ESTABLISHMENT OF 4-TUPLES CONCERNING INTEGERS WITH ELITE FEATURE

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Abstract
In this manuscript, a remarkable 4-tuples $(\alpha, \beta, \gamma, \delta)$ with elements are non-zero integers such that the arithmetic mean of any three elements among the four elements listed in this 4-tuples yields a square number is appraised by applying several techniques in Mathematics.

Keywords: Diophantine quadruples, Ternary quadratic, Diophantine equation

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1. Introduction

"A set of m positive integers $\{a_1, a_2 \dots . a_m\}$ is called a Diophantine m −tuple with the property $D(n)$, $n - \{0\} \in Z$ if $a_i . a_j + n$ is a perfect square for all $1 \le i < j \le m$". In [3,4], the authors initiated how to find Diophantine triples with suitable properties. In [5,6], the authors concentrated on evaluating Diophantine quadruples with an elegant property. For further review of quadruples one can refer [1, 2,7 − 9]. In this paper, an integer quadruples $(\alpha, \beta, \gamma, \delta)$ where α, β, γ and δ are non-zero integers where the average of any three elements amid the four numbers deliver a square number is explained by following different methods.

2. Procedures of Examinations.

Let α, β, γ, δ be distinct non-zero integers such that the average of any three among these four elements stay a perfect square. Originate with the exact postulation as given below:

$$\alpha + \beta + \gamma = 3p^2 \quad (1)$$
$$\alpha + \beta + \delta = 3q^2 \quad (2)$$
$$\alpha + \gamma + \delta = 3r^2 \quad (3)$$
$$\beta + \gamma + \delta = 3s^2 \quad (4)$$

concurrently with the supplementary proposition that

$$\alpha + \beta + \gamma + \delta = z^2 \quad (5)$$

By making simple arithmetical calculations in the simultaneous equations (1), (2), (3) and (4), the values of α, β, γ and δ are declared as follows

$$\alpha = p^2 + q^2 + r^2 - 2s^2 \quad (6)$$
$$\beta = p^2 + q^2 + s^2 - 2r^2 \quad (7)$$
$$\gamma = p^2 + r^2 + s^2 - 2q^2 \quad (8)$$
$$\delta = q^2 + r^2 + s^2 - 2p^2 \quad (9)$$

Addition of all the above four equations from (6) to (9) offers the successive equation

$$\alpha + \beta + \gamma + \delta = p^2 + q^2 + r^2 + s^2 \quad (10)$$

Comparing equation (5) and (10), it is noted that

$$z^2 = p^2 + q^2 + r^2 + s^2 \quad (11)$$

Procedure 1.

Undertaking the new linear modifications $p = m + n$, $q = m + 7n$, $r = m - n$, $s = m - 7n$ where $m \ne n \ne 0$ in (6), (7), (8) and (9), the corresponding values of α, β, γ and δ in two variables are offered subsequently by

$$\alpha = m^2 - 47 n^2 + 42 mn \quad (12)$$
$$\beta = m^2 + 97 n^2 + 6 mn \quad (13)$$
$$\gamma = m^2 - 47 n^2 - 42 mn \quad (14)$$
$$\delta = m^2 + 97 n^2 - 6 mn \quad (15)$$

Relieving the same conversions in (11) ensurethe following famous Pythagoreans equation

$$z^2 = (2m)^2 + (10n)^2 \quad (16)$$

The process of executing the quadruples $(\alpha, \beta, \gamma, \delta)$ wherethe average of three among four elements stand for a square number by solving (16) in two different proposals are enlightened below.

Proposal 1.1.

The Pythagorean equation (16) is fulfilled by

$$m = \frac{(r^2 - s^2)}{2} \quad (17)$$
$$n = \frac{rs}{10} \quad (18)$$

Meanwhile our concern is to treasure the parameters in integers, it is experiential that m and n are integers for the ensuing options of r and s

$r = 2R$ and $s = 10S$

These selections of r and s reduce (17) and (18)as

$m = 2R^2 - 50S^2$
$n = 4RS$

Replacement of the above values of m and n in (12), (13), (14) and (15) convert the preferable values of α, β, γ and δ as follows

$\alpha = 4R^4 + 336R^3S - 952R^2S^2 - 8400RS^3 + 2500S^4$
$\beta = 4R^4 + 48R^3S + 1352R^2S^2 - 1200RS^3 + 2500S^4$

$\gamma = 4R^4 - 336R^3S - 952R^2S^2 + 8400RS^3 + 2500S^4$
$\delta = 4R^4 - 48R^3S + 1352R^2S^2 + 1200RS^3 + 2500S^4$

The table below contains numerical values for a few parameters that support the proposal.

