The Geo Chromatic Number of Strong Product of Graphs

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Abstract

A set $S_c \subseteq V(G)$ is said to be a geo chromatic set of G if S_c is both a geodetic set and a chromatic set of G . The minimum cardinality among all geo chromatic sets of a graph G is the geo chromatic number and is denoted by $\chi_{gc}(G)$. A set of extreme vertices E of G is said to be a weak extreme chromatic set if $Ext(G)$ is a chromatic set of G. A weak extreme chromatic set is denoted by $WExt(G)$ and the number of extreme vertices in $WExt(G)$ is its weak extreme order $\chi_w(G)$. A graph G is an extreme geo chromatic graph if $\chi_{gc}(G) = ex(G)$. Bounds for the geo chromatic number of strong product graphs are obtained.

Keywords: geodetic number, chromatic number, geo chromatic number, weak extreme chromatic set, extreme geo chromatic graph, strong product.

Subject Classification: 05C15, 05C12

1 Introduction

We consider finite simple connected graphs with at least two vertices. For any graph G, the set of vertices is denoted by $V(G)$ and the edge set by $E(G)$. The order and size of G are denoted by p and q respectively. For basic graph theoretic terminology we refer to Harary [4]. The distance $d(u, v)$ between two vertices u and v in a connected graph G is the length of a shortest $u - v$ path in G. An $u - v$ path of length $d(u, v)$ is called an $u - v$ geodesic. For each vertex $v \in V(G)$, the open neighborhood of v is the set $N(v)$ containing al the vertices u adjacent to v and the closed neighborhood of v is the set $N[v] = N(v) \cup v$. If the subgraph induced by its neighbors is complete then a vertex v is called an extreme vertex of a graph G. The set of all extreme vertices of G is denoted by $Ext(G)$ and $|Ext(G)| = ex(G)$. A vertex of G is a stem if it is adjacent to an end vertex. The set of all stem vertices is denoted by $Stem(G)$. A vertex x is said to lie on an $u - v$ geodesic P, if x is a internal vertex of P. The closed interval $I[u,v]$ consists of u,v and all vertices lying on a $u-v$ geodesic of G and for a non-empty set $S \subseteq V(G)$, $I[S] = \bigcup_{u,v\in S} I[u,v]$. A set S of vertices is a geodetic set, if $I[S] = V(G)$. The minimum cardinality among all geodetic sets of G is the geodetic number and is denoted by $g(G)$. Geodetic number was introduced in [4] and further studied in [1, 5-18, 20-23] . The strong product of graph G_1 and G_2, denoted by $G_1 \otimes G_2$, has vertex set $V(G_1) \times V(G_2)$, where two distinct vertices (u_1,v_1) and (u_2,v_2) are adjacent with respect to the strong product if

(i) $v_1 = v_2$ and $u_1u_2 \in E(G_1)$ or
(ii) $u_1 = u_2$ and $v_1v_2 \in E(G_2)$ or
(iii) $u_1u_2 \in E(G_1)$ and $v_1v_2 \in E(G_2)$

The mappings $\pi_{G_1}: (u,v) \to u$ amd $\pi_{G_2}: (u,v) \to v$ from $V(G_1 \otimes G_2)$ onto G_1 and G_2 respectively are called projections. For a set $S \subseteq V(G_1 \otimes G_2)$, we define the G_1–projection in G_1 as $\pi_{G_1}(S) = \{u \in V(G_1) : u,v \in S$ for some $v \in V(G_2)\}$ and the G_2 – projection in G_2 as $\pi_{G_2}(S) = \{v \in V(G_2) : u,v \in S$ for some $u \in V(G_1)\}$. For $u_i \in V(G_1)$, the set $G_2^i = u_{i_i} \times G_2$ is a layer of G_2 and for $vj \in V(G_2)$, the set $G_1^j = G_1 \times v_j$ is a layer of G_1. For references see [19] . A c-vertex coloring of G is an assignment of c colors, $1,2,\ldots,c$ to the vertices of G; the coloring is proper if no two distinct adjacent vertices have the same color. If $\chi(G) = c$, G is said to be c-chromatic. A set $C \subseteq V(G)$ is called a chromatic set if C contains all c vertices of distinct colors in G. Chromatic number of G is the minimum cardinality among all chromatic sets of G. That is $\chi(G)$ = min $\{|C| / C$ is the chromatic set of $G\}$. For references on chromtic sets see [7] .

