bipartite graph with $n+2$ vertices and $n \geq 3$. Let the partite sets of $K_{2,n}$ be $V_1 = \{v_1, v_2\}$ and $V_2 = \{v_3, \dots v_n, v_{n+1}, v_{n+2}\}$. Since $K_{2,n}$ contains $n+2$ vertices, the number of subsets of $K_{2,n}$ with cardinality i is $\binom{n+2}{i}$. Each time $\binom{n}{i}$ number of subsets of $K_{2,n}$ with cardinality i are not hub dominating sets. Hence, $K_{2,n}$ contains $\binom{n+2}{i} - \binom{n}{i}$ number of subsets of hub dominating sets with cardinality i. When $i = n$, the subgraph induced by the vertex set $\{v_3, \dots, v_n, v_{n+1}, v_{n+2}\}$ is also a hub dominating set. Thereore, one more set is hub dominating set when the cardinality is n. Therefore, $K_{2,n}$ contains $\binom{n+2}{i} - \binom{n}{i} + 1$ number of subsets of hub dominating sets with cardinality n.

Hence, $$hd(K_{2,n},\ i) = \begin{cases} \binom{n+2}{i} - \binom{n}{i} & when\ 2 \leq i \leq n+2\ and\ i \neq n \\ \binom{n+2}{i} - \binom{n}{i} + 1 & when\ i = n \end{cases}$$

Theorem 2.4

Let $K_{2,n}$ be the complete bipartite graph with $n+2$ vertices. Then,

(i) $hd(K_{2,n}, i) = hd(K_{2,n-1}, i) + 2$ if $i = 2$

(ii) $hd(K_{2,n}, i) = hd(K_{2,n-1}, i) + hd(K_{2,n-1}, i-1) - 1$ if $i = n-1$

(iii) $hd(K_{2,n}, i) = hd(K_{2,n-1}, i) + hd(K_{2,n-1}, i-1)$

for all $3 \leq i \leq n+2\ and\ i \neq n-1$

Proof:

(i) When $i = 2$

$$hd(K_{2,n}, 2) = \binom{n+2}{2} - \binom{n}{2}$$

$$= \frac{(n+2)(n+1)}{2} - \left[\frac{n(n-1)}{2}\right]$$

$$= \frac{1}{2}[(n^2 + 3n + 2) - (n^2 - n)]$$

$$= \frac{1}{2}[n^2 + 3n + 2 - n^2 + n]$$

$$= \frac{1}{2}[4n + 2]$$

$$hd(K_{2,n}, 2) = 2n + 1$$

Consider, $hd(K_{2,n-1}, 2) = \binom{n+1}{2} - \binom{n-1}{2}$

$$= \frac{(n+1)n}{2} - \left[\frac{(n-1)(n-2)}{2}\right]$$

$$= \frac{1}{2}[(n^2 + n) - (n^2 - 3n + 2)]$$

$$= \frac{1}{2}[n^2 + n - n^2 + 3n - 2]$$

$$= \frac{1}{2}[4n - 2]$$

$$= 2n - 1$$

$$= 2n + 1 - 2$$

$$hd(K_{2,n-1}, 2) = hd(K_{2,n}, 2) - 2$$

Therefore, $hd(K_{2,n}, 2) = hd(K_{2,n-1}, 2) + 2$

Hence, $hd(K_{2,n}, i) = hd(K_{2,n-1}, i) + 2$ if $i = 2$

(ii) When $i = n - 1$

By Theorem 2.3,

we have, $hd(K_{2,n}, n-1) = \binom{n+2}{n-1} - \binom{n}{n-1}$

$hd(K_{2,n-1}, n-1) = \binom{n+1}{n-1} - \binom{n-1}{n-1} + 1$ and

$$hd(K_{2,n-1}, n-2) = \binom{n+1}{n-2} - \binom{n-1}{n-2}$$

Consider,

$$hd(K_{2,n-1}, n-1) + hd(K_{2,n-1}, n-2) = \binom{n+1}{n-1} - \binom{n-1}{n-1} + 1 + \binom{n+1}{n-2} - \binom{n-1}{n-2}$$

$$= \left[\binom{n+1}{n-1} + \binom{n+1}{n-2}\right] + 1 - \left[\binom{n-1}{n-1} + \binom{n-1}{n-2}\right]$$

$$= \binom{n+2}{n-1} - \binom{n}{n-1} + 1$$

$$= hd(K_{2,n}, n-1) + 1$$

$$hd(K_{2,n-1}, n-1) + hd(K_{2,n-1}, n-2) = hd(K_{2,n}, n-1) + 1$$

Therefore, $hd(K_{2,n}, n-1) = hd(K_{2,n-1}, n-1) \;\; hd(K_{2,n-1}, n-2) - 1$

Hence, $hd(K_{2,n}, i) = hd(K_{2,n-1}, i) + hd(K_{2,n-1}, i-1) - 1$ if $i = n-1$

(iii) By Theorem 2.3, we have,

$hd(K_{2,n}, i) = \binom{n+2}{i} - \binom{n}{i}$ for all $3 \leq i \leq n+2$ and $i \neq n-1$

