Section A -Research paper

# Gaussian Graceful Labeling

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**Abstract**

The concept of "Gaussian graceful labeling (or $\alpha^g -$valuation)" is introduced in this paper while labeling graphs. Throughout the paper, we used the Gaussian integers to label the graph. For a graph G to be gracefully labeled with Gaussian integers, for any positive number p, the vertex labeling $\varphi^g : V \to \{0, z, 2z, ...., pz\}$, z is a Gaussian integer, which lies in the first quadrant and the edge labeling $l^g : E \to \{z, 2z, 3z, ...., pz\}$ is defined by $l^g(v_i v_j) = |\varphi^g(v_i) - \varphi^g(v_j)|$ must both be injective and also there exist $a \in \{0, z, 2z, ...., pz\}$ such that for any edge $v_i v_j$ of G holds, either $v_i \leq a,\ v_j > a$ or $v_j \leq a,\ v_i > a$. Let us refer to graph G as a Gaussian graceful graph if it permits graceful labeling. We establish a $\alpha^g -$valuation in this study and examine the valuation for some unique type of graphs $K^g_{1,p}, C^g_p, DW^g_p$, and family of fan graphs such as $K^g_{1,p,1}, F^g_{1,p}, F^g_{2,p}$

**Keywords**: *Graph labeling, Graceful labeling, Gaussian integers, Gaussian graceful labeling, Gaussian graceful graph,* $\alpha^g -$*valuation*

AMS Subject Classification (2020): 05C78, 05C76

## 1. Introduction

### 1.1 Graceful Labeling

In graph theory, a graph's edges and vertices are labeled. There has been significant interest in graph labeling as a subfield of graph theory. Graph theory presents many of its most challenging problems in a straightforward manner. In many cases, issues from this area attract attention because they are applied to real-life situations or because their history explains why they are of interest. Planar maps can, for instance, be coloured using a maximum of four colours so that no two neighbouring regions have the same colour. Computer software was used to check many cases to prove this problem, initially hypothesized in the early 1850s. It has taken some one hundred and eighty years for more elegant proofs to appear, but the issue remains unsolved. The power of graph labeling is extensive in various fields of networking. In addition to circuit design and coding theory, graphs with labels are also helpful in astronomy, radar, and X-rays. We will examine a few simple graphs and their characteristics to see how they can be gracefully labeled. Late in the 1960s, graph labeling became popular. Rosa [2] defined values as functions that generate graceful labeling. After Golomb [1] studied it several years later [1977], he coined the term "graceful labeling". Acharya [1982] has proved that all graphs can be embedded in graceful graphs as induced sub-graphs. The graceful labeling problem identifies graphs with graceful labels. Graphs with finite edges are the only ones we consider when studying graceful labeling.

Graph theory is a part of the field of discrete mathematics, which is distinct from continuous mathematics. In discrete mathematics, we count items rather than measure their sizes. It is important to note that continuous mathematics has always dominated discrete mathematics since man learned how to count. The twenty-first century saw the beginning of this transformation. From the perspective of a continuous approach, it suited discrete mathematics approaches better than continuous mathematics, which shifted the focus from numbers to sets. A second notable development is the increasing social use of computers. Discrete mathematics is used frequently in computer science theory. A part of the discrete mathematics, graph theory have an extensive range of applications in fields such as commerce, engineering, physical, and biological, in addition to computer technology.

Suppose that $G=(V,E)$ seems to be a graph where $|V|=p$ , $|E|=q$ , and label the vertices of $V(G)$ with $\varphi:V(G)\rightarrow N.$ Let $\alpha_i=\varphi(v_i),$ for $1\leq i\leq p.$ Let the vertices of G be labeled by $\{\alpha_1,\alpha_2,\alpha_3,.......,\alpha_{p-1}\},$ for any edge $v_iv_j$ in $E(G)$ is equal to $|\varphi(v_i)-\varphi(v_j)|=|\alpha_i-\alpha_j|.$

Consider the criteria given below while using $\phi$ as a labeling of $G(p,q)$ .

(a) $V\subset\{0,1,2,........,p\}$

(b) $V\subset\{0,1,2,.........,2p\}$

(c) $E = \{1, 2, ........, p\}$

(d) $E = \{a_1, a_2, ...., a_p\}$ with $a_k = k \, or \, a_k = 2p + 1 - k$

(e) For some $a$, where $a \in \{0, 1, 2, ....., p\}$, such that for any edge $v_i v_j$ of G holds, either $\alpha_i \leq a \, and \, \alpha_j > a, or \, \alpha_j \leq a \, and \, \alpha_i > a$ .

Then Rosa [2] goes on to describe the subsequent variations:

- If (a), (c), and (e) satisfied by $\varphi$, then it has α-valuation.
- If (a) and (c) satisfied by $\varphi$, then it has β-valuation.
- If (b) and (c) satisfied by $\varphi$, then it has σ-valuation.
- If (b) and (d) satisfied by $\varphi$, then it has ρ-valuation.

