SOME PROPERTIES OF PROJECTION GRAPHS OF RING OF INTEGERS MODULO N

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Abstract

The projection graph $P_1(R)$ of a commutative ring *R* with identity is an undirected graph with the vertex set as the set of all nontrivial elements of *R* and, *x*, *y* are adjacent (x¯y) iff *xy* equals *x* or *y*. In this paper, projection graphs associated with ring of integers modulo *n* having prime power characteristic and finite Boolean rings are considered. Stability number and connected domination number are computed. Minimum spanning trees containing connected dominating set and maximum number of pendants are obtained. Planarity is also verified. It is found that the projection graphs of rings of integers modulo *n* with prime power characteristic possess unique minimum spanning bistars. It is seen that the stability number is maximum iff *n* is prime. Projection graphs of Boolean rings have multiple minimum spanning strong double brooms. Projection graphs of Boolean rings are proved to be Hamiltonian. Projection graph of Boolean ring of order 16 is found to be Hamiltonian maximal planar graph, in which densely packed hexagons are identified.

Keywords: Stability number, Connected dominating set, Maximal planar graph, Bistar, double broom, Hexagonal chains.

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1. Introduction

To analyze the characteristics of algebraic structures, graphs can be constructed [2] using the vertices as the elements or substructures and the edges as the algebraic properties of a pair of elements or substructures. Connected graphs are used to represent various networks and connected dominating sets find applications in mobile ad hoc networks [1], wireless networks [4], chemical graphs [6], circulant networks [7] etc. Spanning subgraphs are necessary tools for developing algorithms for communication problems on networks. A spanning tree approach to chemical networks is used in many fields, including drug design, materials science, and nanotechnology [6]. Planar graphs are found useful in designing electrical circuits.

Projection graph $P_1(R)$ of commutative ring R was introduced in [8]. The following results were proved.

Theorem 1.1 [8] Let $x \in R\backslash\{0,1\}$.
Then (i) $N(x) = (1 + Ann(x)\backslash\{0\}) \cup (Ann(1-x)\backslash\{0, x\})$.

Theorem 1.2 [8] $ZD(R)\backslash\{0\} = \cup_{d|n, d\neq 1} [d]_{\sim}$, where $x \sim y$ iff $Ann(x) = Ann(y)$ and $[d]_{\sim}$ is equivalence class of d.

Proposition 1.3 [8] The set of units and set of nilpotents are independent in $P_1(R)$.

In this paper, projection graphs of $\mathbb{Z}_{p^k}$ and $\mathbb{Z}_2^k$, $k \geq 3$, are studied. Maximum stable sets (MSSs) stability number $(\alpha(P_1(R)))$, connected domination number $(\gamma_{cs}(P_1(R)))$, minimum spanning trees containing connected dominating set with maximum number of pendants (MSTCDS) for projection graphs are found. Planarity is also verified for $P_1(\mathbb{Z}_{p^k})$ and $P_1(\mathbb{Z}_2^k)$. Idempotents play an important role in connectivity of projection graphs, so the study is carried out for the local rings $\mathbb{Z}_{p^k}$, in which there are only trivial idempotents in first section and in second section projection graphs of Boolean rings are explored.

In the investigation, It is observed that $P_1(\mathbb{Z}_2^4)$ is Hamiltonian maximal planar graph containing densely packed hexagonal chains, which are not commonly encountered in graph theory.

A subset $\mathbb{D}$ of vertex set is called a *dominating set* if every vertex is either in $\mathbb{D}$ or adjacent to a vertex in $\mathbb{D}$. A *connected dominating set* [4] is a dominating set whose induced graph is connected. A connected dominating set with a minimum cardinality is called a *minimum connected dominating set (MCDS)*. A *star* $K_{1,l}$ is a graph with single non-pendant vertex called *center* of $K_{1,l}$ joined to l other vertices. A *double star* is a graph with two stars $K_{1,l}$, $K_{1,m}$ whose centers are joined by an edge called *central edge*. A *bistar* $BS_{l,l}$ is a double star in which the stars have the same number of pendant vertices. A *double broom* $\mathrm{Br}(m, n, n)$ is a tree obtained from a path of order m by joining each vertex of degree 1 to n new vertices. A *strong double broom* is the graph on at least 5 vertices obtained from a union of (at least two) internally vertex disjoint paths with same ends u and v (called *hubs*) by appending leaves at u and v. In particular, $Br(kP_m, n, n)$ is the strong double broom with n leaves at both the hubs and with k internally vertex disjoint paths of order m joining hubs. The stability number $\alpha(G)$ of a graph G is cardinality of maximum stable set (MSS) of G with maximum number of vertices. Connected domination number $\gamma_{cs}(G)$ is the minimum cardinality of MCDSs. A planar graph G is said to be *triangulated* (also called *maximal planar*) if the addition of any edge to G results in a nonplanar graph. [2] and [3] provide definitions, results, and terminologies on graphs and rings.

Throughout this article, R denotes commutative ring with unit element. If I is ideal of R $1 + I = \{1 + x | x \in I\}$ and $A \backslash B$ denotes the set difference between sets A and B.

2. Main Results

Connectedness and stability of $P_1(\text{R})$ are verified for $R = \mathbb{Z}_n$ $(n = p^k)$ and Boolean ring.

Theorem 2.1 (i) $P_1(\mathbb{Z}_n)$ is totally disconnected iff n is prime.
(ii) $P_1(\mathbb{Z}_n)$ contains isolated vertex if n is odd.
(iii) $\phi(n) - 1 \leq \alpha(P_1(\mathbb{Z}_n)) \leq n - 2$. In particular, $\alpha(P_1(\mathbb{Z}_n))$ is maximum iff n is prime.
(iv) $\gamma_{cs}(P_1(\mathbb{Z}_n)) \geq 2$.

