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ON A THEORY OF Γ-FIELD AND IT'S CHARACTERISTICS

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Abstract

This research work introduces the idea by gamma-field as a characterization of a field, studies the qualities by gamma-field, and establishes that N is a gamma-field iff N is a simple, commutative and integral gamma-ring.

Keywords: gamma-field, gamma-ring, gamma-semiring, gamma-group, regular gamma-semigroup, commutative gamma-ring, simple gamma-ring, integral gamma-ring.

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1. INTRODUCTION

Vandiver proposed the first idea of a semiring in 1934. Sen first proposed the idea of a Γ-semigroup as a generalisation of the semigroup in 1981. Nobusawa established the concept of a "gamma-ring" in 1964 as a generalisation of the word "ring." In 1932, Lehmer proposed the idea of a ternary algebraic system. A tripple semiring was first discussed by Dutta and Kar(2003). The term " Γ-semiring" was first used by Murali Krishna Rao [5-8] in 1995 as a generalisation of the terms " Γ-ring," "ring," "ternary semiring," and "semiring." At the beginning of the 20th century, semigroups were formally studied. Studies on the Γ-semigroup were done by Dutta, Adhikari(1993), Sen and Saha(1986,1981). A semigroup, a gamma-semigroup, a semiring, and a gamma-semiring are among the ideals that Murali Krishna Rao [10–13] explored. Neumann [15] described regular rings. M.M. Krishna Rao changed the Specify by gamma-group as a modification of group and created the idea of regular gamma-group and examined the other features of a gamma-group [9]. He also proposed the notions of the identity element of a gamma-semigroup and the reciprocal element of a gamma-semigroup. As a soft semiring, gamma-semiring was researched by Murali Krishna Rao [14]. Since gamma-algebras are a derivation of soft algebras, studying gamma-algebras is nothing more than studying soft algebras. In this essay, the author defines a "gamma-field" as a simplification of a "field" and examines its characteristics.

We review some of the essential ideas and Specifys required for this study in this part.

Specify 1(a). Semigroups are algebraic systems with the form (N, •) where N is a set containing one or more elements and • is an associative binary properties.

Specify 1(b). If an algebraic system (N, •) satisfies the conditions mentioned below, and then it is a group. A set containing one or more elements set N and if it satisfies the associative law ("•").

I. there exists $f \in N$, : $u \cdot f = f \cdot u = u, \forall\, u \in N$,

II. If for each $u \in N\ \exists\ b \in N$, : $u \cdot g = g \cdot u = f$.

Specify 1(c). Semi-ring is a set N and two associative Boolean properties, multiplication and addition (represented by • and +, respectively).

(i) $u + v = v + u$.

Multiplication distributes over addition both from the left and right $u, v, w \in N$,

(ii) There exists $0 \in S$: $u + 0 = u$ and $u \cdot 0 = 0 \cdot u = 0\ \forall\, u \in N$.

Specify 1(d). Suppose N and Γ be the non-empty sets. Then we say N a Γ-semigroup, if there exists a mapping $N \times \Gamma \times N \rightarrow N$ (images of (u, σ, v) will be identified by $u\sigma v$, $u, v \in N$, $\sigma \in \Gamma$) : it satisfies $u\sigma(v\rho w) = (u\sigma v)\rho w\ \forall\, u, v, w \in N$ and $\sigma, \rho \in \Gamma$.

Specify 1(e). Suppose N and gamma (Γ) be the two additive abelian semigroups with identity elements 0 and 0' respectively. If there exists a mapping $N\, u\Gamma u N \rightarrow N$ (images to be denoted $u\lambda v$, $u, v \in N$, $\lambda \in \Gamma$) satisfying $\forall\, u, v, w \in N$, $\lambda, \eta \in \Gamma$

(a) $u\lambda(v\eta w) = (u\lambda v)\eta w$

(b) $u\lambda(v + w) = u\lambda v + u\lambda w$

$(u + v)\lambda w = u\lambda w + v\lambda w$

$u(\lambda + \eta)w = u\lambda w + u\eta w$

(c) $u\lambda 0 = 0\lambda u = 0$ and $u0'v = 0$, then N is called a Γ-semiring.

Specify 1(f). A non-empty set N is called a gamma-ring according to the following conditions.

(i) N and gamma (Γ) are two abelian groups with identity elements 0 and 0' respectively. If there exists a mapping $N\, u\Gamma u N \rightarrow N$ (images to be denoted $u\lambda v$, $u, v \in N$, $\lambda \in \Gamma$) satisfying $\forall\, u, v, w \in N$, $\lambda, \eta \in \Gamma$

(a) $u\lambda(v\eta w) = (u\lambda v)\eta w$

(b) $u\lambda(v + w) = u\lambda v + u\lambda w$

$(u + v)\lambda w = u\lambda w + v\lambda w$

$u(\lambda + \eta)w = u\lambda w + u\eta w$

(c) $u\lambda 0 = 0\lambda u = 0$ and $u0'v = 0$, then N is called a gamma-ring.