A set $S_c \subseteq V(G)$ is said to be a geo chromatic set of G if S_c is both a geodetic set and a chromatic set of . The minimum cardinality among all geo chromatic sets of a graph G is the geo chromatic number and is denoted by $\chi_{gc}(G)$. For references see [2,3].The following theorems are used in sequel.

Theorem 1.1. [19] Let G_1 and G_2 be connected graphs and $P: (u,v) - (u',v')$ geodesic in $G_1 \otimes G_2$ of length p. If $d_{G_1}(u,u') \geq d_{G_2}(v,v')$, then $\pi_{G_1}(P)$ is a $u - u'$ geodesic in G_1 of length p, and if $d_{G_1}(u,u') \leq d_{G_2}(v,v')$, then $\pi_{G_2}(P)$ is a $v - v'$ geodesic in G_2 of length p.

Theorem 1.2. [19] Let G_1 and G_2 be connected graphs. Then $Ext(G_1 \otimes G_2) = Ext(G_1) \times Ext(G_2)$.
Theorem 1.3. [19] Let $S_1 \subseteq V(G_1)$ and $S_2 \subseteq V(G_2)$ for graphs G_1 and G_2, then $I_{G_1}[S_1] \times I_{G_2}[S_2] \subseteq I_{I_{G_1 \otimes G_2}}[S_1 \times S_2]$.
Theorem 1.4.[6] Let G be a connected graph of order p and diameter d. Then $g(G) \leq$
$p - d + 1$.
Theorem 1.5.[2] For the complete graph $K_p (p \geq 2)$, $\chi_{gc}(K_p) = p$.

2 .Weak Extreme Chromatic Set

Definition 2.1. A set of extreme vertices E of G is said to be a weak extreme chromatic set if $Ext(G)$ is a chromatic set of G. A weak extreme chromatic set is denoted by $WExt(G)$ and the number of extreme vertices in $WExt(G)$ is its weak extreme order $\chi_w(G)$.
Example 2.2. Let the path $P_{2n}: v_1, v_2, \dots, v_{2n}$ $(n \geq 1)$. Then $Ext(P_{2n}) = \{v_1, v_{2n}\}$.
Clearly $Ext(P_{2n})$ is a chromatic set and so $WExt(P_{2n}) = \{v_1, v_{2n}\}$. For the path $P_{2n+1}: v_1, v_2, \dots, v_{2n+1}$ $(n \geq 1)$, let $Ext(P_{2n+1}) = \{v_1, v_{2n+1}\}$. It is clear that $Ext(P_{2n+1})$ is not a chromatic set. Therefore P_{2n+1} does not have a weak extreme chromatic set.
Observation 2.3. Let G be a connected graph of order p. Then
1. no cut vertex belongs to a weak extreme chromatic set of G.
2. $WExt$(G) is the set of all extreme vertices of G.
Theorem 2.4. Let G be a connected graph of order p and diameter d. Then $\chi_w(G) \leq p - d + 1$.
Proof. Let $W\,Ext(G) = \{u_1, u_2, \dots, u_{p-d+1}\}$ is a weak extreme chromatic set of G. It is clear that $WExt(G)$ is $Ext(G)$. Then $Ext(G) = \{u_1, u_2, \dots, u_{p-d+1}\}$ is an extreme set of G. Also by Theorem 1.4, we have $ex(G) \leq p - d + 1$ and so $\chi_w(G) \leq p - d + 1$.
Remark 2.5. The upper bound in Theorem 2.4 is sharp. For the graph G given in Figure
2.1, $WExt(G) = \{u_1, u_5, u_6, u_7\}$, $p = 7$ and $d = 4$. Therefore $\chi_w(G) = 4 = p - d + 1$

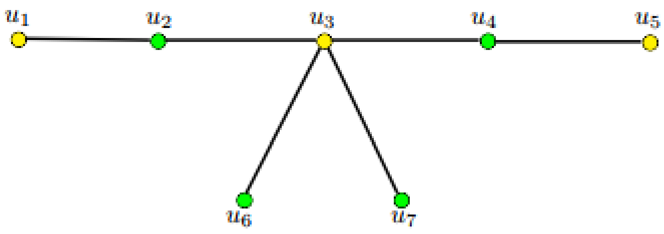

Figure 2.1 : G .

Theorem 2.6. Let T be a tree of order $p \geq 2$. Then one of the following condition hold.

