Total coloring for splitting graph of certain graphs

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ABSTRACT

A total coloring of a graph G is an assignment of colors to all the vertices and edges of the graph for which no two adjacent or incident vertices and edges receives the same color. The total chromatic number of G, denoted by $\chi^{''}(G)$, is the minimum number of colors suffice for a total coloring. Behzad and Vizing was introduced the total coloring conjecture independently and claims that, $\Delta(G)+1 \leq \chi^{''}(G) \leq \Delta(G)+2$. In this paper, we determined total coloring conjecture for $S(L_n), S(S_n), S(H_n), S(T_n), S(TL_n), S(SL_n)$ respectively.

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I. INTRODUCTION

All graphs considered in this paper are finite, simple, undirected and connected. Let $G=(V(G),E(G))$ be a graph. Graph coloring is the one of the most important sub-topic in graph theory with many real time applications. Vertex coloring is an assignment of colors to vertices for which no two adjacent vertices are received the same color. The minimum number of colors needed for vertex coloring is known as chromatic number, denoted by $\chi(G)$. Similarly, a proper edge coloring is the assignment of colors to the edges such that no two adjacent edges received the same colors is called the edge chromatic number of the graph and it is denoted by $\chi^{'}(G)$. Vizing proved that for any graph G, $\chi^{'}(G)$ is either $\Delta(G)$ or $\Delta(G)+1$. The graph having $\Delta(G)$ colors for its edge coloring are called type-I graphs and $\Delta(G)+1$ colors are called type-II graphs.

A total coloring of G is a mapping $f: V(G) \cup E(G) \rightarrow C$, where C is the set of colors and f satisfies as follows:

i) $f(a) \neq f(b)$, for any two adjacent vertices $a,b \in V(G)$

ii) $f(e_1) \neq f(e_2)$, for any two adjacent edges $e_1, e_2 \in E(G)$ and

iii) $f(a) \neq f(e)$, for any vertex $a \in V(G)$ and any edge $e \in E(G)$ incident to a

The total chromatic number of a graph G, denoted by $\chi^{''}(G)$, is the minimum number of colors that are used in a total coloring. It is clearly known that $\chi^{''}(G) \geq \Delta(G)+1$. The total coloring conjectured proposed independently by Bezhad [1] and Vizing [6] that for every graph G, $\chi''(G) \geq \Delta(G)+2.$ The graphs G total colored with $\Delta(G)+1$ colors are known as type -I graphs and have $\Delta(G)+2$ colors are called type-II graphs. The total coloring conjecture is long–standing conjecture and proved several graph families. It is easily observed that TCC is true for complete, bipartite, complete multipartite graphs. The total coloring conjecture has been confirmed for several other classes of graphs. The literature reviews of techniques and other results on total coloring can be found in Yap [7], Borodin [2] and Geetha et.al [3]. Jayaraman and Muthuramakrishnan [4, 5] proved that total coloring of splitting graph path, cycle, star, complete bipartite, fan, wheel and bistar graph. In this paper, we determined the total chromatic number for $S(L_n), S(S_n), S(H_n), S(T_n), S(TL_n), S(SL_n)$.

II. PRELIMINARIES

Definition 2.1: The *splitting graph* $S(G)$ of a graph G is obtained from adding to each vertex v, a new vertex v' such that v' is adjacent to each vertex that is adjacent to v in G, which means $N(v) = N(v')$.

Definition 2.2: The *ladder graph* is obtained from the path P_n with $\{v_1, v_2, \ldots\ldots v_n\}$ and $\{u_1, u_2, \ldots\ldots u_n\}$ vertices by joining the vertices $v_i u_i$ for $1 \leq i \leq n$.

Definition 2.3: The *sunlet graph* S_n is obtained from a cycle C_n attaching a pendant edge at each vertex of a cycle C_n.

Definition 2.4: A *Triangular snake* T_n is obtained from a path by identifying each of the path with an edge of the cycle C_3.