It is clear that there is a hierarchy structure connecting these valuations from their definitions. If $\varphi$ has α-valuation, then it also has β, σ, and ρ-valuation. If $\varphi$ has β-valuation, then it also has σ and ρ-valuation. Finally, if $\varphi$ has σ-valuation, then it also has ρ-valuation. Additionally, G tends to a bipartite graph if there is a α-valuation for it. The sufficient condition for G to have a α-valuation is, G is bipartite or not but G being complete bipartite. From above, another name of α-valuation is bipartite labeling.

## 1.2 Gaussian Integers

Gaussian integers are a natural extension of the integers and have unique properties that make them useful in various fields of mathematics. The paper by S. Klee, et al.[15] has inspired researchers to explore the application of Gaussian integers in “Prime labeling of families of trees with Gaussian integers”. To build the groundwork for our approach, we begin with some understanding of Gaussian integers.

Complex numbers with integer real and imaginary components are known as Gaussian integers. The representation and use of complex numbers, which are widely used in the mathematics of the physical sciences, are the topics of this introduction. An example of a two-dimensional vector is a complex number, which is made up of the so-called real and imaginary part. The imaginary part is a completely mathematical construct, while the real part typically relates to physical quantities.

The applications of the complex numbers are endless. They help to model periodic motions like light and water waves and allow us to understand how certain physical laws, such as gravity and the conservation of energy, determine their behaviour. Because Fourier transforms are routinely used to understand oscillations in both alternating current and signals modulated by electromagnetic waves, they are essential to the modern electronics industry, and complex numbers are widely used in electrical engineering. Mathematicians have used complex analysis to understand fluid dynamic issues, such as how electronic equipment operates on a quantum level, earthquakes shake structures, and oil is pumped on oil rigs and buildings. This is applicable to everyday life as devices shrink.

Gaussian integers, $Z[i] = \{\alpha + \beta i / \alpha, \beta \in Z\}$ where $i^2 = -1$. The unit of Gaussian integers is one of $+1, -1, +i, -i$ and the norm of Gaussian integer is mentioned by $N(\alpha + \beta i) = \alpha^2 + \beta^2$

The fundamentals of graceful graph labelling with Gaussian integers are presented in Section 2, followed by a general discussion of graceful graphs in Section 3 and an paradigm of the grace of some graph classes in Section 4.

## 2. Basic Results on Gaussian Graceful Labeling

During the investigation, we used finite, undirected simple graph, assessing the vertex and edge using the red and blue shades, respectively.

**Definition 2.1**

A graph G to be gracefully labeled with Gaussian integers, for any positive integer p, the vertex labeling $\varphi^g : V \to \{0, z, 2z, ...., pz\}$, z is a Gaussian integer, which lies in the first quadrant and the edge labeling $l^g : E \to \{z, 2z, 3z, ...., pz\}$ is defined by $l^g(v_i v_j) = |\varphi^g(v_i) - \varphi^g(v_j)|$ must both be injective , also there exist “a” with $a \in \{0, z, 2z, ...., pz\}$ such that for some edge $v_i v_j$ of G holds, moreover $v_i \le a, v_j > a$ or $v_j \le a, v_i > a$. Let us refer to graph G as a Gaussian graceful graph if it permits Gaussian graceful labeling or $\alpha^g -$valuation. In this study, we refer to Gaussian graceful labeling as $\alpha^g -$valuation. We show the examples of $\alpha^g -$valuation in Figure 2.1.

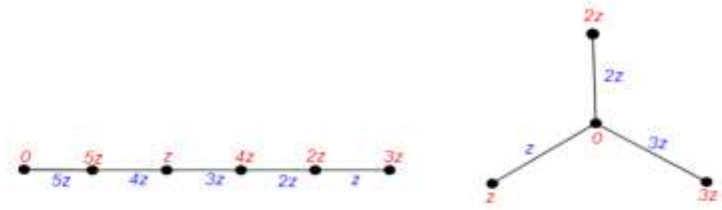


Figure 2.1: $\alpha^g -$valuation of $P_6^g$ and $K_{1,3}^g$

**Theorem 2.2**

The Gaussian path graph $P_n^g, n \geq 1$ is $\alpha^g -$valuation.

**Proof**

Consider a path graph $P_n^g$, and define $V(P_n^g) = \{x_0, x_1, ......x_{n-1}\}$ where $x_k = kz / z \in Z[i]$ as the collection of vertices, so that $x_{k-1}x_k \in E(P_n^g)$ for $0 < k < n$.To ensure that each number in $\{z, 2z, 3z, ....., (n-1)z\}$ as a label on the edge, we can identify the vertices with Gaussian integers within 0 and (n -1)z even though $P_n^g$ contains $m = n-1$ edges. As convenient the only one method to obtain a complete variation equal to $(n-1)z$, which is to have a vertex by make 0 next to a vertex with $(n-1)z$,we begin with edge label $(n-1)z$. So, let's attempt to label $x_0 = 0$ and $x_1 = (n-1)z$. Next aim to obtain an edge label with $(n-2)z$. The possible ways to obtain $(n-2)z$ are the complete difference of $|(n-2)z - 0| or |(n-1)z - z|$.We can only obtain the edge label by labelling $x_2$ with $z$ because there are only labeled adjacent vertices to $x_0$ and only by labeling $x_2$ with z are we able to obtain the edge label $(n-2)z$. If we continue using this method, our labeling will look like this:

$$\varphi^g(x_k) = \begin{cases} 0 & \text{if } k = 0 \\ \dfrac{kz}{2} & \text{if } k \text{ is even and } z \in Z[i] \\ \left(n - \dfrac{k+1}{2}\right)z & \text{if } k \text{ is odd and } z \in Z[i] \end{cases}$$