Proof. (i) Suppose n is prime. Then $\mathbb{Z}_n$ is an integral domain.
Let $x \in \mathbb{Z}_n \backslash \{0,1\}$. If $xy = x$ (respectively, $xy = y$), then $x(1 - y) = 0$ (respectively, $y(1 - x) = 0$), which implies $y = 1$ (respectively, $y = 0$). Hence there is no $y \in \mathbb{Z}_n \backslash \{0,1\}$ adjacent to x. Thus $P_1(\mathbb{Z}_n)$ is totally disconnected.
Conversely, suppose n is not prime and $x_0, y_0 \in \mathbb{Z}_n \backslash \{0,1\}$ be such that $x_0 y_0 = 0$. Then either $x_0 \urcorner (1 + x_0)$ or $x_0 \urcorner (1 + y_0)$. This contradiction completes the proof of (i).
(ii) Suppose n is odd. Then 2 is unit. If $x \in \mathbb{Z}_n \backslash \{0,1\}$ is such that $2x = 2$ (respectively, $2x = x$), then $x = 1$ (respectively, $x = 0$), which shows that 2 is isolated, which proves (ii).
(iii) As mentioned in introduction section, set of units is stable. Therefore the stable number of $P_1(\mathbb{Z}_n)$ must be at least $|U(\mathbb{Z}_n) \backslash \{1\}|$. It is known that there are $\phi(n) - 1$ units in $\mathbb{Z}_n$ and therefore $\alpha(P_1(\mathbb{Z}_n)) \geq \phi(n) - 2$. Also $\phi(n) = n - 1$ iff n is prime. Hence the maximum value of $\alpha(P_1(\mathbb{Z}_n))$ is $n - 2$, which completes proof of (iii).
(iv) From (i) and (ii), it is clear that there is no universal vertex u adjacent to every vertex in $P_1(\mathbb{Z}_n)$, which completes proof.

2.1. Projection graphs of $\mathbb{Z}_n$ for $n = p^k, k \geq 1$

Vertex set is split by using equivalence relation $\sim$ mentioned in [8], which eases the process of finding stable sets.

Proposition 2.1.1 Let $n = p^k$. Then
(i) $V\left(P_1\left(\mathbb{Z}_{p^k}\right)\right) = S \cup T \cup W$, where $S = \cup_{i=1}^{k-1} S_i$; $T = \cup_{i=1}^{k-1} T_i$ and $W = (U(\mathbb{Z}_n) \backslash \{1\}) \backslash T$; $S_i = [p^{i-1}]_\sim \backslash [p^i]_\sim$; $T_i = 1 + S_i$, $[p^i]_\sim$ is the set of nontrivial elements in the ideal generated by p^i.
(ii) $|S_i| = |T_i| = p^{k-i} - p^{k-i-1}$ for $1 \leq i \leq k - 1$.
(iii) $\deg(x) = \begin{cases} p^i - 1 & \text{if } x \in S_i \cup T_i, 1 \leq i \leq k-1 \\ 0 & \text{otherwise} \end{cases}$.
(iv) degree sequence is $p - 1, p^2 - 1, \cdots, p^{k-1} - 1$ with corresponding multiplicities $2(p^{k-1} - p^{k-2}), 2(p^{k-2} - p^{k-3}), \cdots, 2(p - 1)$.
(v) Number of isolated vertices is $n - \frac{2n}{p}$.

Proof. Let $R = \mathbb{Z}_{p^k}$.
Then $V(P_1(R)) = (ZD(R) \backslash \{0\}) \cup (U(R) \backslash \{1\})$, where $(ZD(R) \backslash \{0\}) = \cup_{d|n, d \neq 1} [d]_\sim$.
Here the divisors of n are $p, p^2, \cdots, p^{k-1}$ and

$$[p^i]_\sim = \{y \in \mathbb{Z}_n \backslash \{0\} | (y, n) = p^i\}, \text{ for every } 1 \leq i \leq k - 1$$
$$= \{p^i, 2(p^i), \cdots, (p^{k-i} - 1)p^i\}.$$

Also $\{p^{k-1}\} = [p^{k-1}]_\sim \subset \cdots [p^i]_\sim \subset [p^{i-1}]_\sim \cdots \subset [p]_\sim$.
Now let $S_i = [p^{i-1}]_\sim \backslash [p^i]_\sim$ and $T_i = 1 + S_i$. Then $ZD(R) \backslash \{0\} = \cup_{i=1}^{k-1} S_i$ and
$U(\mathbb{Z}_n) \backslash \{1\} \supseteq 1 + [p]_\sim = \cup_{i=1}^{k-1} T_i$.

Therefore if $S = \cup_{i=1}^{k-1} S_i$; $T = \cup_{i=1}^{k-1} T_i$ and $W = (U(\mathbb{Z}_n)\setminus\{1\})\setminus T$, the proof of (i) follows.
(ii) $|S_i| = \left|[p^{i-1}]_{\sim}\setminus[p^i]_{\sim}\right| = p^{k-i+1} - p^{k-i}$; $|T_i| = |S_i| = p^{k-i+1} - p^{k-i}$.
(iii) If $x \in S_1$, $Ann(x) = \{0, p^{k-1}, 2p^{k-1}, \cdots (p-1)p^{k-1}\}$; $N(x) = 1 + [p^{k-1}]_{\sim}$;
For $2 \le i \le k-2$, if $x \in S_i$, $Ann(x) = \{0, p^{k-i}, 2p^{k-i}, \cdots (p^i - 1)p^{k-i}\}$;
$N(x) = \{1 + p^{k-i}, 1 + 2p^{k-i}, \cdots, 1 + (p^i - 1)p^{k-i}\} = 1 + [p^{k-i}]_{\sim}$;
If $x \in S_{k-1}$, $Ann(x) = \{0, p, 2p, \cdots, (p^{k-1} - 1)p\}$;
$N(x) = \{1 + p, 1 + 2p, \cdots, 1 + (p^{k-1} - 1)p\} = 1 + [p]_{\sim}$.
Thus $P_1(\mathbb{Z}_n)$ contains an induced subgraph on $S \cup T$ as a unique component. The vertex partition and adjacency between vertices is as shown in Figure 2.1.1, in which circles on the left (respectively, right) represent sets S_i (respectively, T_i) and $|S_i| = |T_i| = p^{k-i} - p^{k-i-1}$, for $i = 1$ to $k-1$