Specify 1(g). A gamma-semigroup N is considered to be shifting if $j\sigma g = g\sigma j$, $\forall\, j, g \in N$, $\forall\, \sigma \in \Gamma$.

Specify 1(h). Suppose N be a gamma-semigroup. j is an element considered to be regular element of N if there exist $u \in N$, $\sigma, \rho \in \Gamma$: $j = j\sigma u\rho j$.

Specify 1(i). Suppose N be a gamma-semigroup. Each element of N is a regular element of N then N is defined to be regular gamma-semigroup N.

Specify 1(j). Suppose N be a gamma-semigroup and $\sigma \in \Gamma$. *A* binary operation$*$ on N by $j * g = j\sigma g$, $\forall\, j, g \in N$. Then $(N, *)$ is a semigroup. It is identified by N_σ.

Specify 1(k). A gamma-semigroup N is called a gamma-group, if N_σ is a group and $\sigma \in \Gamma$.

2. Γ-FIELD

We discuss about the idea of gamma-field and their properties.

Specify 2(a). A gamma-semigroup N is clearly gamma-group and satisfies the conditions

(i) if there exists $1 \in N$ and for each $u \in N$ $\exists\, \sigma \in \Gamma$, : $u\sigma 1 = 1\sigma u = u$.

(ii) If for each element $0 \neq j \in N$ there exist $g \in N$, $\sigma \in \Gamma$: $j\sigma g = g\sigma j = 1$.

We discuss a gamma-field.

Specify 2(b). A commutative gamma-ring N is called to be gamma-field if N is a gamma -group.

Specify 2(c). Suppose N be a gamma-ring. An element $1 \in N$ is clearly identity if for each $u \in N$ $\exists\, \sigma \in \Gamma$: $u\sigma 1 = 1\sigma u = u$.

Specify 2(d). In a gamma-ring with identity 1, an element $j \in N$ is clearly left conversable (right conversable) if there exist $g \in N$, $\sigma \in \Gamma$: $g\sigma j = 1$ ($j\sigma g = 1$).

Specify 2(e). In a gamma-ring N, an element $u \in N$ is clearly unit if there exist

$j \in N$ and $\sigma \in \Gamma$, : $j\sigma u = 1 = u\sigma j$.

Specify 2(f). A gamma-ring N is clearly simple gamma-ring if it has no proper ideals of N.

Specify 2(g). A non-zero element j in a gamma-ring N is clearly zero divisor if there exists a non zero element $g \in N$, $\sigma \in \Gamma$: $j\sigma g = g\sigma j = 0$.

Specify 2(h). A gamma-ring N with identity 1 and zero elements 0 is called an integral gamma-ring if it has no zero divisors.

Specify 2(i). A gamma-ring N with zero element 0 is clearly hold cancellation laws if j $\neq 0$, $j\sigma g = j\sigma h$, $g\sigma j = h\sigma j$, where $j, g, h \in N$, $\sigma \in \Gamma$ then $g = h$.

Specify 2(j). A gamma-ring with identity 1 and zero element 0 is called a pre -integral gamma-ring if N defined cancellation laws.

Example 2(k). Suppose N belongs to the sets of all real numbers and Γ belongs to all rational numbers. Then N and Γ are additive abelian groups with respect to usual addition. Determine the triplet operation $N \times \Gamma \times N \to N$ by $(j, \sigma, g) \to j\sigma g$, using the usual multiplication. Then N is a gamma-field.

Assumption 2(l). Suppose N be a Γ-ring with identity 1. If $, g \in N$, and $\varphi, \rho \in \Gamma$:

$j\varphi g$ is ρ-idempotent and j is left invertible, then g is a regular element.

Proof. Suppose $j, g \in N$ and j be left invertible. There exist $d \in N$, *and* $\varphi, \lambda \in \Gamma$: $1\varphi g = g$ and $d\lambda j = 1$

$d\lambda j = 1 \Rightarrow$
$d\lambda j\varphi g = 1\varphi g$

$\Rightarrow$ $d\lambda j\varphi g = g$.

Suppose $j\varphi g$ is $\rho -$ unchanged

$\Rightarrow$ $j\varphi g\rho j\varphi g = j\varphi g$

$\Rightarrow$ $d\lambda j\varphi g\rho j\varphi g = d\lambda j\varphi g$

$\Rightarrow$ $g\rho j\varphi g = g$.