Definition 2.5: The *H-graph* of a path P_n is the graph obtained from two copies of P_n with vertices $v_1, v_2, \ldots\ldots v_n$ by joining the vertices $V_{\frac{n+1}{2}}$ *and* $U_{\frac{n+1}{2}}$ by an edge if n is odd and the vertices $V_{\frac{n}{2}+1}$ and $U_{\frac{n}{2}}$ if n is even.

Definition 2.6: A *Triangular ladder graph* $TL_n, n \geq 2$ is a graph obtained from L_n by adding the edges $v_a u_{a+1}, 1 \leq a \leq n-1$.

Definition 2.7: The *slanting ladder* SL_n is a graph that consists of two copies of P_n having vertex set $\{u_a : 1 \leq a \leq n\} \cup \{v_a : 1 \leq a \leq n\}$ and the edge set is formed by adjoining u_a and v_{a+1} for all $1 \leq a \leq n-1$.

III. RESULTS AND DISCUSSION

Theorem 3.1: For n ≥ 3, $\chi''(SL_n) = 7$.

Proof. Let $V(SL_n) = \{v_a, u_a, v'_a, u'_a : 1 \leq a \leq n\}$ and

$$E(SL_n)=\begin{cases}\{e_a, f_a, g_a : 1\le a\le n-1\}\cup \\ \{e_a^{'}, f_a^{'}, g_a^{'} : 1\le a\le n-1\}\cup \\ \{e_a^{''}, f_a^{''}, g_a^{''} : 1\le a\le n-1\}\end{cases}$$, where $\{e_a, f_a, g_a : 1\le a\le n-1\}$ is the edges $\{v_a v_{a+1}, v_{a+1}v_a^{'}, u_{a+1}u_a^{'} : 1\le a\le n-1\}$, $\{e_a^{'}, f_a^{'}, g_a^{'} : 1\le a\le n-1\}$ is the edges $\{u_a u_{a+1}, v_a v_{a+1}^{'}, u_a u_{a+1}^{'}; 1\le a\le n-1\}$ and $\{e_a^{''}, f_a^{''}, g_a^{''} : 1\le a\le n\}$ is the edges $\{v_a u_a, u_a v_a^{'}, v_a u_a^{'} : 1\le a\le n\}$.

Define a total coloring f, such that $f : \mathrm{V}(\mathrm{SL}_n)\cup \mathrm{E}(\mathrm{SL}_n)\to\{1,2,3,4,5,6,7\}$ as follows:

For $1\le a\le n$

$$f(v_a)=\begin{cases}1, & a\equiv 1\ (\mathrm{mod}\,3)\\ 3, & a\equiv 2\ (\mathrm{mod}\,3)\\ 2, & a\equiv 0\ (\mathrm{mod}\,3)\end{cases}$$

$$f(v_a^{'})=\begin{cases}1, & a\equiv 1\ (\mathrm{mod}\,3)\\ 3, & a\equiv 2\ (\mathrm{mod}\,3)\\ 2, & a\equiv 0\ (\mathrm{mod}\,3)\end{cases}$$

$$f(u_a)=\begin{cases}2, & a\equiv 1\ (\mathrm{mod}\,3)\\ 1, & a\equiv 2\ (\mathrm{mod}\,3)\\ 3, & a\equiv 0\ (\mathrm{mod}\,3)\end{cases}$$

$$f(u_a^{'})=\begin{cases}2, & a\equiv 1\ (\mathrm{mod}\,3)\\ 1, & a\equiv 2\ (\mathrm{mod}\,3)\\ 3, & a\equiv 0\ (\mathrm{mod}\,3)\end{cases}$$

$$f(v_a u_a)=\begin{cases}4, & if\ \ a\ is\ odd\\ 5, & if\ \ a\ is\ even\end{cases}$$

$$f(v_a u_a^{'})=f(u_a v_a^{'})=\begin{cases}5, & if\ \ a\ is\ odd\\ 4, & if\ \ a\ is\ even\end{cases}$$