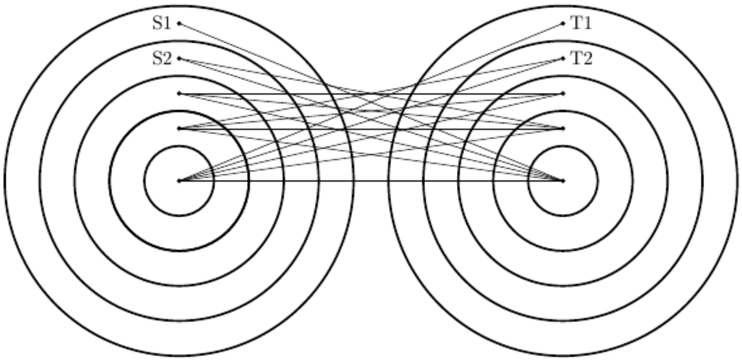


Figure 2.1.1 component of $P_1(\mathbb{Z}_{p^k})$

Let $x \in \mathbb{Z}_n\setminus\{0,1\}$. Then $\deg(x) = p - 1$, if $x \in S_1 \cup T_1$;
$\deg(x) = p^i - 1$, if $x \in S_i \cup T_i$, $2 \le i \le k-2$;
$\deg(x) = p^{k-1} - 1$, if $x \in S_{k-1} \cup T_{k-1}$.
(iv) From (iii), degree sequence is $p-1, p^2-1, \cdots, p^{k-1}-1$ with corresponding multiplicities $2|S_1|, 2|S_2| \cdots, \cdots, 2|S_i| \cdots, 2|S_{k-1}|$, namely
$2(p^{k-1} - p^{k-2}), 2(p^{k-2} - p^{k-3}), \cdots, 2(p^{k-i} - p^{k-i-1}), \cdots, 2(p-1)$.
(v) $|W| = n - 2 - 2(|[p]_{\sim}|) = n - 2 - 2(p^{k-1} - 1) = n - 2p^{k-1} = n - \frac{2n}{p}$

Notation 2.1.2 If $n = p^k$, then $S = ZD(\mathbb{Z}_n)\setminus\{0\} = [p]_{\sim}$; $T = U(\mathbb{Z}_n)\setminus\{1\} = 1 + [p]_{\sim}$; $W = (U(\mathbb{Z}_n)\setminus\{1\})\setminus(1 + (ZD(\mathbb{Z}_n)\setminus\{0\}))$.

Proposition 2.1.3 Let $n = p^2$. Then
(i) $P_1(\mathbb{Z}_n) \cong K_{(p-1),(p-1)} \cup mK_1, m = n - 2p$.
(ii) $P_1(\mathbb{Z}_n)$ contains unique triangle-free component as induced subgraph on $S \cup T$ with $2(p-1)$ vertices and $(p-1)^2$ edges.
(iii) $\deg(x) = \begin{cases} p-1 & \text{if } x \in S \cup T \\ 0 & \text{otherwise} \end{cases}$.
(iv) $P_1(\mathbb{Z}_n)$ is planar iff $p \le 3$.

Proof. Let $n = p^2$. Then $V(P_1(\mathbb{Z}_n)) = S \cup T \cup W$ by using Notation 2.1.2.
Let $x \in V(P_1(\mathbb{Z}_n))$.
If $x \in S$, $Ann(x) = \{0, p, 2p, \cdots, (p-1)p\}$; $Ann(1-x) = \{0\}$; $N(x) = 1 + [p]_{\sim} = T$.
If $x \in T$, $Ann(x) = \{0\}$; $Ann(1-x) = [p]_{\sim} \cup \{0\}$; $N(x) = [p]_{\sim} = S$.
If $x \in W$, $Ann(x) = Ann(1-x) = \{0\}$.
Therefore every element in S is adjacent to every element in T and vice versa; W gives the set of isolated vertices.

Hence $P_1(\mathbb{Z}_n)$ contains an induced subgraph on $S \cup T$, which is complete bipartite together with the set W of isolated vertices.
Thus $P_1(\mathbb{Z}_n) \cong K_{|[p]_\sim|,|1+[p]_\sim|} \cup |W|K_1$, where $|[p]_\sim| = p-1$,
$|W| = p^2 - 2 - 2(p-1) = p^2 - 2p$, which completes the proof of (i).
(ii) $P_1(\mathbb{Z}_n)$ contains unique complete bipartite induced subgraph $K_{p-1,p-1}$ on
$[p]_\sim \cup (1+[p]_\sim)$, in which $[p]_\sim = ZD(\mathbb{Z}_n)\backslash\{0\}$. Thus (ii) is proved since $K_{l,l}$ is triangle-free.
(iii) If $x \in \mathbb{Z}_n\backslash\{0,1\}$, then x is either isolated or a vertex in $K_{p-1,p-1}$, which is an induced subgraph on $S \cup T$. Hence $\deg(x)$ is 0 or $p-1$.
(iv) Suppose $p = 2$. Then $P_1(\mathbb{Z}_n) \cong K_{1,1}$, which is planar.
Suppose $p = 3$. Then $P_1(\mathbb{Z}_n) \cong K_{2,2} \cup 3K_1$, which is planar. Suppose $p > 3$. Then $P_1(\mathbb{Z}_n)$ contains $K_{p-1,p-1}$ and $p-1 > 3$. Hence $P_1(\mathbb{Z}_n)$ is not planar.

Proposition 2.1.4 Let $n = 2^k$, $k \geq 2$.
(i) $V\left(P_1\left(\mathbb{Z}_{2^k}\right)\right) = S \cup T$, where $S = ZD(\mathbb{Z}_n)\backslash\{0\}$; $T = 1 + (ZD(\mathbb{Z}_n)\backslash\{0\})$
(ii) $|S_i| = |T_i| = 2^{k-i} - 2^{k-i-1}$ for $1 \leq i \leq k-1$.
(iii) All the elements in S_i and T_i have the same degree $p^i - 1$, $1 \leq i \leq k-1$.
(iv) degree sequence is $1, 3, \cdots, 2^{k-1} - 1$ with corresponding multiplicities
$2(2^{k-1} - 2^{k-2}), 2(2^{k-2} - 2^{k-3}), \cdots, 2$.
(v) $P_1\left(\mathbb{Z}_{2^k}\right)$ is connected.
(vi) $ZD(\mathbb{Z}_n)\backslash\{0\}$ and $U(\mathbb{Z}_n)\backslash\{1\}$ are MSSs.

Proof. (i) If n is 2^k, the proofs of (i) to (v) follow from Proposition 2.2.1 and Figure 2.1.2.