Hence g is a regular element.

Consequence 2(m). Suppose N be a Γ-ring with identity 1. If j, g ∈ N , φ, ρ ∈ Γ :

jφg is ρ-unchanged and g is right invertible, then j is regular.

Assumption 2(n). If N is a gamma-ring with identity 1 and j ∈ N is left invertible, then

j is a regular.

Proof. Suppose N be a gamma-ring with identity 1. Suppose $j \in N$ is left invertible, there exist $g \in N$, $\sigma \in \Gamma$, : $g\sigma j = 1$. Since 1 is identity.

there exists $\varphi \in \Gamma$: $j\varphi 1 = 1\varphi j = j$.

$j\varphi 1 = j$

$\Rightarrow j\varphi(g\sigma j) = j.$

$\Rightarrow j\varphi g\sigma j = j.$

Hence j is a regular element.

Consequence 2(o). If N is a gamma-ring with identity 1 and j ∈ N is invertible, then j is regular.

Assumption 2(p). If N is a gamma-field, then N is a regular.

Proof. Suppose N be a gamma-field. Then each non-zero element is invertible.

Consequence 2(p), every element is a regular. Therefore N is a regular gamma-field.

Assumption 2(q). A gamma-field continues cancellative laws.

Proof. Suppose N be a gamma-field. Suppose $j \neq 0$ and $j\sigma g = j\sigma h$, where $j, g, h \in N$, $\sigma \in \Gamma$. There exist $u \in N$, $\varphi \in \Gamma$, : $u\varphi j = 1$.

$j\sigma g = j\sigma h,$

$\Rightarrow \quad u\varphi j\sigma g = u\varphi j\sigma h$

$\Rightarrow \quad (u\varphi j)\sigma g = (u\varphi j)\sigma h$

$\Rightarrow \quad 1\sigma g = 1\sigma h$

$\Rightarrow \quad g = h.$

Hence the Assumption.

Assumption 2(r). If N is a gamma-field, then the equation $j\sigma u = g$ has a individual solution for any non-zero elements j, g ∈ N and for σ ∈ Γ.

Proof. Suppose N be a gamma-field and the equation $j\sigma u = g$ for any non-zero elements $j, g \in N$ and for $\sigma \in \Gamma$. Then there exist $h \in N$, $\rho \in \Gamma$: $1\rho g = g$ and $j\sigma h = 1$.

Now $j\sigma h = 1$

$\Rightarrow \quad j\sigma h\rho g = 1\rho g$

$\Rightarrow \quad j\sigma(h\rho g) = g$

Suppose there exist $u, v \in N$: $j\sigma u = g$ and $j\sigma v = g$. Then $j\sigma u = j\sigma v$. Therefore by Assumption 2(q), $u = v$. Now the proof is complete.

Assumption 2(s). Any shifting finite pre-integral gamma-ring N is a gamma-field N.

Proof. Suppose $N = \{j_1, j_2, \ldots, j_n\}$ and $0 \neq j \in N$, $\sigma \in \Gamma$. We consider the n products $j\sigma j_1, j\sigma j_2 \ldots j\sigma j_n$. These products are all distinct. Since $j\sigma j_i = j\sigma j_j \Rightarrow j_i = j_j$. Since $1 \in N$, $\exists\ j_i \in N$: $j\sigma j_i = 1$. Therefore j has reciprocal. Hence any shifting finite pre-integral gamma-ring N is a gamma -field.

Assumption 2(t). Suppose N be a gamma-ring with zero element 0 and identity element. If I is an ideal of a gamma-ring N containing a unit element then I = N .

Proof. Suppose I be an ideal of the gamma-ring N containing a unit element u and $u \in N$. Then there exists $\sigma \in \Gamma$: $u\sigma 1 = u$ and $u\sigma u \in I$, since I is an ideal. Since u is a unit element, there exist $\varphi \in \Gamma$, $t \in N$: $u\varphi t = 1$ $\Rightarrow u\sigma u\varphi t = u\sigma 1 = u \in I$. Hence $I = N$.

Assumption 2(u). Every gamma-field is zero divisors free.

Proof. Suppose N be a Γ-field, $j, g \in N$ and $j\sigma g = 0$, $\sigma \in \Gamma$ and $j \neq 0$. Since $j \neq 0$ $\exists$ $\rho \in \Gamma : j^{-1}\rho j = 1$.

$$j\sigma g = 0 \Rightarrow \quad {}^{-1}\rho(j\sigma g) = \quad {}^{-1}\rho 0$$

a

$$\Rightarrow (j^{-1}\rho j)\sigma g = 0$$

$$\Rightarrow 1\sigma g = 0 = 1\sigma 0.$$

Therefore $g = 0$. Hence proved N is zero divisors free.