1. $\chi_{gc}(T) = g(T)$
2. $\chi_{gc}(T) = g(T) + 1$

Proof. For any tree T, $Ext(T) = L(T)$ is a unique geodetic set of T . Let $V(T) = \{v_1, v_2, \dots, v_p\}$. We consider two cases.

Case 1. Suppose that $Ext(T)$ is $WExt(T)$.

Clearly, $Ext(T)$ is a unique minimum geo chromatic set of T. Hence $\chi_{gc}(T) = g(T)$.

Case 2. Suppose that $Ext(T)$ is not a $WExt(T)$.

It is clear that $Ext(T)$ is not a chromatic set of T. Clearly $\chi_{gc}(T) > g(T)$. Since $\chi(T) = 2$, the vertices from one color class, say C_{l1} belong to S and no vertex from another color class, say C_{l2} belongs to S. For obtaining S as a chromatic set, choose at least one vertex from C_{l2}. Let $v_{p-1} \in C_{l2}$. If $v_{p-1} \in S$, then $S_c = S \cup \{v_{p-1}\}$ is a geo chromatic set of T and $\chi_{gc}(T) \leq g(T) + 1$. It is clear that the removal of at least one vertex from $S_c = S \cup \{v_{p-1}\}$ is not a geo chromatic set of T. Hence $\chi_{gc}(T) = g(T) + 1$.

3. Extreme Geo Chromatic Graph

Definition 3.1. A graph G is an extreme geo chromatic graph if $\chi_{gc}(G) = ex(G)$.

Example 3.2. For the graph G given in Figure 2.2, $Ext(G) = \{v_1, v_8\}$. Therefore $ex(G) = 2$. Also $S_c = \{v_1, v_8\}$ is a minimum geo chromatic set of G and so $\chi_{gc}(G)$ = $ex(G) = 2$. Hence G is an extreme geo chromatic graph.

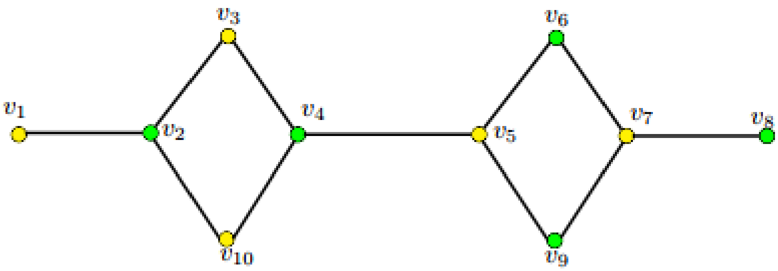

Figure 2.2 : G .

Remark 3.3. Let G_1 be the graph obtained from Figure 2.2 by joining one vertex, say v_9 to v_8. The graph G_1 is shown in Figure 2.3 . Clearly $ex(G_1) = 2$. But $S_c = \{v_1, v_8, v_9\}$ is a minimum geo chromatic set of G_1 and so $\chi_{gc}(G_1) = 3$. Therefore $\chi_{gc}(G_1) \neq ex(G_1)$ and G_1 is not a extreme geo chromatic graph.

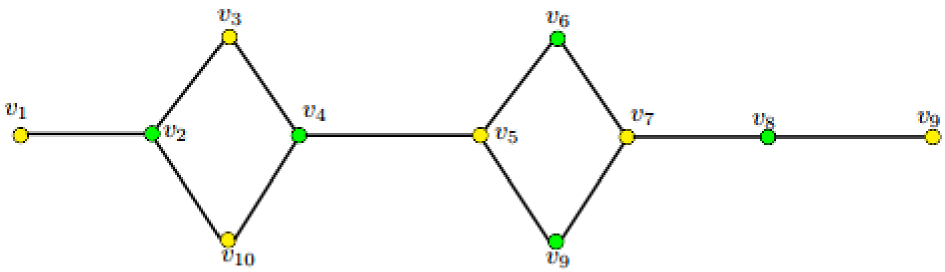

Figure 2.3 : G_1 .

Theorem 3.4. Let G be an extreme geo chromatic graph of order p. Then $WExt(G) \subseteq S_c$.