For $1\le a\le n-1$

$$f(v_a v_{a+1})=\begin{cases}2, & a\equiv 1\ \ (\text{mod } 3)\\ 1, & a\equiv 2\ \ (\text{mod } 3)\\ 3, & a\equiv 0\ \ (\text{mod } 3)\end{cases}$$

$$f(u_a u_{a+1})=\begin{cases}3, & a\equiv 1\ (\text{mod } 3)\\ 2, & a\equiv 2\ (\text{mod } 3)\\ 1, & a\equiv 0\ (\text{mod } 3)\end{cases}$$

$$f(v_a v_{a+1}^{'})=f(u_a u_{a+1}^{'})=7$$

$$f(v_{a+1}v_a^{'})=f(u_{a+1}u_a^{'})=6$$

The sequence of vertices $\{v_1, v_2, ...\}$ and $\{v_1', v_2', ...\}$ is of the form $\{1,3,2,1,3,2...\}$ and its end $\{...v_{n-2}, v_{n-1}, v_n\}$ and $\{...v_{n-2}', v_{n-1}', v_n'\}$ is of the form $\{...2,1,3,2\}$ if $n \equiv 0(\text{mod}\,3)$, $\{...2,1,3,2,1\}$ if $n \equiv 1(\text{mod}\ 3)$, $\{...1,3,2,1,3\}$ $n \equiv 2(\text{mod}\,3)$. The sequence of vertices $\{u_1, u_2, ...\}$ and $\{u_1', u_2', ...\}$ is of the form $\{2,1,3,2,1,3...\}$ and its end $\{...u_{n-2}, u_{n-1}, u_n\}$ and $\{...u_{n-2}', u_{n-1}', u_n'\}$ is of the form $\{...1,3,2,1,3\}$ if $n \equiv 0(\text{mod}\,3)$, $\{...3,2,1,3,2\}$ if $n \equiv 1(\text{mod}\,3)$, $\{...2,1,3,2,1\}$ if $n \equiv 2(\text{mod}\,3)$. The sequence of edges $\{e_1, e_2, ...\}$ is of the form $\{2,1,3,2,1,3,...\}$ and its ends $\{...e_{n-2}, e_{n-1}, e_n\}$ is of the from $\{...2,1,3,2,1\}$ if $n \equiv 0(\text{mod}\,3)$, $\{...2,1,3\}$ if $n \equiv 1(\text{mod}\,3)$, $\{...3,2,1,3,2\}$ if $n \equiv 2(\text{mod}\,3)$. The sequence of edges $\{f_1, f_2, ..., f_{n-1}, f_n\}$ and $\{g_1, g_2, ..., g_{n-1}, g_n\}$ is of the form $\{6,6,6,..,6,6\}$ and $\{6,6,6,..,6,6\}$. The sequence of edges $\{e_1', e_2', ...\}$ is of the form $\{3,2,1,3,2,1...\}$ and its ends $\{...e_{n-2}', e_{n-1}', e_n'\}$ is of the from $\{...2,1,3,2\}$ if $n \equiv 0(\text{mod}\,3)$, $\{...2,1,3,2,1\}$ if $n \equiv 1(\text{mod}\,3)$, $\{....3,2,1,3\}$ if $n \equiv 2(\text{mod}\,3)$. The sequence of edges $\{f_1', f_2', ... f_{n-1}', f_n'\}$ and $\{g_1', g_2', ...g_{n-1}', g_n'\}$ are of the form $\{7,7,7,..7,7\}$ and $\{7,7,7,..7,7\}$. The sequence of edges $\{e_1'', e_2'', ...\}$ is of the form $\{4,5,4,5...\}$ and its end $\{...e_{n-2}'', e_{n-1}'', e_n''\}$ is of the form $\{...5,4\}$ if $n \equiv 1(\text{mod}\,2)$ and $\{...5,4,5\}$ if $n \equiv 0(\text{mod}\,2)$. The edges $\{f_1'', f_2'', ...\}$ and $\{g_1'', g_2'', ...\}$ are of the form $\{5,4,5,4...\}$ and its ends $\{...f_{n-2}'', f_{n-1}'', f_n''\}$ and $\{...g_{n-2}'', g_{n-1}'', g_n''\}$ are of the form $\{...4,5\}$ if $n \equiv 1(\text{mod}\,2)$ and $\{...5,4\}$ if $n \equiv 0(\text{mod}\,2)$. Hence f is a total 7-coloring of $S(L_n)$ and therefore $\chi''(SL_n) \le 7$. By conjecture, $\chi''(SL_n) \ge \Delta(SL_n) + 1 = 6 + 1 \ge 7$ and so $\chi''(SL_n) = 7$.