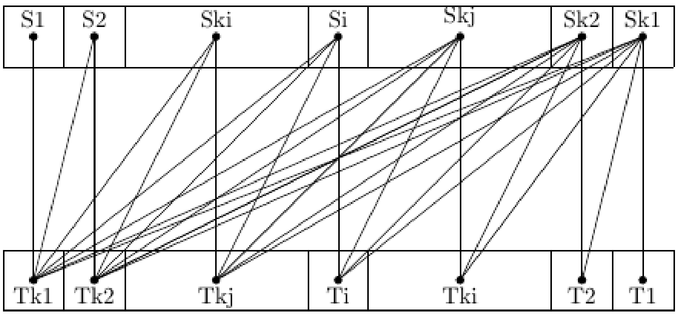


Figure 2.1.2 : $P_1\left(\mathbb{Z}_{2^k}\right)$

(v) Note that (ii) $|W| = n - \frac{2n}{2} = 0$. Therefore $P_1\left(\mathbb{Z}_{2^k}\right)$ has no isolated vertices.
Also if $x_0 = 2^{k-1}$, $N(x) = 1 + [2]_\sim = T$;
if $1 + x_0 = 1 + 2^{k-1}$, $Ann(1 - x_0)\backslash\{0\} = N(x) = \{2, 2^2, \cdots, (2^{k-1} - 1)2\} = [2]_\sim = S$ and $x_0 \urcorner (1 + x_0)$. Hence $P_1\left(\mathbb{Z}_{2^k}\right)$ is connected.
(vi) As $\mathbb{Z}_n$ is local, $ZD(\mathbb{Z}_n)\backslash\{0\} = Nil(\mathbb{Z}_n)\backslash\{0\}$, which is stable as mentioned in [8].
Also $1 + (ZD(\mathbb{Z}_n)\backslash\{0\}) = U(\mathbb{Z}_n)\backslash\{1\}$, which is stable, as desired.

Remark 2.1.5 The remaining MSSs are found by using principle of exclusion and inclusion as below:
Consider the maximal stable set $ZD(\mathbb{Z}_n)\backslash\{0\} = S_1 \cup S_2 \cup \cdots \cup S_{k-1}$. Exclude the vertices in S_{k-1}, which has maximum degree (that is, $2^i - 1$) and include T_1, whose elements have minimum degree (that is, 1) to get maximal stable set $S_1 \cup S_2 \cup \cdots \cup T_1$; Repeat excluding vertices with maximum degree and include vertices with minimum degree to get maximal stable set $S_1 \cup S_2 \cup \cdots \cup T_2 \cup T_1$ etc.

The algorithm for finding maximal stable sets is given below:
Input : $S_i, T_i, 1 \leq i \leq k-1$;
Step 1: Set $i = 0$ and $M_i = S_1 \cup S_2 \cup \cdots \cup S_{k-1}$;
Step 2: $i = i + 1$;
Step 3: Exclude S_{k-i} and include T_i; set $M_{1+1} = S_1 \cup S_2 \cup \cdots \cup T_1$;
Step 4: Print M_i; $|M_i|$;
Step 5: Repeat Step 2 to Step 4 till $M_i = T_1 \cup T_2 \cup \cdots \cup T_{k-1}$.
Illustration 2.1.6 The process of finding MSSs is illustrated for both even and odd values of k.

Enumeration of stable sets is given in Table 2.2.1 (a) for $P_1(\mathbb{Z}_{2^6})$; Table 2.2.1 (b) for $P_1(\mathbb{Z}_{2^7})$.

Maximal stable set	Cardinality
$S_1 \cup S_2 \cup S_3 \cup S_4 \cup S_5$	31
$S_1 \cup S_2 \cup S_3 \cup S_4 \cup T_1$	46
$S_1 \cup S_2 \cup S_3 \cup T_2 \cup T_1$	52
$S_1 \cup S_2 \cup T_3 \cup T_2 \cup T_1$	52
$S_1 \cup T_4 \cup T_3 \cup T_2 \cup T_1$	46
$T_5 \cup T_4 \cup T_3 \cup T_2 \cup T_1$	31

Maximal stable set	Cardinality
$S_1 \cup S_2 \cup S_3 \cup S_4 \cup S_5 \cup S_6$	63
$S_1 \cup S_2 \cup S_3 \cup S_4 \cup S_5 \cup T_1$	94
$S_1 \cup S_2 \cup S_3 \cup S_4 \cup T_2 \cup T_1$	108
$S_1 \cup S_2 \cup S_3 \cup T_3 \cup T_2 \cup T_1$	112
$S_1 \cup S_2 \cup T_4 \cup T_3 \cup T_2 \cup T_1$	108
$S_1 \cup T_5 \cup T_4 \cup T_3 \cup T_2 \cup T_1$	94
$T_6 \cup T_5 \cup T_4 \cup T_3 \cup T_2 \cup T_1$	63

Table 2.2.1 (a) $P_1(\mathbb{Z}_{2^6})$ Table 2.2.1 (b) $P_1(\mathbb{Z}_{2^7})$

Proposition 2.1.7 Let n be 2^k, $k \geq 3$. Then
(i) $\gamma_{cs}\big(P_1(\mathbb{Z}_n)\big) = 2$.
(ii) $P_1(\mathbb{Z}_n)$ contains $BS_{l,l}$ as its unique MSTCDS with $2^k - 4$ pentants.
(iii) $P_1(\mathbb{Z}_n)$ is planar iff $k \leq 3$.
Proof. (i) Write $\mathbb{Z}_n \backslash \{0,1\} = S \cup T$ as earlier.
Let $x_0 = 2^{k-1}$. Then $x \urcorner x_0 \urcorner (1 + x_0) \urcorner y$ for every $x \in T$ and $y \in S$. Therefore $\mathbb{D} = \{x_0, 1 + x_0\}$ is MCDS, which completes the proof of (i).
(ii) Let $BS_{l,l}$ be the bistar where $x_0 = 2^{k-1}$ and $1 + x_0$ are the centers of stars with pendant set T at x_0 (respectively, S at $1 + x_0$). Then $l = |S| - 1 = |T| - 1 = 2^{k-1} - 2$ and $BS_{l,l}$ is the unique MSTCDS since $\mathbb{D}$ is unique MCDS.
(iii) For $k = 3$, $P_1(\mathbb{Z}_n) \cong BS_{2,2}$, which is planar.
Suppose $k \geq 4$. Then the induced subgraph on
$\{2^{k-2}, 2(2^{k-2}), 3(2^{k-2})\} \cup \{1 + 2^{k-2}, 1 + 2(2^{k-2}), 1 + 3(2^{k-2})\}$ contains $K_{3,3}$, showing that $P_1(\mathbb{Z}_n)$ is not planar.