Now, it is sufficient to demonstrate the edge label z, which appeared on the last edge $x_{n-2}x_{n-1}$, appears to demonstrate that it is in fact graceful labeling of $P_n^g$ . If n is even, then $\varphi^g(x_{n-1}) = \frac{nz}{2}$ and $\varphi^g(x_{n-2}) = \left(\frac{n-2}{2}\right)z.$

For this reason, $l^g(x_{n-1}x_{n-2}) = \frac{nz}{2} - \left(\frac{n-2}{2}\right)z = z.$ If n is odd, an equivalent argument establish the edge label z. Hence, the theorem proved.

**Theorem 2.3**

*The Gaussian complete graph $K_n^g$ is Gaussian graceful graph if, and only if ,* $n \le 4$

**Proof**

Here look for a thoughtful way to label the entire Gaussian complete graph $K_n^g$ . $K_1^g$ *and* $K_2^g$ are graceful because they have path graphs as well. Figure 2.2 represents the $\alpha^g$ −valuation of $K_3^g$ *and* $K_4^g$ .

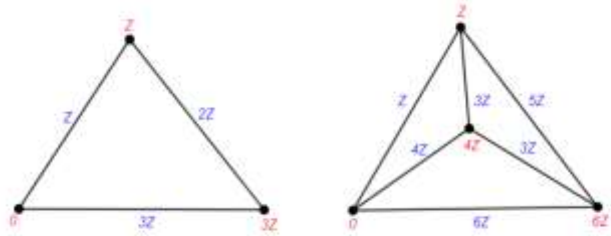


Figure 2.2: Gaussian Graceful labeling of $K_3^g$ and $K_4^g$

Let us first introduce a property of $\alpha^g$ −valuation before we analyse the general situation. If we exchange each vertex label from $kz$ *to* $(m-k)z$ in a graph with $\alpha^g$ − valuation, the consequential labeling is also graceful because the edge values remain the same.

When an edge has labels *az* and *bz*, where *a*, *b* are some positive integers, its end vertices change to $(mz-az)$ *and* $(mz-bz)$ respectively, and $|az-bz| = |(mz-az)-(mz-bz)|$ . The Gaussian complementarily property is what makes this possible.

]

In order to obtain the edge label $mz$ for $K_n^g$ by $n > 4$, as previous to, we need a vertex by make 0 next to a vertex with label $mz$. The difference is that every vertex in this instance is close to every other vertex. As a result, we can simplify the labels of any vertex as 0, and any adjacent vertex as $mz$. The only two ways to obtain the edge label $(m-1)z$ as an absolute difference are $(m-1)z = |(m-1)z - 0| = |mz - z|$. Nevertheless, the complimentary property enables us to select either one exclusive of losing generality. When a vertex is given the label z, we obtain the edge labels $z \; and \; (m-1)z$.

The edge value $(mz-2z) = |(mz-2z)-0| = |(mz-z)-z| = |mz-2z|$ must now be obtained. Because, it would result in a duplicate edge label, we are unable to label a vertex with the value of $(mz-z) \, or \, 2z$.

Thus, the only option we have is to assign a vertex the value $(m-2)z$ thereby generating the edge labels $.2z, \; (mz-3z) \; and \; (mz-2z)z$. Given that $mz-3z$ has previously appear on an edge, the subsequently edge label will be ,

$(mz-4z) = |(mz-4z)-0| = |(mz-3z)-z| = |(mz-2z)-2z| = |(mz-z)-3z| = |mz-4z|$.

Once again, there is only one option to label a vertex with 4z, to get $3z, 4z, (m-6)z \, and \, (m-4)z$ to avoid creating duplicate edge labels.

After the labeling of five vertices, we can move on to the next. The edge label would be replacement for $K_5^g$ if we had $(m-6)z = 4z$ .If $n > 6$ , you will get $(m-5)z$ as the subsequently edge value. A duplicate edge label results from all five methods of obtaining $(m-5)z$. The theorem proves by the label $(m-5)z$, which cannot be on an edge.

## 3. General Results On Gaussian Graceful Labeling

**Theorem 3.1**

If $G^g$ have a $\alpha^g -$valuation, followed by $G^g$ is a bipartite graph.

**Proof**

Let $G^g$ has a $\alpha^g -$ valuation and partition of the vertex set $V(G^g) = \{0, z, 2z, ...., pz\}$ as follows.