Theorem 3.2: For n ≥ 3, $\chi''(S(S_n)) = \Delta(S(S_n)) + 1$.

Proof: Let $V(S(S_n)) = \{v_a u_a : 1 \le a \le n\} \bigcup \{v_a' u_a' : 1 \le a \le n\}$ and $E(S(S_n)) = \{e_a, f_a, g_a : 1 \le a \le n\} \bigcup \{e_a', f_a', g_a' : 1 \le a \le n\}$, where $\{e_a, f_a, g_a : 1 \le a \le n\}$ are the edges $\{u_a v_a, u_a' u_{a+1}, u_a v_a' : 1 \le a \le n\}$, $\{e_a' f_a', g_a' : 1 \le a \le n\}$ are the edges $\{u_a u_{a+1(\text{mod}\,n)}, u_a u_{a+1}', v_a u_a' : 1 \le a \le n\}$.

Define a total coloring f, such that $f : V(S(S_n)) \bigcup E(S(S_n)) \to \{1,2,3,4,5,6,7\}$, as follows:

Case (i): - n is even

For $1 \le a \le n$

$$f(v_a) = 7 \ , \ f(v_a') = 5$$

$$f(u_a) = \begin{cases} 1, & if \text{ a } is\ odd \\ 2, & if \text{ a } is\ even \end{cases}$$

$$f(u_a') = \begin{cases} 1, & if\ a\ is\ odd \\ 2, & if\ a\ is\ even \end{cases}$$

$$f(v_a u_a') = 3, \ f(u_a v_a') = 7,$$

$$f(u_a u_{a+1(\text{mod n})}') = 6,$$

$$f(u_a' u_{a+1(\text{mod n})}) = 5$$

$$f(u_a u_{a+1(\text{mod n})}) = \begin{cases} 3, & if\ a\ is\ odd \\ 4, & if\ a\ is\ even \end{cases}$$

$$f(v_a u_a) = \begin{cases} 2, & if \ a \ is \ odd \\ 1, & if \ a \ is \ even \end{cases}$$

Case (ii): - n is odd

In this case, the vertices and edges are colored as in case (i), except those vertices and edges as follows

For $1 \le a \le n$

$$f(u_a) = \begin{cases} 1, & if \ a \ is \ odd \\ 2, & if \ a \ is \ even \\ 3, & a = n \end{cases}$$

$$f(u_a^{'}) = \begin{cases} 1, & if \ a \ is \ odd \\ 2, & if \ a \ is \ even \\ 4, & a = n \end{cases}$$

$$f(v_a u_a) = \begin{cases} 4, & a = 1 \\ 1, & if \ a \ is \ even \\ 2, & if \ a \ is \ odd, \ 3 \le a \le n-2 \end{cases}$$

$$f(u_a u_{a+1}) = \begin{cases} 3, & if \ a \ is \ odd \\ 4, & if \ a \ is \ even \\ 2, & if \quad a = n \end{cases}$$

Hence f is a total 7-coloring of $S(S_n)$ and therefore $\chi^{"}(S(S_n)) \le 7$. By conjecture, $\chi^{"}(S(S_n)) \ge \Delta(S(S_n)) + 1 = 6 + 1 \ge 7$ and so $\chi^{"}(S(S_n)) = 7$.

Theorem 3.3: For n ≥ 3, $\chi^{"}(ST_n) = \Delta(ST_n) + 1$.