2.2. Projection graphs of $\mathbb{Z}_2^k$, $k \geq 3$

Projection graphs of finite Boolean rings are investigated in this section. Every element in Boolean ring R is an idempotent and if e is nontrivial idempotent, then e is adjacent to every nontrivial element in eR. Also if $V_e = \cup I, I \supseteq eR$, where I is ideal, then e is adjacent to every vertex in V_e. Consequently maximal ideal containing e is split into smaller sets, which enables to enumerate MSSs, MCDSs easily.

The following proposition is due to the fact the elements of $\mathbb{Z}_2^k$ can be paired as orthogonal complements.

Proposition 2.2.1 Let $e = (0,1,1) \in \mathbb{Z}_2^3$. Then $V\big(P_1(\mathbb{Z}_2^3)\big) = S \cup S'$, where $S = eR\backslash\{(0,0,0)\}$, $S' = (1,1,1) + S$ and $P_1(\mathbb{Z}_2^3)$ is a cycle.

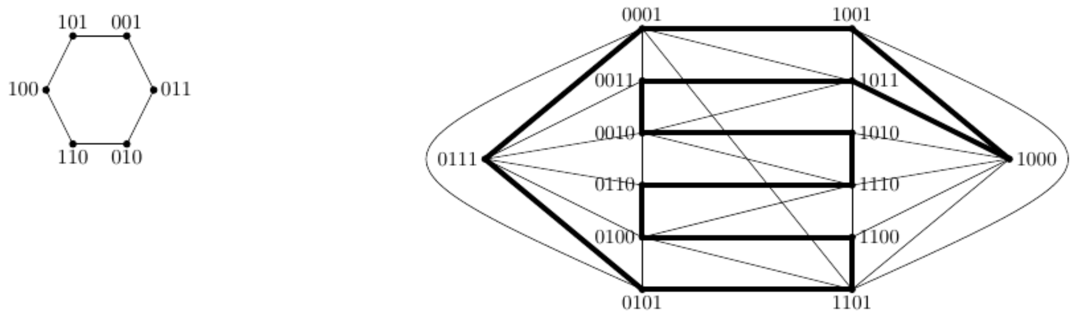
Proof. $\mathbb{Z}_2^3 = \{(0,0,0),(1,1,1),(0,0,1),(0,1,0),(1,0,0),(1,1,0),(1,0,1),(0,1,1)\}$;
$eR\backslash\{(0,0,0)\} = \{(0,0,1),(0,1,0),(0,1,1)\}$ and $(1,1,1) + eR = \{(1,1,0),(1,0,1),(1,0,0)\}$
Now let $S = eR\backslash\{(0,0,0)\}$ and $S' = (1,1,1) + S$.
Then $V\big(P_1(\mathbb{Z}_2^3)\big) = \mathbb{Z}_2^3\backslash\{(0,0,0),(1,1,1)\} = S \cup S'$,
Also $P_1(\mathbb{Z}_2^3)$ is drawn in Figure 2.2.1 (a), which is a cycle.

Notation 2,2,2 (i) For the sake of convenience, the set of elements in $\mathbb{Z}_2^k$ is written as $\{(0,\boldsymbol{x}) \,|\, \boldsymbol{x} \in \mathbb{Z}_2^{k-1}\}$ so that $V\left(P_1\big(\mathbb{Z}_2^k\big)\right) = \{(0,\boldsymbol{x}) \,|\, \boldsymbol{x} \in \mathbb{Z}_2^{k-1}\}\backslash(\{(0,\mathbf{0}),(1,\mathbf{1})\})$.

(ii) The induced subgraph isomorphic to a hexagon with maximum degree m is denoted as H_m. Note that H_4 are chemical graphs.

Proposition 2.2.3 Let e be $(0,\mathbf{1}) \in \mathbb{Z}_2^4$. Then
(i) The induced subgraphs on $S\backslash\{e\}$ and $S'\backslash\{1-e\}$ are H_2, where $S = e\mathbb{Z}_2^4$, $S' = (1.\mathbf{1}) + S$.
(ii) The induced subgraphs on S and S' are $K_1 + H_2$.
(iii) The induced subgraph on $(S \cup S')\backslash\{e,e'\}$ is isomorphic to edge disjoint union of three H_2 and six K_2.
(iv) $P_1(\mathbb{Z}_2^4)$ is triangulated.

Proof. $V\big(P_1(\mathbb{Z}_2^4)\big) = \{(0,\boldsymbol{x}) \,|\, \boldsymbol{x} \in \mathbb{Z}_2^3\}\backslash(\{(0,\mathbf{0}),(1,\mathbf{1})\})$.
Now for every $\boldsymbol{x} \neg \boldsymbol{y}$ in $P_1(\mathbb{Z}_2^3)$, $P_1(\mathbb{Z}_2^4)$ contains



$(0,\mathbf{1})\neg(0,\boldsymbol{x})\neg(1,\boldsymbol{x})\neg\ (1,\mathbf{0})\neg(1,\boldsymbol{y})\neg(0,\boldsymbol{y})\neg(0,\mathbf{1})$; $(0,\boldsymbol{x})\neg(0,\boldsymbol{y})$, $(1,\boldsymbol{x})\neg\ (1,\boldsymbol{y})$ and $(0,\boldsymbol{x})\neg(1,\boldsymbol{y})$ or $(1,\boldsymbol{x})\neg(0,\boldsymbol{y})$ and $P_1(\mathbb{Z}_2^4)$ is drawn in Figure 2.2.1 (b).

Figure 2.2.1 (a) : $P_1(\mathbb{Z}_2^3)$ (b) : $P_1(\mathbb{Z}_2^4)$

(i) Let $e = (0,\mathbf{1})$ and $S = e\mathbb{Z}_2^4$, $S' = 1 + S$. Then the induced subgraph G_0 on $S\backslash\{e\}$ (respectively, G_0' on $S'\backslash\{1-e\}$) is as given in Figure 2.2.2 (a) (respectively, Figure 2.2.2 (b)), which is H_2.