Proof. Let S_c, $Ext(G)$ and $WExt(G)$ be a geo chromatic set, extreme set and weak extreme chromatic set of G respectively. Since $S \subseteq S_c, Ext(G) \subseteq S_c$. Also by Observation 2.3(2) , $WExt(G)$ is $Ext(G)$. It follows that $WExt(G) \subseteq S_c$. ■

Theorem 3.5. Let G be an extreme geo chromatic graph of order $p \geq 2$. Then $\chi_{gc}(G) = p$ if and only if $G = K_p$.

Proof. If $G = K_p$, then $\chi_{gc}(G) = p$ (Theorem 1.5). Conversely, suppose that $\chi_{gc}(G) = p$. Since G is an extreme geo chromatic graph, $\chi_{gc}(G) = ex(G)$. It is clear that $ex(G) = p$. Therefore, every vertex of G is an extreme vertex. Hence $G = K_p$. ■

4 .The Geo Chromatic Number of a strong product graph

Theorem 4.1. For a connected graph $P_k \otimes P_2$ $(k < p)$,

$$\chi_{gc}(P_k \otimes P_2) = \begin{cases} 4 \; if \; k \; is \; even \\ 6 \; if \; k \; is \; odd \end{cases}$$

Proof. Let $G = P_k \otimes P_2$ and $\{(a_i, b_j) : 1 \leq i \leq k, j = 1,2\}$ be the set of all vertices of G. It is clear that $S = \{(a_1, b_1), (a_1, b_2), (a_k, b_1), (a_k, b_2)\}$ is a geodetic set of G, where $(a_1, b_1), (a_1, b_2)$ are the vertices of the first layer of G and $(a_k, b_1), (a_k, b_2)$ are the vertices of the k^{th} layer of G. Let us consider two cases.

Case 1. Suppose that k is even.

Define a proper coloring of G such that different vertices of S receive distinct colors, say color 1 , color 2 , color 3 , color 4 and the vertices of $G\backslash S$ repeated by the four colors of S. Therefore $\chi(G) = 4$. It is clear that S is a chromatic set of G. Thus S is a geo chromatic set S_c of G and $\chi_{gc}(G) \leq 4$. Removal of at least one vertex from S is not a geo chromatic set of G. Hence $\chi_{gc}(G) = 4$.

Case 2. Suppose that k is odd.

Define a proper coloring of G such that $(a_1, b_1), (a_k, b_1)$ receive color 1, $(a_1, b_2), (a_k, b_2)$ receive color 2 and the vertices of $G\backslash S$ repeated by four color, say color 1 , color 2 , color 3 , color 4 . Therefore $\chi(G) = 4$. Let the vertices which receive color 1 , color 2 , color 3 and color 4 belong to the color classes, namely C_{l1}, C_{l2}, C_{l3} and C_{l4}. Since no vertex from C_{l3}, C_{l4} belongs to S, S is not a chromatic set of G. Let $(a_2, b_1) \in C_{l3}$ and $(a_2, b_2) \in C_{l4}$. If $(a_2, b_1), (a_2, b_2) \in S$, then $S_c = S \cup \{(a_2, b_1), (a_2, b_2)\}$ is a geo chromatic set of G and $\chi_{gc}(G) \leq 6$. But, the removal of at least one from $S \cup \{(a_2, b_1), (a_2, b_2)\}$ is not a geo chromatic set of G. Hence $\chi_{gc}(G) = 6$.

■

Theorem 4.2. Let G_1 and G_2 be non - trivial connected graphs. Then

$\chi_{gc}(G_1 \otimes G_2) \geq 4$.

Proof. Since K_4 is an induced subgraph of $G_1 \otimes G_2$, by assigning a proper coloring, at least four colors are needed for $G_1 \otimes G_2$. It is clear that at least four vertices must belong to a geo chromatic set of $G_1 \otimes G_2$. Hence $\chi_{gc}(G_1 \otimes G_2) \geq 4$.

■

Remark 4.3. The bound in Theorem 4.2 is sharp. For example, if $G_1 = P_2$ and $G_2 = P_2$, then $G_1 \otimes G_2 = 4$.

Theorem 4.4. Let G_1 and G_2 be connected graphs and S_c a geo chromatic set of $G_1 \otimes G_2$. If $xt(G_1) \neq \phi$, then $\pi_{G_2}(S_c)$ is a geo chromatic set of G_2.

Proof. Let S_c be a geo chromatic set of $G_1 \otimes G_2$ and $Ext(G_1) \neq \phi$. We have to prove that $\pi_{G_2}(S_c)$ is a geo chromatic set of G_2.