R	S	α	β	γ	δ	$\frac{\alpha+\beta+\gamma}{3}$	$\frac{\alpha+\beta+\delta}{3}$	$\frac{\alpha+\gamma+\delta}{3}$	$\frac{\beta+\gamma+\delta}{3}$
2	3	−277244	187588	613828	314884	$(418)^2$	$(274)^2$	$(466)^2$	$(610)^2$
3	2	−177404	62788	189508	115204	$(158)^2$	$(14)^2$	$(206)^2$	$(350)^2$
4	1	−23804	23428	388	26884	$(2)^2$	$(94)^2$	$(34)^2$	$(130)^2$
6	2	−349952	203008	166144	276736	$(80)^2$	$(208)^2$	$(176)^2$	$(464)^2$

Proposal 1. 2.
Let us designate an additional solution of (16) be
$m = rs$ (19)
$n = \frac{(r^2 - s^2)}{10}$ (20)
For receiving an integer solution of equation (16), let us prefer the fortuitous of r and s as given below.
$r = 10R$ and
$s = 10S$
Hence,(19) and (20) are developed into
$m = 100RS$
$n = 10R^2 - 10\,S^2$
Substituting the above-mentioned values of m and n in (12), (13), (14) and (15)), the comparable values of α , β, γ and δ are turned into
$\alpha = -4700R^4 + 42000R^3S + 19400R^2S^2 - 42000RS^3 - 4700S^4$
$\beta = 9700R^4 + 6000R^3S - 9400R^2S^2 - 6000RS^3 + 9700S^4$
$\gamma = -4700R^4 - 42000R^3S + 19400R^2S^2 + 42000RS^3 - 4700S^4$
$\delta = 9700R^4 - 6000R^3S - 9400R^2S^2 + 6000RS^3 + 9700S^4$

Algebraic Computations of an essential triples are enumerated below for few chances of the newly introduced parameters *R* and *S*.

R	S	α	β	γ	δ	$\frac{\alpha+\beta+\gamma}{3}$	$\frac{\alpha+\beta+\delta}{3}$	$\frac{\alpha+\gamma+\delta}{3}$	$\frac{\beta+\gamma+\delta}{3}$
2	3	−1017500	422500	1502500	782500	$(550)^2$	$(250)^2$	$(650)^2$	$(950)^2$
3	2	1502500	782500	−1017500	422500	$(650)^2$	$(950)^2$	$(550)^2$	$(250)^2$
4	1	1622500	2702500	−3417500	1982500	$(550)^2$	$(1450)^2$	$(250)^2$	$(650)^2$
6	2	12755200	13676800	−19500800	9068800	$(1520)^2$	$(3440)^2$	$(880)^2$	$(1040)^2$

Procedure 2.
A tactic of an alternative translations $p = m + n$, $q = m - n$, $r = m + 3n$ and $s = m - 3n$ where $m \neq n \neq 0$ in (6), (7), (8) and (9)deliver the assessment of α, β , γ and δas follows
$\alpha = m^2 - 7\,n^2 + 18\,mn$ (21)
$\beta = m^2 - 7\,n^2 - 18\,mn$ (22)
$\gamma = m^2 + 17\,n^2 + 6\,mn$ (23)
$\delta = m^2 + 17\,n^2 - 6\,mn$ (24)
Reservation of the same alterations in (11)leads to an equation of the form as below
$z^2 = (2m)^2 + 20n^2$ (25)
Applying few methods of solving (25), an estimation of a gorgeous integer quadruple fulfilling the condition that the average of three quantities stays a square number is established as follows.

Proposal 2. 1.