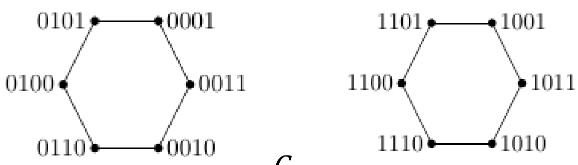
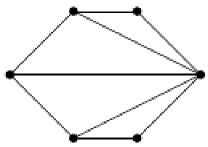
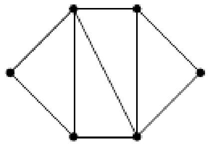


Figure 2.2.2 (a) : G_0 (b) : G_0'





(c) : H_5 of type G_3 (b) : H_4 of type G_4

(ii) From Figure 2.2.1 (b), it is clear that the induced subgraph on $S = e\mathbb{Z}_2^4\backslash\{(0,0,0,0)\} = \{0\} \times \mathbb{Z}_2^4\backslash\{(0,0,0,0)\}$ is $K_1 + H_2$.
Similarly, the induced subgraph on $S' = \{1\} \times \mathbb{Z}_2^4\backslash\{(0,0,0,0)\}$ is $K_1 + H_2$.
(iii) The induced subgraph on $(S \cup S')\backslash\{e, e'\}$ consists of the following:
(a) induced subgraph on S and S' given in (i);
(b) induced subgraph on $S^{(0)} = \{(0,0,0,1), (1,0,1,1), (0,0,1,0), (1,1,1,0), (0,1,0,0), (1,1,0,1)\}$, which is H_2.
(c) Induced subgraphs on $(0,0,0,1), (1,0,0,1)$;$(0,0,1,1), (1,0,1,1)$; $(0,0,1,0), (1,0,1,0)$; $(0,1,1,0), (1,1,1,0)$;$(0,1,0,0), (1,1,0,0)$; $(0,1,0,1), (1,1,0,1)$, which are K_2.
Thus the proof of (iii) is completed.
(iv) From Figure 2.2.1 (b), it is clear that $\left|E\left(P_1(\mathbb{Z}_2^4)\right)\right| = 3\left|V\left(P_1(\mathbb{Z}_2^4)\right)\right| - 6$, which shows $P_1(\mathbb{Z}_2^4)$ is maximal planar graph and hence is triangulated.

Remark 2.2.4 (i) The induced subgraphs on $(S \cup S')\backslash\{e, 1-e\}$ contain chemical graphs.
(ii) $P_1(\mathbb{Z}_2^4)$ also contains induced subgraphs H_4 of type G_4, which are chemical graphs.
(iii) The induced subgraphs on $(S \cup S')\backslash\{e, e'\}$ also contains H_5 of type G_3.
The induced subgraphs on $N(0,0,0,1)$, $N(1,0,1,1)$, $N(0,0,1,0)$, $N(1,1,1,0)$, $N(0,1,0,0)$ and $N(1,1,0,1)$ contained in $(S \cup S')\backslash\{e, e'\}$ are H_5 of type G_3.

In what follows, Proposition 2.2.1 is inductively generalized for $\mathbb{Z}_2^k$, $k \geq 4$.
Proposition 2.2.5 Let $R = \mathbb{Z}_2^k$, $k \geq 3$ and $e = (0, \mathbf{1})$. Then
(i) $eR = \{0\} \times \mathbb{Z}_2^{k-1}$; $(1, \mathbf{1}) + eR = \{1\} \times \mathbb{Z}_2^{k-1}$.
(ii) $V\left(P_1(\mathbb{Z}_2^k)\right)$ has a partition $S \cup S'$, where S is the set of nontrivial elements of maximal ideal generated by e and $S' = (1, \mathbf{1}) + S$. Also, $|S| = |S'| = 2^k - 1$.
Proof. (i) The proof of (i) follows from the fact that $\mathbf{1}$ is the identity element in $\mathbb{Z}_2^{k-1}$.
(ii) Let $S = eR\backslash\{(0, \mathbf{0})\}$ and $S' = (1, \mathbf{1}) + eR\backslash\{(0, \mathbf{0})\}$.
Then $S' = (1, \mathbf{1}) + \left(\{0\} \times \mathbb{Z}_2^{k-1}\right) = \{1\} \times \mathbb{Z}_2^{k-1}$.
Thus $V\left(P_1(\mathbb{Z}_2^k)\right) = R\backslash\{(0, \mathbf{0}), (1, \mathbf{1})\} = \left(\{0\} \times \mathbb{Z}_2^{k-1}\right) \cup (\{1\} \times \mathbb{Z}_2^{k-1}) = S \cup S'$.
Also $\left|\{0\} \times \mathbb{Z}_2^{k-1}\right| = 2^{k-1}$ and hence $|S| = |S'| = 2^{k-1} - 1$.
One edge in $P_1(\mathbb{Z}_2^{k-1})$ induces multiple blocks in $P_1(\mathbb{Z}_2^k)$ hereditarily, which is proved in the following proposition.
Proposition 2.2.6 Let $R = \mathbb{Z}_2^k$, $k \geq 3$. Then
(i) One edge in $P_1(\mathbb{Z}_2^{k-1})$ induces a block of type G_4 in $P_1(\mathbb{Z}_2^k)$.
(ii) $P_1(R)$ contains three copies of subgraphs isomorphic to $P_1(\mathbb{Z}_2^{k-1})$.
(iii) $P_1(R)$ contains a subgraph isomorphic to $(k-1)K_2$.
(iv) $P_1(R)$ contains two copies of a subgraph isomorphic to of $K_1 + P_1(\mathbb{Z}_2^{k-1})$.
(v) $P_1(R)$ is Hamiltonian.
(vi) $P_1(R)$ is not planar for $k \geq 5$.

Proof. (i) Let $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{Z}_2^{k-1}$ be nontrivial element and $\boldsymbol{x} \urcorner \boldsymbol{y}$ in $P_1(\mathbb{Z}_2^{k-1})$.

Then $(0,\boldsymbol{x}),(0,\boldsymbol{y}),(1,\boldsymbol{x}),(1,\boldsymbol{y}) \in V(P_1(R))$, in which $(0,\boldsymbol{x}),(0,\boldsymbol{y}) \in S$ and $(1,\boldsymbol{x}),(1,\boldsymbol{y}) \in S'$. Also $P_1(R)$ contains $(0,\boldsymbol{1})\neg(0,\boldsymbol{x})\neg(1,\boldsymbol{x})\neg\ (1,\boldsymbol{0})\neg(1,\boldsymbol{y})\neg(0,\boldsymbol{y})\neg(0,\boldsymbol{1})$ and $(0,\boldsymbol{x})\neg(0,\boldsymbol{y})$, $(1,\boldsymbol{x})\neg\ (1,\boldsymbol{y})$. Thus $P_1(R)$ contains an induced subgarph isomorphic to H_4, which is a block of type G_4.