First we prove that $\pi_{G_2}(S_c)$ is a chromatic set of G_2. Suppose not, $\pi_{G_2}(S_c)$ is not a chromatic set of G_2. Then there exists a color class, say C_l such that no vertex from C_l belongs to $\pi_{G_2}(S_c)$. Clearly S_c is not a chromatic set of $G_1 \otimes G_2$, which is a contradiction. Hence $\pi_{G_2}(S_c)$ is a chromatic set of G_2.

Again we prove that $\pi_{G_2}(S_c)$ is a geodetic set of G_2. Let $u \in WExt(G_1)$ and $v \in V(G_2)$. Since S_c is a χ_{gc}-set of $G_1 \otimes G_2$, the vertex (u, v) lies on a geodesic $P': (x_0, y_0), (x_1, y_1), \dots, (x_i, y_i) = (u, v), \dots, (x_p, y_p)$ of length p with (x_0, y_0), $(x_p, y_p) \in S_c$. First, suppose that $d_{G_1}(x_0, x_p) \leq d_{G_2}(y_0, y_p)$. Then it follows that that $\pi_{G_2}(P')$ is a $y_0 - y_p$ geodesic in G_2 containing the vertex v, with $y_0, y_p \in \pi_{G_2}(S_c)$. Similarly assume that $d_{G_1}(x_0, x_p) > d_{G_2}(y_0, y_p)$. Then, as above, by Proposition 1.1, $\pi_{G_1}(P')$ is a $x_0 - x_p$ geodesic in G_1 containing u. Since u is the weak extreme vertex, either $u = x_0$ or $u = x_p$ and it follows that either $v = y_0$ or $v = y_p$ so that $\pi_{G_2}(S_c)$ is a geodetic set of G_2. Hence $\pi_{G_2}(S_c)$ is a geo chromatic set of G_2. ■

Theorem 4.5. Let G_1 and G_2 be connected graphs such that $WExt(G_1) \neq \phi$. Then $\chi_{gc}(G_2) \leq \chi_{gc}(G_1 \otimes G_2)$.

Proof. Let $WExt(G_1) \neq \phi$ and S_c be a geo chromatic set of $G_1 \otimes G_2$. Then by Theorem 4.4 , $\pi_{G_2}(S_c)$ is a geo chromatic set of G_2 so that $\chi_{gc}(G_2) = |\pi_{G_2}(S_c)| \leq |S_c| = \chi_{gc}(G_1 \otimes G_2)$.
■

Corollary 4.6. Let G_1 and G_2 be connected graphs such that $WExt(G_1) \neq \phi$ and $WExt(G_2) \neq \phi$. Then $max\, \{\chi_{gc}(G_1), \chi_{gc}(G_2)\} \leq \chi_{gc}(G_1 \otimes G_2)$.

Proof. This follows from Theorem 4.5. ■

Theorem 4.7. Let G_1 and G_2 be connected graphs. If S_{c_1} and S_{c_2} are geo chromatic sets of G_1 and G_2 respectively, then $S_{c_1} \times S_{c_2}$ is a geo chromatic set of $G_1 \otimes G_2$.

Proof. Let S_{c_1} and S_{c_2} be a geo chromatic sets of G_1 and G_2 respectively. Let S_1 and S_2 be a geodetic sets of G_1 and G_2. Then $I_{G_1}[S_1] = V(G_1)$ and $I_{G_2}[S_2] = V(G_2)$. By Theorem 1.3 , $I_{G_1}[S_1] \times I_{G_2}[S_2] \subseteq I_{G_1 \otimes G_2}[S_1 \times S_2]$. Also, let C_1 and C_2 be a chromatic sets of G_1 and G_2. Clearly, $C_1 \times C_2$ is a chromatic set of $G_1 \otimes G_2$. Hence $S_{c_1} \times S_{c_2}$ is a geo chromatic set of $G_1 \otimes G_2$.

■

Theorem 4.8. Let G_1 and G_2 be connected graphs. Then $\chi_{gc}(G_1 \otimes G_2) \leq \chi_{gc}(G1).\chi_{gc}(G_2)$.