$$hd(K_{2,n-1}, i) = \binom{n+1}{i} - \binom{n-1}{i}$$

and $hd(K_{2,n-1}, i-1) = \binom{n+1}{i-1} - \binom{n-1}{i-1}$

Consider,

$$hd(K_{2,n-1}, i) + (K_{2,n-1}, i-1) = \binom{n+1}{i} - \binom{n-1}{i} + \binom{n+1}{i-1} - \binom{n-1}{i-1}$$

$$= \left[\binom{n+1}{i} + \binom{n+1}{i-1}\right] - \left[\binom{n-1}{i} + \binom{n-1}{i-1}\right]$$

$$= \binom{n+2}{i} - \binom{n}{i}$$

$$hd(K_{2,n-1}, i) + (K_{2,n-1}, i-1) = hd(K_{2,n}, i)$$

Therefore,

$hd(K_{2,n}, i) = hd(K_{2,n-1}, i) + hd(K_{2,n-1}, i-1)$ for all $3 \leq i \leq n+2$ and $i \neq n-1$

II. HUB DOMINATION POLYNOMIALS OF THE COMPLETE BIPARTITE GRAPH $K_{2,n}$

Definition 3.1

Let $K_{2,n}$ denotes the complete bipartite graph with $n+2$ vertices and $HD(K_{2,n}, i)$ denotes the family of hub dominating sets of $K_{2,n}$ with cardinality i. Then, the polynomial,

$$HD(K_{2,n}, x) = \sum_{i=\hbar d(k_{2,n})}^{|V(K_{2,n})|} hd(K_{2,n}, i)x^i$$

is called the hub domination polynomial of $K_{2,n}$ where $hd(K_{2,n}, i)$ is the number of hub dominating sets of $K_{2,n}$ with cardinality i and $\hbar d(k_{2,n})$ is the hub domination number of $K_{2,n}$.

Theorem 3.2

Let $K_{2,n}$ be the complete bipartite graph with $n+2$ vertices. Then, the hub domination polynomial of $K_{2,n}$ is $HD(K_{2,n}, x) = (1+x)HD(K_{2,n-1}, x) + 2x^2 - x^{n-1}$

with initial value $HD(K_{2,3}, x) = 7x^2 + 10x^3 + 5x^4 + x^5$.

Proof:

From the definition of hub domination polynomial, we have,

$$HD(K_{2,n}, x) = \sum_{i=2}^{n+2} hd(K_{2,n}, i)x^i$$

$$= hd(K_{2,n}, 2)x^2 + hd(K_{2,n}, n-1)x^{n-1}$$

$$+ \sum_{\substack{i=3 \\ i \neq n-1}}^{n+2} hd(K_{2,n}, i)x^i$$

$$= [hd(K_{2,n-1}, 2) + 2]x^2 \quad + [hd(K_{2,n-1}, n-1)$$

$$+hd\left(K_{2,n-1}, n-2\right)-1]x^{n-1}$$

$$+\sum_{\substack{i=3 \\ i\neq n-1}}^{n+2}\left[hd\left(K_{2,n-1}, i\right)+hd\left(K_{2,n-1}, i-1\right)\right]x^{i}$$

$$= hd\left(K_{2,n-1}, 2\right)x^{2}+2x^{2}+\left[hd\left(K_{2,n-1}, n-1\right)+hd\left(K_{2,n-1}, n-2\right)\right]x^{n-1}$$

$$-x^{n-1}+\sum_{\substack{i=3 \\ i\neq n-1}}^{n+2}\left[hd\left(K_{2,n-1}, i\right)+hd\left(K_{2,n-1}, i-1\right)\right]x^{i}$$

$$=\sum_{i=2}^{n+2} hd\left(K_{2,n-1}, i\right)x^{i}+\sum_{i=2}^{n+2} hd\left(K_{2,n-1}, i-1\right)x^{i}+2x^{2}-x^{n-1}$$

$$=\sum_{i=2}^{n+2} hd\left(K_{2,n-1}, i\right)x^{i}+x\sum_{i=2}^{n+2} hd\left(K_{2,n-1}, i-1\right)x^{i-1}+2x^{2}-x^{n-1}$$

$$= HD\left(K_{2,n-1}, x\right)+xHD\left(K_{2,n-1}, x\right)+2x^{2}-x^{n-1}$$

$$HD\left(K_{2,n}, x\right)=(1+x)HD\left(K_{2,n-1}, x\right)+2x^{2}-x^{n-1}$$

Hence, $HD\left(K_{2,n}, x\right)=(1+x)HD\left(K_{2,n-1}, x\right)+2x^{2}-x^{n-1}$

with initial value $HD\left(K_{2,3}, x\right)=7x^{2}+10x^{3}+5x^{4}+x^{5}$.