Identify the two sub sets of $V(G^g) = \{V_1(G^g), V_2(G^g)\}$ , where

$V_1(G^g) = \{v_i / v_i > kz, 1 < k < p\}$

$V_2(G^g) = \{v_j / v_j \leq kz, 1 < k < p\}$

Clearly, $V_1 \cap V_2 = \phi$. This implies that, for some edge $e = v_i v_j$, the ending vertices of e are in dissimilar partition. Hence, $G^g$ is a bipartite graph. Figure 3.1 gives an explanation of the theorem.

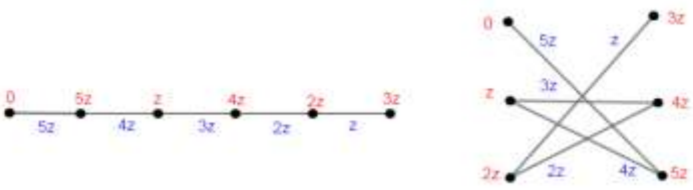


Figure 3.1 Gaussian Path Graph $P_6^g$ isomorphic to Gaussian Bipartite Graph $B_{3,3}^g$

**Theorem 3.2**

Every Gaussian star graph $K_{1,p}^g$ has $\alpha^g -$valuation with $p \geq 1$.

**Proof**

A star graph $K_{1,p}^g$ has one universal vertex, connected to all other pendant vertices. Let, assign the value of the universal vertex as $\varphi^g(x_0) = 0$ and the remaining p pendant vertices as $\varphi^g(x_k) = kz \quad if\ k = 1,2,3,....., p$. Figure 3.2 gives an example of $K_{1,p}^g$.

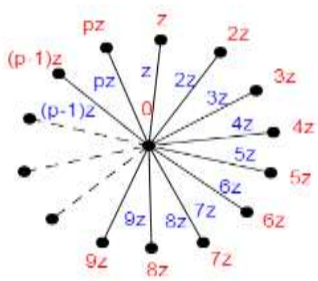


Figure 3.2 Gaussian Star Graph $K_{1,p}^g$ with $\alpha^g -$valuation

**Definition 3.3**

A clique is a subset of undirected graph vertex set so that each two distinct vertices are connected**.**

**Definition 3.4**

A split graph is composed of an independent vertex set and a clique. In other words, Spl(G) is a graph formed by replace all vertex v of G with two distinct vertices, v and v', and connecting the new vertices in the same way that they were connected in G.

**Proposition 3.5**

Every Gaussian split of a Gaussian star graph $K_{1,p}^{g}$ has $\alpha^{g}-$valuation with $p \geq 1$.

**Proof**

The split graph of the Gaussian star graph has the partition of the vertex set as $\{U^{g}(G), V^{g}(G)\} = \{\{u_0, u_1, u_2, ....., u_p\}, \{v_0, v_1, v_2, ....., v_p\}\}$ such as, $N(u_k) = N(v_k)$ where $k = 0,1,2,3,...., p$ and the valuation of the vertices is as below.

$$\varphi(u_k) = \begin{cases} 0 & if\ k = 0 \\ (2p+k)z & if\ k = 1,2,3,...., p \end{cases}$$

$$\varphi(v_k) = \begin{cases} z & if\ k = 0 \\ 2kz & if\ k = 1,2,3,...., p \end{cases}$$

The example of Gaussian Star Graph $Spl(K_{1,9}^{g})$ is given in Figure 3.3

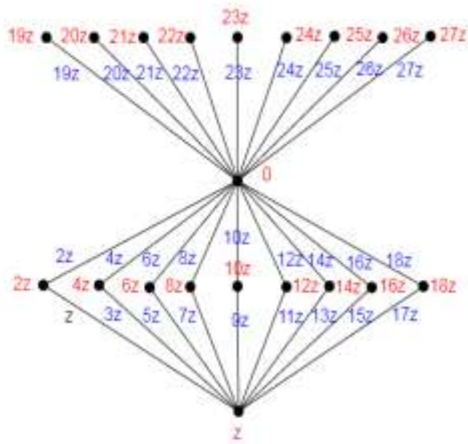


Figure 3.3 Gaussian Star Graph $Spl(K_{1,9}^{g})$ with $\alpha^{g}-$valuation

**Proposition 3.6**

The Gaussian Cycle graph $C_p^g$ has $\alpha^g -$valuation if, and only if,

$p \equiv 0 \pmod 4$ *or* $p \equiv 3 \pmod 4$ [17]

**Proof**

Cycle graphs are a part of Euler graphs since they contain vertices of even degree. According to the similarity criteria, if $|E| \equiv 1, 2 (\text{mod } 4)$, then graph $C_p^g$ is ungraceful since the condition on an odd cycles characterises these graphs. Otherwise, let us take $V(C_p^g) = \{x_0, x_1, ......, x_{p-1}\}$ such that $x_k x_{k+1} \in E(C_p^g)$ for $0 \le k \le p-1$ and $x_p = x_0$.