By considering the general solutions to the equation of the form $z^2 = Dx^2 + y^2$ where D is a square free integer, the choice of m, n and z attained from (25) are as given below

$$\left.\begin{aligned} m &= \frac{1}{2}[a^2 - 20\,b^2] \\ n &= 2ab \\ z &= a^2 + 20\,b^2 \end{aligned}\right\} \quad (26)$$

In order to find out an integer solution, striking the replacement as $a = 2A$ and b is an arbitrary value in (26) , the equivalent values m, n and z are revealed by

$m = 2A^2 - 10b^2$
$n = 4Ab$
$z = 4A^2 + 20b^2$

Exchanging the values of m and n in (21), (22), (23) and (24), the exact chances of the non-zero parameters α, β, γ and δ are turned out to be

$\alpha = 4A^4 + 144A^3b - 152A^2b^2 - 720Ab^3 + 100b^4$
$\beta = 4A^4 - 144A^3b - 152A^2b^2 + 720Ab^3 + 100b^4$
$\gamma = 4A^4 + 48A^3b + 232A^2b^2 - 240Ab^3 + 100b^4$
$\delta = 4A^4 - 48A^3b + 232A^2b^2 + 240Ab^3 + 100b^4$

Numerical illustrations for some positive values of *A* and *b* filling the supposition are presented in the below table.

A	b	α	β	γ	δ	$\frac{\alpha+\beta+\gamma}{3}$	$\frac{\alpha+\beta+\delta}{3}$	$\frac{\alpha+\gamma+\delta}{3}$	$\frac{\beta+\gamma+\delta}{3}$
2	3	-32732	38116	4708	28324	$(58)^2$	$(106)^2$	$(10)^2$	$(154)^2$
3	2	-13052	5956	7108	13444	$(2)^2$	$(46)^2$	$(50)^2$	$(94)^2$
4	1	5028	-7644	6948	2724	$(38)^2$	$(6)^2$	$(70)^2$	$(26)^2$
6	2	12544	-42752	49408	30976	$(80)^2$	$(16)^2$	$(176)^2$	$(112)^2$

Proposal 2. 2.

Subcase 2. 2. 1.

Factorization of (25) provides the following equation

$$(z + 2m)(z - 2m) = 20n^2 \quad (27)$$

The overhead equation can be inscribed in the fraction form as

$$\frac{(z-2m)}{(20n)} = \frac{(n)}{(z+2m)} = \frac{a}{b}, b \neq 0 \quad (28)$$

Modify (28) into the subsequent form of double equations

$$zb - 2bm - 20an = 0 \quad (29)$$
$$az + 2am - bn = 0 \quad (30)$$

Resolving (29) and (30) by cross multiplication rule, the values of $m, n$and z needed for accomplishing an integer solution of (25) are furnished as

$m = -20a^2 + b^2$
$n = 4ab$
$z = 40\,a^2 + 2b^2$

Transmitting the values of m and n in (21), (22), (23) and (24), the precise options of α, β, γ and $\delta$are converted into

$\alpha = 400a^4 - 144a^3b - 152a^2b^2 + 72ab^3 + b^4$
$\beta = 400a^4 + 144a^3b - 152a^2b^2 - 72ab^3 + b^4$
$\gamma = 400a^4 - 480a^3b + 232a^2b^2 + 24ab^3 + b^4$
$\delta = 400a^4 + 480a^3b + 232a^2b^2 - 24ab^3 + b^4$

Examples for few positive values of a and b gratifying the proposition are given in the succeeding table.

a	b	α	β	γ	δ	$\frac{\alpha+\beta+\gamma}{3}$	$\frac{\alpha+\beta+\delta}{3}$	$\frac{\alpha+\gamma+\delta}{3}$	$\frac{\beta+\gamma+\delta}{3}$
2	3	-29663	31681	4609	25057	$(47)^2$	$(95)^2$	$(1)^2$	$(143)^2$
3	5	-168575	166225	29425	141025	$(95)^2$	$(215)^2$	$(25)^2$	$(335)^2$
4	1	8097	191841	75489	136737	$(303)^2$	$(335)^2$	$(271)^2$	$(367)^2$
6	2	-122096	1115152	3456616	758032	$(668)^2$	$(764)^2$	$(572)^2$	$(860)^2$

Subcase $2.2.2$.