(ii) Suppose $\boldsymbol{x}\neg\boldsymbol{y}$ in $P_1(\mathbb{Z}_2^{k-1})$.

Then the following are equivalent:

(a) $\boldsymbol{x}\neg\boldsymbol{y}$ in $P_1(\mathbb{Z}_2^{k-1})$

(b) $(0,\boldsymbol{x})\neg(0,\boldsymbol{y})$ in $P_1(\mathbb{Z}_2^{k})$.

(c) $(1,\boldsymbol{x})\neg(1,\boldsymbol{y})$ in $P_1(\mathbb{Z}_2^{k})$

(d) $(0,\boldsymbol{x})\neg(1,\boldsymbol{y})$ or $(1,\boldsymbol{x})\neg(0,\boldsymbol{y})$ in $P_1(\mathbb{Z}_2^{k})$.

Hence one edge $\boldsymbol{x}\neg\boldsymbol{y}$ in $P_1(\mathbb{Z}_2^{k-1})$ gives rise to three copies of $P_1(\mathbb{Z}_2^{k-1})$ in $P_1(R)$.

Therefore $P_1(R)$ contains three copies of $P_1(\mathbb{Z}_2^{k-1})$, which completes (i).

(iii) As $\boldsymbol{x}^2=\boldsymbol{x}$ for every $\boldsymbol{x}\in\mathbb{Z}_2^{k-1}$, $(0,\boldsymbol{x})\neg(1,\boldsymbol{x})$ and $P_1(R)$ contains a copy of $(k-1)K_2$.

(iv) $(0,\boldsymbol{1})\neg(0,\boldsymbol{x})$, $(1,\boldsymbol{0})\neg(1,\boldsymbol{x})$ for every $\boldsymbol{x}\in\mathbb{Z}_2^{k-1}$, which implies $P_1(R)$ contains two copies of subgraphs isomorphic to $K_1+P_1(\mathbb{Z}_2^{k-1})$.

(v) Suppose $k=3$. Then $P_1(\mathbb{Z}_2^3)$ itself is a cycle.

Suppose $k=4$. Then $P_1(\mathbb{Z}_2^4)$ contains Hamiltonian cycle as shown in bold lines in Figure 2.2.1 (b).

Suppose $k\geq 5$ and assume that $P_1(\mathbb{Z}_2^{k-1})$ contains Hamiltonian cycle C_{k-1}.

Let $(0,\boldsymbol{x})$ be nontrivial element in R. Then $(0,\boldsymbol{1})\neg(0,\boldsymbol{x})$, $(1,\boldsymbol{0})\neg(1,\boldsymbol{x})$.

Suppose $\boldsymbol{x}^-\neg\boldsymbol{x}\neg\boldsymbol{x}^+$ is an $\boldsymbol{x}$ −path in C_{k-1}, in which $\boldsymbol{x}^-$, $\boldsymbol{x}^+$ denote predecessor and successor of $\boldsymbol{x}$ in C_{k-1} respectively. Now $(0,\boldsymbol{x})\neg(0,\boldsymbol{x}^+)\neg\cdots\neg(0,\boldsymbol{x}^-)\neg(0,\boldsymbol{x})$ (respectively, $(1,\boldsymbol{x})\neg(1,\boldsymbol{x}^+)\neg\cdots\neg(1,\boldsymbol{x}^-)\neg(1,\boldsymbol{x})$) spans induced subgraph on S (respectively, T) in $P_1(R)$.

Now $(1,\boldsymbol{x}^-)\neg(1,\boldsymbol{0})\neg(1,\boldsymbol{x})\neg(0,\boldsymbol{x}^+)\neg\cdots\neg(0,\boldsymbol{x}^-)\neg(0,\boldsymbol{x})\neg(1,\boldsymbol{x}^+)\neg(0,\boldsymbol{1})\neg(1,\boldsymbol{x}^-)$ spans $P_1(R)$, as desired.

(v) For $k=3$, $P_1(R)$ is a cycle and hence it is planar.

For $k=4$, $P_1(R)$ is planar as it can be drawn as in Figure 2.2.3 with no crossings of edges.

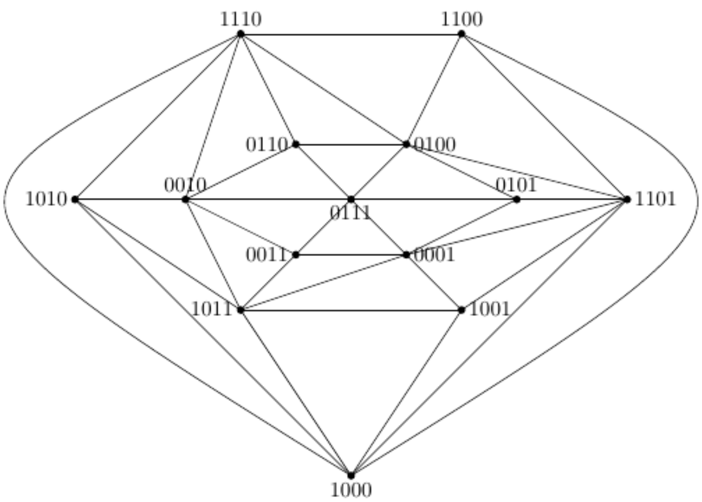


Figure 2.2.3 : $P_1(\mathbb{Z}_2^4)$

Suppose $k=5$. Then $\{(0,1,1,1,1),(1,0,0,1,1),(1,0,1,1,1)\}\cup$ $\{(0,0,0,1,0),(0,0,0,1,1),(0,0,0,0,1)\}$ contains $K_{3,3}$ and hence $P_1(\mathbb{Z}_2^5)$ is not planar.

Therefore inductively, $P_1(\mathbb{Z}_2^k)$ is not planar as it contains copies of $P_1(\mathbb{Z}_2^{k-1})$.