Example 3.3

Consider the complete bipartite graph $K_{2,6}$ with order 8 given in Figure 1.

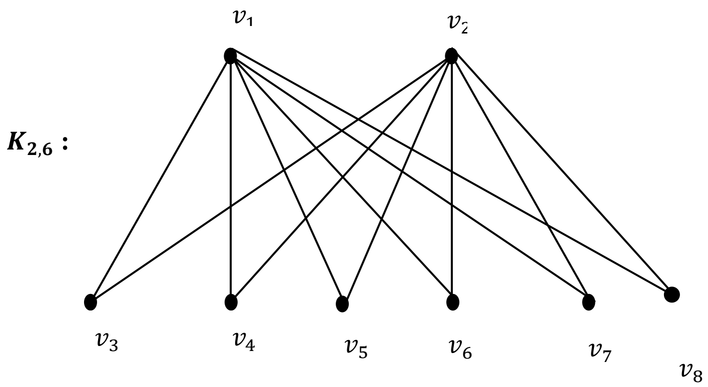

Figure 1

$$HD\left(K_{2,5}, x\right)=11x^{2}+25x^{3}+30x^{4}+21x^{5}+7x^{6}+x^{7}$$

By Theorem 3.2, we have,

$$HD(K_{2,6}, x) = (1 + x)HD(K_{2,5}, x) + 2x^2 - x^5$$

$$= (1 + x)(11x^2 + 25x^3 + 30x^4 + 21x^5 + 7x^6 + x^7) + 2x^2 - x^5$$

$$= 11x^2 + 25x^3 + 30x^4 + 21x^5 + 7x^6 + x^7 + 11x^3 + 25x^4 + 30x^5$$

$$+21x^6 + 7x^7 + x^8 + 2x^2 - x^5$$

$$HD(K_{2,6}, x) = 13x^2 + 36x^3 + 55x^4 + 50x^5 + 28x^6 + 8x^7 + x^8$$

Theorem 3.4

Let $K_{2,n}$ be the complete bipartite graph with $n \geq 3$.Then

$$HD(K_{2,n}, x) = \sum_{i=2}^{n+2} \binom{n+2}{i} x^i - \sum_{i=2}^{n+2} \binom{n}{i} x^i + x^n.$$

Proof:

Proof follows from Theorem 2.3, Theorem 2.4 and the definition of Hub Domination Polynomial.

We obtain $hd(K_{2,n}, i)$ for $3 \leq n \leq 10$ and $2 \leq i \leq 12$ as shown in Table 1.

Table 1

$HD(k_{2,n}, i)$, Hub Dominating Sets of $K_{2,n}$ with cardinality i.

n \ i	2	3	4	5	6	7	8	9	10	11	12
$K_{2,3}$	7	10	5	1							
$K_{2,4}$	9	16	15	6	1						
$K_{2,5}$	11	25	30	21	7	1					
$K_{2,6}$	13	36	55	50	28	8	1				
$K_{2,7}$	15	49	91	105	77	36	9	1			
$K_{2,8}$	17	64	140	196	182	112	45	10	1		
$K_{2,9}$	19	81	204	336	378	294	156	55	11	1	
$K_{2,10}$	21	100	285	540	714	672	450	210	66	12	1

In the following Theorem, we obtain some properties of $HD(K_{2,n}, i)$.

Theorem 3.5

The following properties hold for the coefficients of $HD(K_{2,n}, i)$ for all n.

(i) $hd\left(K_{2,n}, 2\right) = 2n + 1$, for every $n \geq 3$.

(ii) $hd\left(K_{2,n}, n + 2\right) = 1$, for every $n \geq 3$.

(iii) $hd\left(K_{2,n}, n + 1\right) = n + 2$, for every $n \geq 3$.

(iv) $hd\left(K_{2,n}, n\right) = \frac{1}{2}(n^2 + 3n + 2)$, for every $n \geq 3$.

(v) $hd\left(K_{2,n}, n - 1\right) = \frac{1}{6}(n^3 + 3n^2 - 4n)$, for every $n \geq 4$.

(vi) $hd\left(K_{2,n}, n - 2\right) = \frac{1}{24}(n^4+2n^3 - 13n^2 + 10n)$, for every $n \geq 4$.