Rewrite (27) in the ratio as

$$\frac{(z-2m)}{n}=\frac{20n}{(z+2m)}=\frac{a}{b}, b\neq 0$$

Applying the same technique as explained in subcase 2.2.1, the particular selections of the elements in the quadruple are expressed by

$$\alpha = a^4 - 72a^3b - 152a^2b^2 + 1440ab^3 + 400b^4$$
$$\beta = a^4 + 72a^3b - 152a^2b^2 - 1440ab^3 + 400b^4$$
$$\gamma = a^4 - 24a^3b + 232a^2b^2 + 480ab^3 + 400b^4$$
$$\delta = a^4 + 24a^3b + 232a^2b^2 - 480ab^3 + 400b^4$$

Some models are acknowledged in the subsequent table.

a	b	α	β	γ	δ	$\frac{\alpha+\beta+\gamma}{3}$	$\frac{\alpha+\beta+\delta}{3}$	$\frac{\alpha+\gamma+\delta}{3}$	$\frac{\beta+\gamma+\delta}{3}$
2	3	102976	-49088	66112	15424	$(200)^2$	$(152)^2$	$(240)^2$	$(104)^2$
3	5	746161	-314399	479041	125521	$(551)^2$	$(431)^2$	$(671)^2$	$(311)^2$
4	1	-624	-2928	4752	3984	$(20)^2$	$(12)^2$	$(52)^2$	$(44)^2$
6	2	23824	-52208	53776	28432	$(92)^2$	$(4)^2$	$(188)^2$	$(100)^2$

Subcase $2.2.3$.

Redraft (27) as

$$\frac{(z+2m)}{20n}=\frac{n}{(z-2m)}=\frac{a}{b}, b\neq 0$$

By manipulating the identical procedure as clarified in subcase 2.2.1, the specific collections of the elements in an essential quadruples are articulated by

$$\alpha = 400a^4 + 1440a^3b - 152a^2b^2 - 72ab^3 + b^4$$
$$\beta = 400a^4 - 1440a^3b - 152a^2b^2 + 72ab^3 + b^4$$
$$\gamma = 400a^4 + 480a^3b + 232a^2b^2 - 24ab^3 + b^4$$
$$\delta = 400a^4 - 480a^3b + 232a^2b^2 + 24ab^3 + b^4$$

The following table offered numerical values for limited choice of the parameters rewarding the hypothesis.

a	b	α	β	γ	δ	$\frac{\alpha+\beta+\gamma}{3}$	$\frac{\alpha+\beta+\delta}{3}$	$\frac{\alpha+\gamma+\delta}{3}$	$\frac{\beta+\gamma+\delta}{3}$
2	3	17169	-5583	11121	3537	$(87)^2$	$(71)^2$	$(103)^2$	$(55)^2$
3	2	102976	-49088	66112	15424	$(200)^2$	$(152)^2$	$(248)^2$	$(104)^2$
3	4	152464	-130928	113296	18832	$(212)^2$	$(116)^2$	$(308)^2$	$(20)^2$
6	3	1390689	-452223	900801	286497	$(783)^2$	$(639)^2$	$(927)^2$	$(495)^2$

Subcase 2.2.4.
Alternate (27) as

$$\frac{(z+2m)}{n} = \frac{20n}{(z-2m)} = \frac{a}{b}, b \neq 0$$

As in subcase 2.2.1, the gathering of the elements in the crucial quadruple are expressed by

$\alpha = a^4 + 72a^3b - 152a^2b^2 - 1440ab^3 + 400b^4$
$\beta = a^4 - 72a^3b - 152a^2b^2 + 1440ab^3 + 400b^4$
$\gamma = a^4 + 24a^3b + 232a^2b^2 - 480ab^3 + 400b^4$
$\delta = a^4 - 24a^3b + 232a^2b^2 + 480ab^3 + 400b^4$

Numerical characters for selected choices of a and b gratifying the proposal are presented in the below table.

a	b	α	β	γ	δ	$\frac{\alpha+\beta+\gamma}{3}$	$\frac{\alpha+\beta+\delta}{3}$	$\frac{\alpha+\gamma+\delta}{3}$	$\frac{\beta+\gamma+\delta}{3}$
2	3	-2496	2112	576	2112	$(8)^2$	$(24)^2$	$(8)^2$	$(40)^2$
3	2	-29663	31681	4609	25057	$(47)^2$	$(95)^2$	$(1)^2$	$(143)^2$
3	4	-188111	349297	46321	225457	$(263)^2$	$(359)^2$	$(167)^2$	$(455)^2$
6	3	-202176	171072	46656	171072	$(72)^2$	$(216)^2$	$(72)^2$	$(360)^2$