Proposition 2.2.7 Let R be $\mathbb{Z}_2^k$, $k\geq 3$ Then

(i) $\gamma_{cs}\left(P_1(\mathbb{Z}_2^k)\right) = 4$ and there are at least $k-1$ MCDSs.
(ii) $P_1(R)$ contains $Br\left(2P_4, \frac{n-8}{2}, \frac{n-8}{2}\right)$ as its spanning subgraph.
(iii) $\alpha\left(P_1(\mathbb{Z}_2^k)\right) \geq \frac{n}{2} - 1$.

Proof . (i) Suppose $k = 3$. Then $\{(0,1,1), (1,0,0)\}$ is a minimum dominating set in $P_1(\mathbb{Z}_2^3)$ and $(0,1,1) \neg (0,1,0) \neg (1,1,0) \neg (1,0,0)$ is a shortest path between $(0,1,1)$ and $(1,0,0)$. Therefore $\mathbb{D} = \{(0,1,1), (0,1,0), (1,1,0), (1,0,0)\}$ is a MCDS and hence $\gamma_{cs}\left(P_1(\mathbb{Z}_2^3)\right) = 4$.
Note that $(0,1,1) \neg (0,0,1) \neg (1,0,1) \neg (1,0,0)$ is an alternate shortest path between $(0,1,1)$ and $(1,0,0)$. Therefore $\{(0,1,1), (0,0,1), (1,0,1), (1,0,0)\}$ is also an MCDS. Hence $P_1(\mathbb{Z}_2^3)$ has at least 2 MCDSs.
Suppose $k \geq 4$ and $u \in V\left(P_1(\mathbb{Z}_2^k)\right)$. Then either $u \in S$ or $u \in S'$ by Proposition 2.2.1.
Suppose $u \in S$. Then $u = (0, \boldsymbol{x})$, where $\boldsymbol{x} \in \mathbb{Z}_2^{k-1}$.
Now $P_1(\mathbb{Z}_2^k)$ contains $e \neg u \neg u' \neg e'$, where $e = (0, \mathbf{1})$, $u' = (1, \boldsymbol{x})$, $e' = (1, \mathbf{0})$.
Therefore $\mathbb{D} = \{e, u, u', e'\}$ forms a MCDS for every $u \in S$. Hence there are at least $k-1$ MCDSs.
Dual proof is carried out for the case $u \in S'$
(ii) Now let $\boldsymbol{x} \in V\left(P_1(\mathbb{Z}_2^{k-1})\right)$. If $\boldsymbol{x} \in S$, then
$\{(0,\mathbf{1}), (0,\boldsymbol{x}), (1,\boldsymbol{x}), (1,\mathbf{0})\}$ is an MCDS in $P_1(\mathbb{Z}_2^k)$. Hence $\gamma_{cs}\left(P_1(\mathbb{Z}_2^k)\right) = 4$ and $P_1(\mathbb{Z}_2^k)$ contains at least $k-1$ MCDSs.
(ii) Let $\mathbb{D}$ be a connected dominating set containing $(0,\mathbf{1}), (1,\mathbf{0})$. Now having $(0,\mathbf{1}), (1,\mathbf{0})$ as hubs draw $(0,\mathbf{1}) \neg (0,\boldsymbol{x})$ for every $(0,\boldsymbol{x}) \in S \backslash \mathbb{D}$ and $(1,\mathbf{0}) \neg (1,\boldsymbol{y})$ for every $(1,\boldsymbol{y}) \in \boldsymbol{S}' \backslash \mathbb{D}$, which establishes a strong double broom $Br\left(2P_4, \frac{n-8}{2}, \frac{n-8}{2}\right)$ that spans $P_1(R)$.
(iii) S and S' are MSSs with $|S| = |S'| = 2^{k-1} - 1$ vertices and so $\alpha\left(P_1(\mathbb{Z}_2^k)\right) \geq 2^{k-1} - 1 = \frac{n}{2} - 1$.

Remark 2.2.8 (i) Densely packed hexagons are identified as subgraphs in $P_1(\mathbb{Z}_2^4)$ as shown in Figure 2.2.4 (a).

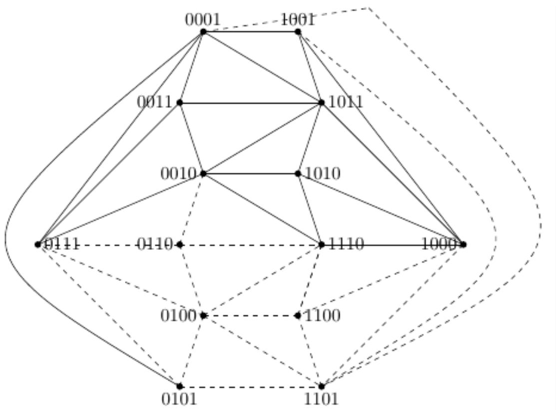
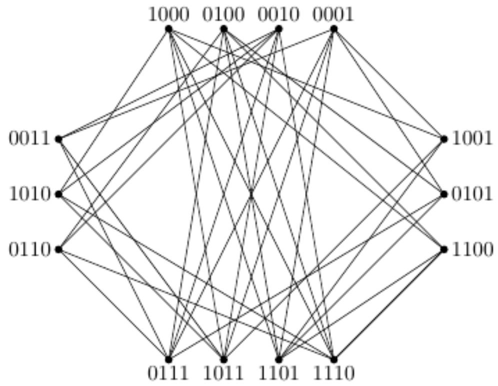


Figure 2.2.4 (a) : Hexagons in $P_1(\mathbb{Z}_2^4)$



(b) :$P_1(\mathbb{Z}_2^4)$

(ii) $P_1(\mathbb{Z}_2^4)$ is redrawn in Figure 2.2.4 (b), which exhibits stable sets explicitly.

3. Conclusion

Characteristic of $\mathbb{Z}_n$ determines connectedness of $P_1(\mathbb{Z}_n)$. $\phi(n)$ describes extreme values of stability number of $P_1(\mathbb{Z}_n)$. $P_1(\mathbb{Z}_{2^k})$, $k \geq 3$ contains spanning bistars and $P_1(\mathbb{Z}_2^k)$, $k \geq 3$ contains spanning strong double brooms. $P_1(\mathbb{Z}_2^4)$ is proved to be Hamiltonian maximal planar graph, in which densely packed hexagons are identified. It is observed that one edge in $P_1(\mathbb{Z}_2^{k-1})$ induces multiple blocks in $P_1(\mathbb{Z}_2^k)$ hereditarily. Further study on projection graphs can be carried out to explore visual representations of isomorphic substructures of rings.

4. References

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