Proof

(i) From Theorem 2.3, we have**,**

$$hd\left(K_{2,n}, 2\right) = \binom{n+2}{2} - \binom{n}{2}$$

$$= \frac{(n+2)(n+1)}{2} - \left[\frac{n(n-1)}{2}\right]$$

$$= \frac{1}{2}[(n^2 + 3n + 2) - (n^2 - n)]$$

$$= \frac{1}{2}[n^2 + 3n + 2 - n^2 + n]$$

$$= \frac{1}{2}[4n + 2]$$

$$= 2n + 1$$

Therefore $hd\left(K_{2,n}, 2\right) = 2n + 1$, for every $n \geq 3$.

(ii) Since, $HD\left(k_{2,n}, n + 2\right) = [n + 2]$, we have the result.

(iii) Since,$HD\left(k_{2,n}, n + 1\right) = \{[n + 2] - x/x \in [n + 2]\}$,

we have the result.

(iv)To prove $hd\left(K_{2,n}, n\right) = \frac{1}{2}(n^2 + 3n + 2)$, for every $n \geq 3$,

we apply induction on n.

When $n = 3$,

L.H.S = $hd(K_{2,3}, 3) = 10$ (from the Table 1)

R.H.S = $\frac{1}{2}(3^2 + 9 + 2)$ =10

Therefore, the result is true for $n = 3$.

Now, suppose that the result is true for all numbers less than n and we prove it for n.

$$hd(K_{2,n}, n) = hd(K_{2,n-1}, n) + hd(K_{2,n-1}, n-1)$$

$$= (n-1) + 2 + \frac{1}{2}[(n-1)^2 + 3(n-1) + 2$$

$$= n + 1 + \frac{1}{2}(n^2 - 2n + 1 + 3n - 3 + 2)$$

$$= \frac{1}{2}(2n + 2 + n^2 + n)$$

$$hd(K_{2,n}, n) = \frac{1}{2}(n^2 + 3n + 2)$$

Hence, the result is true for all n.

(v) To prove $hd(K_{2,n}, n-1) = \frac{1}{6}(n^3 + 3n^2 - 4n)$, for every $n \geq 4$,

we apply induction on n.

When $n = 4$,

L.H.S = $hd(K_{2,4}, 3) = 16$ (from the Table 1)

R.H.S = $\frac{1}{6}[4^3 + 3(4)^2 - 4(4)]$

$$= \frac{1}{6}(64 + 48 - 16) = 16$$

Therefore, the result is true for $n = 4$.

Now, suppose that the result is true for all numbers less than n and we prove it for n.

$$hd(K_{2,n}, n-1) = hd(K_{2,n-1}, n-1) + hd(K_{2,n-1}, n-2) - 1$$

$$= \frac{1}{2}[(n-1)^2 + 3(n-1) + 2]$$

$$+\frac{1}{6}[(n-1)^3+3(n-1)^2-4(n-1)]-1$$

$$=\frac{1}{2}[n^2+n]+\frac{1}{6}[n^3-7n+6]-1$$

$$=\frac{1}{6}[3n^2+3n+n^3-7n+6]-1$$

$$hd\left(K_{2,n},n-1\right)=\frac{1}{6}(n^3+3n^2-4n)$$

Hence, the result is true for all n.

(vi) To prove $hd\left(K_{2,n},n-2\right)=\frac{1}{24}(n^4+2n^3-13n^2+10n)$, for every $n\geq 4$.

we apply induction on n.

When $n=4$,

L.H.S = $hd\left(K_{2,4},2\right)=9$ (from the Table)

R.H.S = $\frac{1}{24}[4^4+2(4)^3-13(4)^2+10(4)]$

$=\frac{1}{24}[256+128-208+40]=9$

Therefore, the result is true for $n=4$.

Now, suppose that the result is true for all numbers less than n and we prove it for n

$$hd\left(K_{2,n},n-2\right)=hd\left(K_{2,n-1},n-2\right)+hd\left(K_{2,n-1},n-3\right)$$

$$=\frac{1}{6}[(n-1)^3+3(n-1)^2-4(n-1)]$$

$$+\frac{1}{24}[(n-1)^4+2(n-1)^3-13(n-1)^2+10(n-1)]$$

$$=\frac{1}{6}[n^3-7n+6]+\frac{1}{24}[n^4-2n^3-13n^2+38n-24]$$

$$=\frac{1}{24}[4n^3-28n+24+n^4-2n^3-13n^2+38n-24]$$

$$hd\left(K_{2,n},n-2\right)=\frac{1}{24}[n^4+2n^3-13n^2+10n]$$

Hence, the result is true for all n.

CONCLUSION

This article deduces the hub domination polynomials of the complete bipartite graph $K_{2,n}$ by identifying its hub dominating sets. We can also use cardinality i to characterise the hub dominating sets. Any complete bipartite graph $K_{m,n}$ can be used as a generalisation of this research, and some intriguing properties can be discovered.

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