Proof. Let S_{c_1} and S_{c_2} be a minimum geo chromatic sets of G_1 and G_2 respectively. Clearly $\chi_{gc}(G_1) = |S_{c_1}|$ and $\chi_{gc}(G_2) = |S_{c_2}|$. By Theorem 4.7 , $S_{c_1} \times S_{c_2}$ is a geo chromatic set of $G_1 \otimes G_2$. Hence $\chi_{gc}(G_1 \otimes G_2) \leq |S_{c_1} \times S_{c_2}| = |S_{c_1}| \cdot |S_{c_2}| = \chi_{gc}(G_1) \cdot \chi gc(G2)$. ■

Remark 4.9. The bound in Theorem 4.8 is sharp. If $G_1 = Km$ and $G_2 = Kn$, then $\chi_{gc}(K_m \otimes K_n) = mn = \chi_{gc}(K_m) \cdot \chi_{gc}(K_n)$.

Theorem 4.10. Let G_1 and G_2 be connected graphs. Then $W\,Ext(G_1 \otimes G_2) = WExt(G_1) \times WExt(G_2)$.

Proof. By Observation 2.3(2) , $WExt(G_1 \otimes G_2)$ is $Ext(G_1 \otimes G_2)$ and by Theorem 1.2, $Ext(G_1 \otimes G_2) = Ext(G_1) \times Ext(G_2)$. It follows that $WExt(G_1 \otimes G_2) = WExt(G_1) \times WExt(G_2)$. ■

Theorem 4.11. Let G_1 and G_2 be extreme geo chromatic graphs. Then $WExt(G_1)$ and $WExt(G_2)$ are geo chromatic sets of G_1 and G_2 if and only if $WExt(G_1 \otimes G_2)$ is a geo chromatic set of $G_1 \otimes G_2$.

Proof. Suppose that $W\,Ext(G_1)$ and $WExt(G_2)$ are geo chromatic sets of G_1 and G_2 respectively. By Theorems 4.7 and 4.10, $WExt(G_1 \otimes G_2) = WExt(G_1) \times W\,Ext(G_2)$ is a geo chromatic set of $G_1 \otimes G_2$. Conversely, suppose that $WExt(G_1 \otimes G_2)$ is a geo chromatic set of $G_1 \otimes G_2$. It is clear that $WExt(G_1) \neq \phi$ and $WExt(G_2) \neq \phi$. Then by Theorem 4.5 and by Theorem 4.7 , $WExt(G_1)$ and $WExt(G_2)$ are geo chromatic sets of G_1 and G_2 respectively. Now, we proceed to characterize graphs G_1 and G_2 for which $\chi_{gc}(G_1 \otimes G_2) = \chi_w(G_1) \cdot \chi_w(G_2)$.

■

Theorem 4.12. Let G_1 and G_2 be connected graphs. Then G_1 and G_2 are extreme geo chromatic graphs if and only if $G_1 \otimes G_2$ is a extreme geo chromatic graph.

Proof. Suppose that G_1 and G_2 be extreme geo chromatic graphs. Clearly, $WExt(G_1)$ and $WExt(G_2)$ are geo chromatic sets of G_1 and G_2 respectively. Then by Theorems 4.10 and 4.11 , $WExt(G_1 \otimes G_2) = WExt(G_1) \times WExt(G_2)$ is a geo chromatic set of $G_1 \otimes G_2$. Hence $G_1 \otimes G_2$ is an extreme geo chromatic graphs.

Conversely, Suppose that $G_1 \otimes G_2$ is an extreme geo chromatic graph. Then $WExt(G_1 \otimes G_2)$ is a geo chromatic set of $G_1 \otimes G_2$. Then by Theorem 4.11 , $WExt(G_1)$ and $WExt(G_2)$ are geo chromatic sets of G_1 and G_2 respectively. Hence G_1 and G_2 are extreme geo chromatic graphs. ■

Theorem 4.13. Let G_1 and G_2 be connected graphs. Then G_1 and G_2 are extreme geo chromatic graphs if and only if $\chi_{gc}(G_1 \otimes G_2) = \chi_w(G_1) \cdot \chi_w(G_2)$.

Proof. This follows from Theorems 4.8, 4.11 and 4.12 .

Remark 4.14. If G_1 and G_2 are not an extreme geo chromatic graph, then $\chi_{gc}(G_1 \otimes G_2) \neq ex(G_1) \cdot ex(G_2)$. For example, if $G_1 = P_3$ and $G_2 = P_3$, then $ex(G_1) = ex(G_2) = 2$. But $\chi_{gc}(G_1 \otimes G_2) = 6$.

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