Case1- If $p \equiv 0 (\text{mod } 4)$, then the valuation of the vertices as follows:

$$\varphi^g(x_k) = \begin{cases} 0 & \text{if } k = 0 \\ \dfrac{kz}{2} & \text{if } k = 2,4,6,....., p-2 \\ pz - \left(\dfrac{k-1}{2}\right)z & \text{if } k = 1,3,5,....., \dfrac{p}{2} - 1 \\ pz - \left(\dfrac{k-1}{2}\right)z - z & \text{if } k = \left(\dfrac{p}{2}+1\right), \left(\dfrac{p}{2}+3\right), ...., (p-1) \end{cases}$$

Case 2- If $p \equiv 3 (\text{mod } 4)$, then the valuation of the vertices as follows:

$$\varphi^g(x_k) = \begin{cases} 0 & \text{if } k = 0 \\ \dfrac{kz}{2} & \text{if } k = 2,4,6,....., p-1 \\ pz - \left(\dfrac{k-1}{2}\right)z & \text{if } k = 1,3,5,....., \dfrac{p+1}{2} - 1 \\ pz - \left(\dfrac{k-1}{2}\right)z - z & \text{if } k = \left(\dfrac{p+1}{2}+1\right), \left(\dfrac{p+1}{2}+3\right), ...., (p-2) \end{cases}$$

The gracefulness of cycle graphs, as characterized by Rosa[2], was acknowledged here.

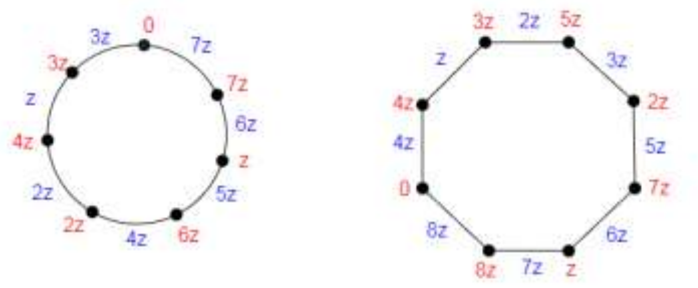

Figure 3.4 Gaussian Cycle Graph $C_7^g$ and $C_8^g$

**Definition 3.7**

When one universal vertex $K_1$ is adjacent to every vertex of the cycle $C_n$, is called a wheel graph $W_n$. i.e., $W_n = C_n + K_1$ .All wheels are graceful, according to Frucht [3]

**Proposition 3.8**

The Gaussian Wheel graph $W_p^g$ has $\alpha^g -$valuation for all $p \geq 3$

**Proof**

Let $V(W_p^g) = \{x_{0,}, x_1, x_2, ....., x_p\}$,

Where $x_0$ is the universal vertex adjacent to every vertex of the Gaussian cycle $C_p^g$ and reflect on the next two cases.

Case 1- If $p \equiv 0 \pmod 2$

$$\varphi^g(x_k) = \begin{cases} 0 & if\ k = 0 \\ 2pz & if\ k = p \\ 2z & if\ k = p-1 \\ kz & if\ k = 1,3,5,...., p-3 \\ (2p-k-1)z & if\ k = 2,4,6,..., p-2 \end{cases}$$

Case 2- If $p \equiv 1 \pmod 2$

$$\varphi^g(x_k) = \begin{cases} 0 & if\ k = 0 \\ 2pz & if\ k = p \\ 2z & if\ k = 1 \\ (p+k)z & if\ k = 2,4,6,...., p-1 \\ (p-k+1)z & if\ k = 3,5,7,...., p-2 \end{cases}$$

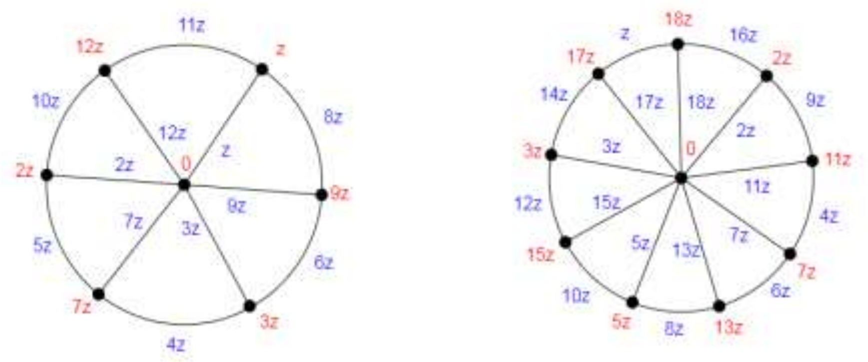


Figure 3.5 Gaussian Wheel Graph $W_7^g$ and $W_{10}^g$

**Definition 3.9[3,4]**

A double wheel graph $DW_n$ , consisting double cycles of size n and have a general universal vertex which is connected to all vertices of two cycles. In general, $DW_n = 2C_n + K_1$

**Proposition 3.10**

The Gaussian double wheel graph $DW_p^g$ has $\alpha^g$ – valuation for all $p \geq 4$

**Proof**

The three subsets of the vertex set of $DW_p^g$ as follows.