Assumption 2(v). N is a gamma-field iff N is an integral, simple and commutative gamma-ring.

Proof. Suppose I be a proper ideal of the gamma-field N. Every non zero element of N is a unit. By Assumption 3.21, we have $I = N$. Therefore gamma-field N contains no proper ideals. Hence gamma-field is a simple Γ-ring. By Assumption 2(v), N is an integral gamma-ring. Conversely, Let N is an integral, simple and commutative gamma-ring. Suppose $0 \neq j \in N$, $\sigma \in \Gamma$. Consider $j\sigma N$, $j\sigma N \neq \{0\}$, since N is an integral gamma-ring. Clearly $j\sigma N$ is a proper ideal of $N \Rightarrow j\sigma N = N$, since N is a simple gamma -ring. Therefore, there exists $g \in N : j\sigma g = 1$. Hence the Assumption.

Assumption 2(w). *Suppose N be a commutative gamma-ring. N satisfies the condition,for each, $0 \neq j \in N$, $\sigma \in \Gamma$ and $d \in N$. Then there exist $g \in N$, $\rho \in \Gamma : j\sigma g\rho d = d$ iff N is a gamma-field.*

Verify. Suppose N be a commutative gamma-ring. Suppose N is a gamma-field, $0 \neq j \in N$ and $c \in N$. Since N is a gamma-field, there exist $g \in N$, $\sigma \in \Gamma$ such that $j\sigma g = 1$. Since 1 is the identity element, there exists $\rho \in \Gamma$: $1\rho h = h$. Therefore $j\sigma g\rho h = 1\rho h \Rightarrow j\sigma g\rho h = h$. Hence N is a gamma-field. Conversely suppose that N is a commutative gamma-ring satisfies the condition,for each, 0 $j \in N$, $\sigma \in \Gamma$, then there exist $g \in N$, $\rho \in \Gamma : j\sigma g\rho d = d$, $\forall$ $d \in N$. Suppose $0 \neq j \in N$, $\sigma \in \Gamma$ and $d \in N$. Then there exists $\rho \in \Gamma : j\sigma g\rho d = d$. Therefore $j\sigma g = 1$. Prove each element is not equal to zero and N has inverse then N is a gamma-field.

Assumption 2(x). Suppose N is a zero element with gamma ring, later N is a gamma field iff commutative gamma-ring. $N \setminus \{0\}$ and gamma-ring N $\setminus \{0\}$ has no proper ideals.

Verify. Suppose N is a gamma-field. By Assumption 3.21, N is Zero divisors free. Suppose I be an ideal of the gamma-field $N \setminus \{0\}$ and $a \in I$. Since $0 \neq j \in N$, there exist $\sigma \in \Gamma$, $u \in N$ such that $j\sigma u = 1$. Therefore $1 \in I$. Suppose $u \in N \setminus \{0\}$. Then $u\sigma 1 \in I$, $\forall$ $\sigma \in \Gamma \Rightarrow u \in I$. Therefore $N \setminus \{0\} = I$. Thus gamma-field $N \setminus \{0\}$ has no proper ideals. Conversely suppose that gamma-ring N is Zero divisors free and gamma-ring $N \setminus\{0\}$ has no proper ideals. Suppose 0 $j \in N$, $\sigma \in \Gamma$. Consider $j\sigma N \neq \{0\}$.Then $j\sigma N = N$. Therefore there exists $g \in N : j\sigma g = 1$. Hence N is a gamma-field.

Assumption 2(y). N is a gamma-field iff N $_\sigma$ is a field for σ ∈ Γ, then N $_\rho$ is a field ∀ ρ ∈ Γ.

Proof. Suppose N be a gamma-field. Suppose N_σ is a field for some $\sigma \in \Gamma$, $j \in N \setminus \{0\}$ and $\sigma \in \Gamma$. Suppose $g \in N \setminus \{0\}$, $\rho \in \Gamma$, Then $j\rho g \neq 0$. By Specify of the field, we have

$$(j\rho g)\sigma h = 1,\ h \in N$$

$$\Rightarrow j\rho(g\sigma h) = 1.$$

Hence N_ρ is a field. Converse is obvious.

3. Conclusion

This paper identified a "gamma-field," a "regular gamma-field," and examined its characteristics. The author established that iff N is an simple, integral and commutative gamma-ring and N is a gamma-field iff N_σ is a field for $\sigma \in \Gamma$, then N_ρ is a field $\forall$ $\rho \in$

Γ.

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