The Python Program for the authentication of the needed quadruples fulfilling our statement is epitomized below.

```
import math
while True:
T = input("Enter your choice part A of B : ")
if T in ('A'):
c = input("Enter choice(1/2): ")
if c in ('1','2'):
r = int(input('Enter a Number r : '))
s = int(input('Enter the second number s : '))

if (c == '1'):
m = 2 * r ** 2 - 50 * s ** 2
n = 4 * r * s

elif(c == '2'):

m = 100 * r * s
n = 10 * r ** 2 - 10 * s ** 2
else:
print('Invalid Input')
break

p = m ** 2 - 47 * n ** 2 + 42 * m * n
q = m ** 2 + 97 * n ** 2 + 6 * m * n
r = m ** 2 - 47 * n ** 2 - 42 * m * n
s = m ** 2 + 97 * n ** 2 - 6 * m * n
a1 = (p + q + r)/3
a2 = (p + q + s)/3
a3 = (p + r + s)/3
a4 = (q + r + s)/3
if (a1 < 0):
a11 = -1 * a1
x = pow(a11,1/2)
```

```
else:
x = pow(a1,1/2)

if (a1 < 0):
x = -x

if (a2 < 0):
a21 = -1 * a2
y = pow(a21,1/2)
else:
y = pow(a2,1/2)

if (a2 < 0):
y = -y

if (a3 < 0):
a31 = -1 * a3
z = pow(a31,1/2)
else:
z = pow(a3,1/2)
if (a3 < 0):
z = -z


if (a4 < 0):
a41 = -1 * a4
xx = pow(a41,1/2)
else:
xx = pow(a4,1/2)

if (a4 < 0):
xx = -xx

print('p : ',p)
print('q : ',q)
print('r : ',r)
print('s : ',s)
print('(p + q + r)/3 :  ',int(x),'^2')
print('(p + q + s)/3 :  ',int(y),'^2')
print('(p + r + s)/3 :  ',int(z),'^2')
print('(q + r + s)/3 :  ',int(xx),'^2')

elif T in ('B'):
c =  input('Enter your choice 3/4/5/6/7   : ')

if c in ('3','4','5','6','7'):
a =  int(input('Enter a Number a :  '))
b =  int(input('Enter the second number b :  '))

if(c  == '3'):

m  =  -10 *  b ** 2  + 2  *  a ** 2
n  =  4 * a * b

elif(c == '4'):
```

```
m = b ** 2 - 20 * a ** 2
n = 4 * a * b

elif(c == '5'):

m = -a ** 2 + 20 * b ** 2
n = 4 * a * b
elif(c == '6'):

m = -20 * a ** 2 + b ** 2
n = -4 * a * b

elif(c == '7'):

m = -a ** 2 + 20 * b ** 2
n = -4 * a * b
else:
print('Invalid Input')
break

p = m ** 2 - 7 * n ** 2 + 18 * m * n
q = m ** 2 - 7 * n ** 2 - 18 * m * n
r = m ** 2 + 17 * n ** 2 + 6 * m * n
s = m ** 2 + 17 * n ** 2 - 6 * m * n
a1 = (p + q + r)/3
a2 = (p + q + s)/3
a3 = (p + r + s)/3
a4 = (q + r + s)/3
if (a1 < 0):
a11 = -1 * a1
x = pow(a11,1/2)
else:
x = pow(a1,1/2)

if (a1 < 0):
x = -x

if (a2 < 0):
a21 = -1 * a2
y = pow(a21,1/2)
else:
y = pow(a2,1/2)
if (a2 < 0):
y = -y

if (a3 < 0):
a31 = -1 * a3
z = pow(a31,1/2)
else:
z = pow(a3,1/2)
if (a3 < 0):
z = -z

if (a4 < 0):
a41 = -1 * a4
xx = pow(a41,1/2)
```

```
else:
xx = pow(a4,1/2)
if (a4 < 0):
xx = -xx

print('p : ',p)
print('q : ',q)
print('r : ',r)
print('s : ',s)
print('(p + q + r)/3 : ',int(x),'^2')
print('(p + q + s)/3 : ',int(y),'^2')
print('(p + r + s)/3 : ',int(z),'^2')
print('(q + r + s)/3 : ',int(xx),'^2')
```

Conclusion.

In this communication, the quadruple $(\alpha,\ \beta,\ \gamma,\ \delta)$in which the average of any three numbers remains a square of an integer is examined. To conclude, one can explore stimulating quadruples and quintuples such that geometric mean of two or three numbers is a square of an integer.

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