$V(DW_p^g) = \{X, Y, u_0\}$, such as X,Y being the vertex set of the outer cycle and inner cycle respectively, and $u_0$ being the universal vertex. $u_0$ is connected to every vertex in X and Y, creating a double wheel structure. Based on the length of circle , consider the following two cases:

In both cases $z_0 = 0$

Case 1- If $p \equiv 0 \pmod 2$

$$\varphi^g(y_k) = \begin{cases} (4p-2)z & if\ k = p \\ 4pz & if\ k = p-1 \\ kz & if\ k = 1,3,5,....,p-3 \\ (4p-k-3)z & if\ k = 2,4,6,...,p-2 \end{cases}$$

$$\varphi^g(x_k)=\begin{cases}2pz & if\ k=p\\(4p-1)z & if\ k=p-1\\4(p-1)z & if\ k=1\\(p+k)z & if\ k=3,5,7,\ ....,\ p-3\\(3p-k-1)z & if\ k=2,4,6,...,\ p-2\end{cases}$$

Case 2- If $p\equiv 1(\bmod 2)$

$$\varphi^g(y_k)=\begin{cases}(2p+4)z & if\ k=p\\2pz & if\ k=p-1\\kz & if\ k=1,3,5,....,\ p-2\\(4p-k+1)z & if\ k=2,4,6,...,\ p-3\end{cases}$$

$$\varphi^g(x_k)=\begin{cases}(2p+2)z & if\ k=p\\(3p+2)z & if\ k=p-1\\4pz & if\ k=1\\(2p+k+2)z & if\ k=3,5,7,\ ....,\ p-2\\(2p-k+1)z & if\ k=2,4,6,...,\ p-3\end{cases}$$

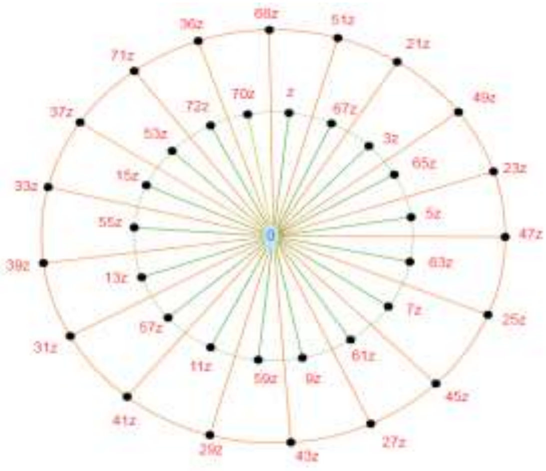


Figure 3.6 Gaussian Double  Wheel Graph  $DW_{18}^{g}$

**Definition 3.11**

A tree whose vertices are all located adjacent to the central path is known as a caterpillar tree or a caterpillar graph.

**Proposition 3.12 [2]**

All Gaussian caterpillar trees having $n+1$ vertices satisfy $\alpha^g -$valuation.

**Proof**

Based on the proof of the Theorem 2.1, the caterpillar tree can be drawn as a planar bipartite form by first dividing the tree into two subsets of vertices and labeling them as in Figure 3.7. This drawing scheme's viability may be easily verified.

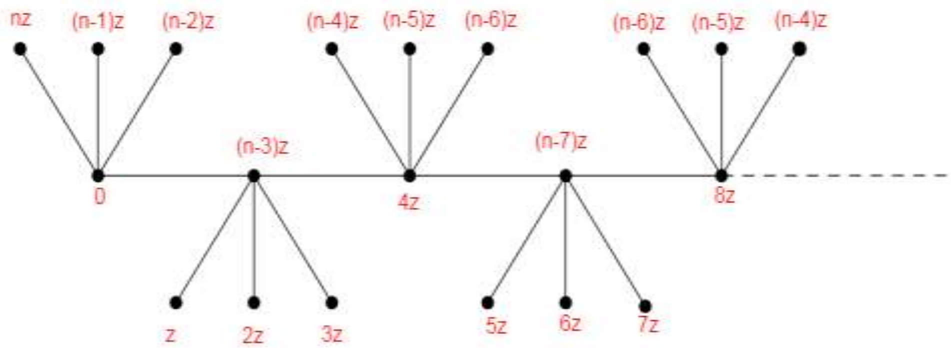


Figure 3.7 $\alpha^g -$valuation of Gaussian Caterpillar Tree with n+1 vertices

**Definition 3.13**

The Graph $G(V(G), E(G))$ is said to be a complete bipartite graph if $V$ can be partitioned into $A$ and $B$ satisfies $A \cup B = V$ and $A \cap B = \phi$ such that each edge $e \in E(G)$ has its end points in $A$ and $B$. If $|A| = m$ and $|B| = n$ ,then the graph G is denoted by $K_{m,n}$.

In particular $K_{1,n}$ is a star.

Golomb [1] and Rosa [2] proves that the $K_{m,n}$ graphs are graceful for all $m, n > 0$.

**Theorem 3.14**

The graph $K^g_{m,n}$ has $\alpha^g -$valuation.