Proof: Let $V(ST_n) = \{v_a, u_a, v_a^{'}, u_a^{'} : 1 \le a \le n\}$ and

$$E(ST_n) = \begin{Bmatrix} e_a, f_a, g_a : 1 \le a \le n-1 \bigcup \\ e_a^{'}, f_a^{'}, g_a^{'} : 1 \le a \le n-1 \bigcup \\ e_a^{"}, f_a^{"}, g_a^{"} : 1 \le a \le n-1 \end{Bmatrix},$$

where $\{e_a, f_a, g_a : 1 \le a \le n-1\}$ is the edges $\{v_a v_{a+1}, \ v_a u_a^{'}, \ v_a^{'} v_{a+1} : 1 \le a \le n-1\}$, $\{e_a^{'}, f_a^{'}, g_a^{'} : 1 \le a \le n-1\}$ is the edges $\{u_a v_a, \ v_{a+1} u_a^{'}, \ v_{a+1}^{'} v_a : 1 \le a \le n-1\}$, $\{e_a^{"}, f_a^{"}, g_a^{"} : 1 \le a \le n-1\}$ is the edges $\{u_a v_{a+1}^{'}, u_a v_{a+1}, u_a v_a^{'} : 1 \le a \le n-1\}$.

Define a total coloring f, such that $f : V(S(T_n)) \bigcup E(S(T_n)) \to \{1,2,3,4,5,6,7,8,9\}$ as follows:

For $1 \le a \le n$

$$f(v_a)=\begin{cases}1, & if\ \ a\ is\ odd\\ 2, & if\ \ a\ is\ even\end{cases}$$

For $1\le a\le n-1$

$$f(v_a^{'})=8,\ f(u_a^{'})=5,\ f(v_{a+1}^{'}v_a)=9,$$
$$f(u_a)=\ 9\ \ f(u_av_a)=8,$$
$$f(u_a^{'}v_a)=7,\ \ f(u_av_{a+1}^{'})=6$$

$$f(v_av_{a+1})=\begin{cases}3, & if\ \ \mathrm{a}\ is\ odd\\ 4, & if\ \ \mathrm{a}\ is\ even\end{cases}$$
$$f(u_av_{a+1})=5,\ \ f(u_av_a^{'})=7$$
$$f(v_a^{'}v_{a+1})=\begin{cases}1, & if\ \ \mathrm{a}\ is\ even\\ 2, & if\ \ \mathrm{a}\ is\ odd\end{cases}$$

Hence f is a total 9-coloring of $S(T_n)$ and therefore $\chi^{''}(ST_n)\le 9$. By conjecture, $\chi^{''}(ST_n)\ge\Delta(ST_n)+1=8+1\ge 9$ and so $\chi^{''}(ST_n)=9$.

Theorem 3.4: For n $\ge$ 3, $\chi^{''}(SH_n)=\Delta(SH_n)+1$.

Proof: Let $V(SH_n)=\{v_a,u_a,v_a^{'},u_a^{'}:1\le a\le n\}$ and

$E(\mathrm{H}L_n)=\{e_a,f_a,g_a:1\le a\le n-1\}\cup\{e_a^{'},f_a^{'},g_a^{'}:1\le a\le n-1\}$

$\cup\{e_a^{''},f_a^{''},g_a^{''}:1\le a\le n\}$, where $\{e_a,f_a,g_a:1\le a\le n-1\}$ is the edges $\{v_av_{a+1},v_a^{'}v_{a+1},u_a^{'}u_{a+1}:1\le a\le n-1\}$, $\{e_a^{'},f_a^{'},g_a^{'}:1\le a\le n-1\}$ is the edges $\{u_au_{a+1},v_av_{a+1}^{'},u_au_{a+1}^{'}:1\le a\le n-1\}$ and the edges $e_a^{''}=v_au_a,\ f_a^{''}=v_au_a^{'},g_a^{''}=u_av_a^{'}$, for $a=\left\lceil\frac{n}{2}\right\rceil$

Define a total coloring f, such that $f:V(S(H_n))\cup E(S(H_n))\to\{1,2,3,4,5,6,7\}$ as follows:

For $1\le a\le n$

$$f(v_a)=\begin{cases}2, & if\ \ \mathrm{a}\ is\ odd\\ 1, & if\ \ \mathrm{a}\ is\ even\end{cases}$$
$$f(u_a)=\begin{cases}1, & if\ \ \mathrm{a}\ is\ odd\\ 4, & if\ \ \mathrm{a}\ is\ even\end{cases}$$
$$f(u_a^{'})=5,\ \ f(v_a^{'})=4$$