**Proof**

Let $A$ and $B$ be the two partitions of $V$ of the graph, and let $V(A) = \{a_1, a_2, ....., a_m\}$ and $V(B) = \{b_1, b_2, ....., b_n\}$. Then the valuation of the complete bipartite graph with

$\phi^g(a_i) = (i-1)z, i = 1,2,3,.....m$ and $\varphi^g(b_j) = jmz, j = 1,2,3,.....n$ is a $\alpha^g -$valuation. It is obvious that $K^g_{m,n}$ has $mn$ edges. We may assume that $m \le n$, without losing generality. Now $\phi^g$ maps the vertex set of $K^g_{m,n}$ into $\{0, z, 2z,....., mnz\}$ and using a fixed value $u = (m-1)z$, we have that for all edges $a_i b_j, \varphi^g(a_i) \le u$ and $\varphi^g(b_j) > u$ Moreover $\left|\varphi^g(b_j) - \varphi^g(a_i)\right| = \varphi^g(b_j) - \varphi^g(a_i) = (jm - i + 1)z.$ Since $i \in \{1,2,3,....., m\}$, for a fixed j, $\left\{\left|\varphi^g(b_j) - \varphi^g(a_i)\right|\right\} \in \{((j-1)m+1)z, ((j-1)m+2))z,....., jmz\}$, which gives that the edge set of $K^g_{m,n} = \{z, 2z, 3z, ....., mnz\}$.

Thus $\varphi^g$ is a $\alpha^g -$valuation.

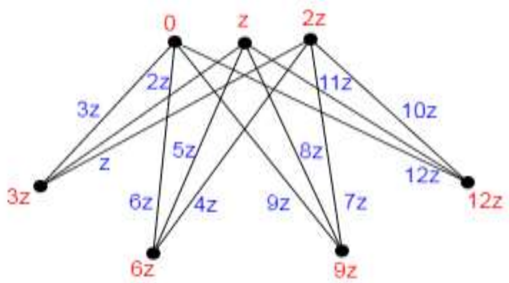


Figure 3.8 Gaussian Complete Bipartite Graph $K^g_{3,4}$ with $\alpha^g -$valuation

## IV. Results On Fan Related Graphs

**Proposition 4.1**

The Gaussian complete multipartite graph $K^g_{1,p,1}$ has a $\alpha^g -$valuation with $p \ge 1$.

**Proof**

Let the vertex set of Gaussian complete multipartite graph being, $V(G) = \{A, B, C\}$ where $A = \{a\}$, $B = \{b_i, i = 1,2,3,...., p\}$, $C = \{c\}$.The $\alpha^g -$valuation of $K^g_{1,p,1}$ as follows:

$$\varphi^g(a) = 0$$
$$\varphi^g(b_i) = (2i+1)z \quad where \quad i = 1,2,3,...., p$$
$$\varphi^g(c) = z$$

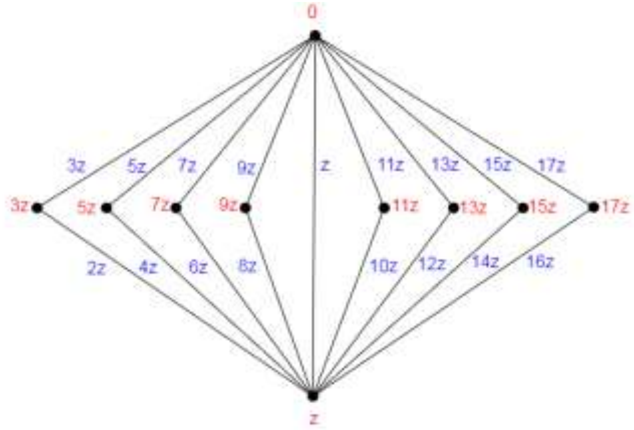


Figure 4.1 Gaussian Complete Bipartite Graph $K^{g}_{1,8,1}$ with $\alpha^{g}-$valuation

**Definition 4.2**

A graph is said to be a fan graph $F_{1,n}$, which join where the path graph of length n and the empty graph are both on nodes. In general $F_{1,n}=P_n+\overline{K}_1$

**Theorem 4.3**

The Gaussian Fan graph $F^{g}_{1,p}$ has $\alpha^{g}-$valuation with $p\geq 1$.

**Proof**

The empty graph vertex $x_0$ has the valuation as $x_0=0$ and the path graph valuation of the fan graph $F^{g}_{1,p}$ is as follows.

$$\varphi^{g}(x_k)=\begin{cases} kz & if\ \ 1\leq k\leq p\ \ and\ \ k\,is\,odd \\ (2p-k+1)z & if\ \ 2\leq k\leq p\ \ and\,k\,is\,even \end{cases}$$

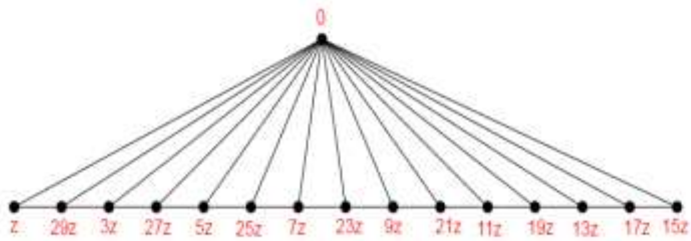


Figure 4.2 Gaussian Fan Graph $F^{g}_{1,15}$ with $\alpha^{g}-$valuation

**Definition 4.4**

A double fan graph $DF_{2,n}$ has $n+2$ vertices and $3n-1$ edges. In general, $DF_{2,n} = P_n + \overline{K}_2$, which join where the path graph and the empty graph on two vertices are both on nodes.