For $1\le a\le n-1$

$$f(v_av_{a+1})=\begin{cases}4, & a\equiv 1(m\mathrm{od}\,2)\\ 3, & a\equiv 0(m\mathrm{od}\,2)\end{cases}$$

$$f(u_a u_{a+1}) = f(v_a v_{a+1}^{'}) = \begin{cases} 3, & a \equiv 1(\bmod 2) \\ 2, & a \equiv 0(\bmod 2) \end{cases}$$

$$f(u_a u_{a+1}^{'}) = \begin{cases} 2, & a \equiv 1(\bmod 2) \\ 3, & a \equiv 0(\bmod 2) \end{cases}$$

$$f(v_a^{'} v_{a+1}) = f(u_{a+1} u_a^{'}) = 6$$

For a = $\left\lceil \frac{n}{2} \right\rceil$

$$f(v_a u_a^{'}) = f(u_a v_a^{'}) = 7 , \; f(v_a u_a) = 5$$

Hence f is a total 7-coloring of $S(H_n)$ and therefore $\chi^{''}(SH_n) \le 7$. By conjecture, $\chi^{''}(SH_n) \ge \Delta(SH_n) + 1 = 6 + 1 \ge 7$ and so $\chi^{''}(SH_n) = 7$.

Theorem 3.5: For n ≥ 3, $\chi^{''}(S(SL_n)) = \Delta(S(SL_n)) + 1$.

Proof: Let $V(SL_n) = \{v_a, u_a, v_a^{'}, u_a^{'} : 1 \le a \le n\}$ and

$E(SL_n) = \left\{ \begin{array}{c} e_a, f_a, g_a : 1 \le a \le n-1 \bigcup \\ e_a^{'}, f_a^{'}, g_a^{'} : 1 \le a \le n-1 \bigcup \\ e_a^{''}, f_a^{''}, g_a^{''} : 1 \le a \le n-1 \end{array} \right\}$, where $\{e_a, f_a, g_a : 1 \le a \le n-1\}$ is the edges $\{v_a v_{a+1}, v_a v_{a+1}^{'}, u_a u_{a+1}^{'} : 1 \le a \le n-1\}$, $\{e_a^{'}, f_a^{'}, g_a^{'} : 1 \le a \le n-1\}$ is the edges $\{u_a u_{a+1}, v_{a+1} v_a^{'}, u_{a+1} u_a^{'} : 1 \le a \le n-1\}$, $\{e_a^{''}, f_a^{''}, g_a^{''} : 1 \le a \le n-1\}$ is the edges $\{v_a u_a, u_a v_{a+1}^{'}, v_{a+1} u_a^{'} : 1 \le a \le n-1\}$.

Define a total coloring f, such that $f : V(S(SL_n)) \bigcup E(S(SL_n)) \to \{1,2,3,4,5,6,7\}$ as follows:

For $1 \le a \le n$

$$f(v_a) = \begin{cases} 1, & a \equiv 1(\bmod 3) \\ 3, & a \equiv 2(\bmod 3) \\ 2, & a \equiv 0(\bmod 3) \end{cases}$$

$$f(u_a) = \begin{cases} 2, & a \equiv 1(\bmod 3) \\ 1, & a \equiv 2(\bmod 3) \\ 3, & a \equiv 0(\bmod 3) \end{cases}$$

$$f(v_a^{'}) = 6, \; f(u_a^{'}) = 5$$

For $1 \le a \le n-1$

$$f(v_a v_{a+1}) = \begin{cases} 2, & a \equiv 1(\bmod 3) \\ 1, & a \equiv 2(\bmod 3) \\ 3, & a \equiv 0(\bmod 3) \end{cases}$$

$$f(u_a u_{a+1}) = \begin{cases} 3, & a \equiv 1 (\bmod 3) \\ 2, & a \equiv 2 (\bmod 3) \\ 1, & a \equiv 0 (\bmod 3) \end{cases}$$

$$f(u_a u'_{a+1}) = f(v_{a+1} u_a) = 4.$$

$$f(v_{a+1} u'_a) = 6, \ f(v_{a+1} v'_a) = f(u_a v'_{a+1}) = 5$$

$$f(v_a v'_{a+1}) = f(u_{a+1} u'_a) = 7$$

Hence f is a total 7-coloring of $S(SL_n)$ and therefore $\chi''(S(SL_n)) \le 7$. By conjecture, $\chi''(S(SL_n)) \ge \Delta(S(SL_n)) + 1 = 6 + 1 \ge 7$ and so $\chi''(S(SL_n)) = 7$.