**Theorem 4.5**

The Gaussian double Fan graph $DF_{2,p}^{g}$ has $\alpha^{g}-$valuation if $p \geq 1$.

**Proof**

The empty graph vertices $x_1$ and $x_2$ has the valuations as $x_1 = 0$, $x_2 = z$.

The path graph valuation of the double fan graph $DF_{2,p}^{g}$ is as follows:

$$\varphi^{g}(y_k) = \begin{cases} \left(\dfrac{6p-3k+1}{2}\right) z & \text{if } k \text{ is odd and } 1 \leq k \leq p \\ \left(\dfrac{3k-2}{2}\right) z & \text{if } k \text{ is even and } 2 \leq k \leq p \end{cases}$$

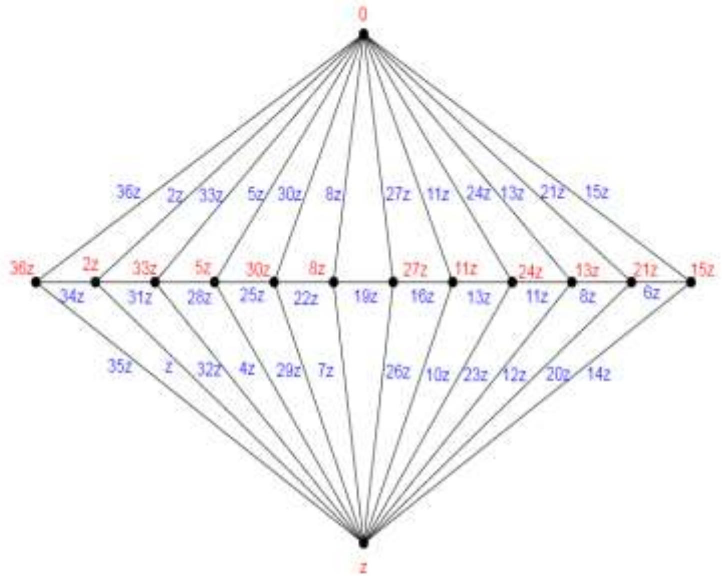


Figure 4.3 Gaussian Double Fan Graph $DF_{2,12}^{g}$ with $\alpha^{g}-$valuation

**Corollary 4.6**

All Gaussian fan graphs $F_{m,n}^{g}$ has $\alpha^{g}-$valuation if, for all positive integers $m,n$.

**Proof**

The fan graph $F^g_{m,n}$ with $m+n$ vertices and $mn+n-1$ edges has m empty graph vertices and n path graph vertices. Let the set of m empty graph vertices are $\{x_1, x_2, x_3, ...., x_m\}$ and let the valuations of the vertices are $\{0, z, 2z, 3z, ....., (m-1)z\}$. Let the path graph vertices are $\{y_1, y_2, y_3, ...., y_n\}$. Then the valuations of the path graph vertices are,

$$\varphi^g(y_k) = \begin{cases} \left[\left(\frac{m+1}{2}\right)(2n-k+1)-1\right]z & \textit{if } k \textit{ is odd and } 1 \le k \le n \\ \left[\left(\frac{m+1}{2}\right)k-1\right]z & \textit{if } k \textit{ is even and } 2 \le k \le n \end{cases}$$

Some examples of the fan graphs is given below.

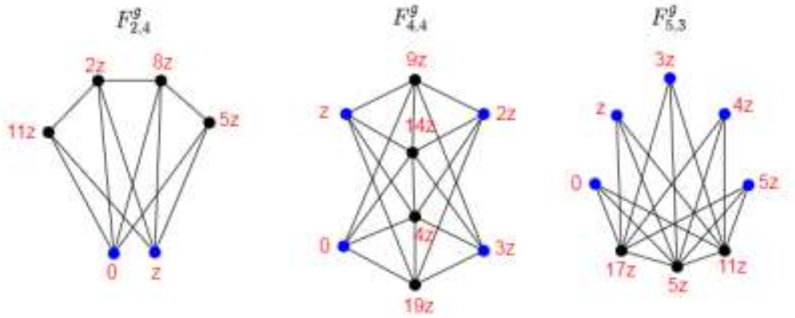


Figure 4.4 Gaussian Fan Graphs $F^g_{2,4}$, $F^g_{4,4}$ and $F^g_{5,3}$ with $\alpha^g$ −valuation

## V. Conclusion and Further Work

By introducing Gaussian integers, a novel idea in the field of graph labelling served as the foundation for this article. The work conducted in this article provided a successful strategy for graph labeling and can provide insight into other problems that involve graphs and integers. Finally, it can become a never-ending task as researchers continue to strive for better solutions and further develop their knowledge of graph labeling with Gaussian integers.

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