Theorem 3.6: For n $\ge$ 3, $\chi''(S(TL_n)) = \Delta(S(TL_n)) + 1$.

Proof: Let $V(S(TL_n)) = \{v_a, u_a, v'_a, u'_a : 1 \le a \le n\}$ and

$$E(S(TL_n)) = \begin{cases} \{e_a, e'_a, e''_a : 1 \le a \le n-1\} \bigcup \{f_a, f'_a, f''_a : 1 \le a \le n-1\} \\ \{g_a, g'_a, g''_a : 1 \le a \le n-1\} \bigcup \{h_a, h'_a, h''_a : 1 \le a \le n\} \end{cases},$$

where $\{e_a, e'_a, e''_a : 1 \le a \le n-1\}$ is the edges $\{v_a v_{a+1}, v_a u_{a+1}, u_a u_{a+1} : 1 \le a \le n-1\}$, $\{f_a, f'_a, f''_a : 1 \le a \le n-1\}$ is the edges $\{v_a v'_{a+1}, v'_a v_{a+1}, v'_a u_{a+1} : 1 \le a \le n-1\}$, $\{g_a, g'_a, g''_a : 1 \le a \le n-1\}$ is the edges $\{u_a u'_{a+1}, u'_a u_{a+1}, v_a u'_{a+1} : 1 \le a \le n\}$ and $\{h_a, h'_a, h''_a : 1 \le a \le n\}$ is the edges $\{v_a u_a, v_a u'_a, v'_a u_a : 1 \le a \le n\}$.

Define a total coloring f, such that $f : V(S(TL_n)) \bigcup E(S(TL_n)) \to \{1,2,3,4,5,6,7,8,9\}$ as follows:

For $1 \le a \le n$

$$f(v_a) = \begin{cases} 1, & a \equiv 1 (\bmod 3) \\ 3, & a \equiv 2 (\bmod 3) \\ 2, & a \equiv 0 (\bmod 3) \end{cases}$$

$$f(u_a) = \begin{cases} 3, & a \equiv 1 (\bmod 3) \\ 2, & a \equiv 2 (\bmod 3) \\ 1, & a \equiv 0 (\bmod 3) \end{cases}$$

$$f(v'_a) = f(u'_a) = 4$$

$$f(v_a u_a) = 4, \ f(u_a v'_a) = 5$$

$$f(v_a u'_a) = 7$$

For $1 \le a \le n-1$

$$f(v_a v_{a+1}) = \begin{cases} 2, & a \equiv 1 (\bmod 3) \\ 1, & a \equiv 2 (\bmod 3) \\ 3, & a \equiv 0 (\bmod 3) \end{cases}$$

$$f(u_a u_{a+1}) = \begin{cases} 1, & a \equiv 1 (\bmod 3) \\ 3, & a \equiv 2 (\bmod 3) \\ 2, & a \equiv 0 (\bmod 3) \end{cases}$$

$$f(v_a u'_{a+1}) = f(v'_a u_{a+1}) = 9,$$

$$f(u_a u'_{a+1}) = f(v'_a v_{a+1}) = 6,$$

$$f(v_a v'_{a+1}) = f(u'_a u_{a+1}) = 8$$

$$f(v_a u_{a+1}) = 7$$

Hence f is a total 9-coloring of $S((TL_n))$ and therefore $\chi''(S(TL_n)) \le 9.$ by conjecture, $\chi''(S(TL_n)) \ge \Delta(S(TL_n)) + 1 = 8 + 1 \ge 9$ and so $\chi''(S(TL_n)) = 9.$

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