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A TWO-WAREHOUSING INVENTORY MODEL FOR NON-INSTANTANEOUS DETERIORATING ITEMS WITH STOCK DEPENDENT DEMAND UNDER TRADE CREDIT POLICY AND PARTIAL BACKLOGGING

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ABSTRACT:

A deterministic inventory model with two levels of storage (own warehouse and rented warehouse) for non-instantaneous deteriorating items with stock dependent demand is studied. The supplier offers the retailer a trade credit period to settle the amount. Moreover, shortage at Own Warehouse (OW) is also allowed subject to partial backlogging. The backlogging rate varies inversely as the waiting time for the next replenishment. Different Cases based on the trade credit period have been considered. The objective of this work is to minimize the total inventory cost and to find the optimal length of replenishment and the optimal order quantity. Computational algorithms for this model are designed to find the optimal order quantity and the optimal cycle time. The solution methodology provided in this model helps to decide the feasibility of renting a warehouse and to avail a trade credit period. The results have been elucidated with numerical examples. Numerical illustrations and managerial insights obtained demonstrate the application and the performance of the proposed theory.

Keywords: Stock dependent demand, Non-instantaneous deterioration, Permissible delay in payment, Two warehouses, Partial backlogging.

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Introduction

Deterioration has received a considerable attention in the past. It has been generally assumed that items start deteriorating as soon as they arrive in the warehouse. Deterioration is defined as decay, damage, obsolescence, evaporation, spoilage, loss of utility, or loss of marginal value of a commodity which decreases the original quality of the product. Many researchers such as Ghare and Schrader (1963), Philip (1974), Goyal and Giri (2001), Li and Mao (2009), Geetha and Udayakumar (2015) and Mahata (2015) assume that the deterioration of the items in inventory starts from the instant of their arrival.

But in real life there are various products, like fresh fruits, vegetables, milk, meat, medicine, volatile liquids, and blood banks etc., that have a shelf-life and start deteriorating after a time lag. This underlines the fact that for some initial period of time, there is no deterioration in items. This phenomenon is termed as non-instantaneous deterioration and the items are termed as non-instantaneous deteriorating items. Wu et al. (2006) defined the term ‘‘non-instantaneous’’ for such deteriorating items. He gave an optimal replenishment policy for non-instantaneous deteriorating items with stock-dependent demand and partial backlogging. Large quantity of goods displayed in market lure the customers to buy more. If the stock is insufficient, customer may prefer some other brand, as a result it will fetch loss to the supplier. Chung-yuan Dye 2002 developed an Inventory model for deteriorating items with stock dependent demand and partial backlogging under the conditions of permissible delay in payments. Kun-ShanWu, Liang-YuhOuyang, Chih-Te Yang (2006) developed an optimal replenishment policy for non-instantaneous deteriorating items with stock-dependent demand and partial backlogging.

Another important aspect associated with inventory management is to decide where to stock the goods. There are many such situations requiring additional storage facility, for instance when one has to procure a larger stock that can’t be accommodated in one’s Own Warehouse (OW) because of its limited capacity. This additional storage capacity may be a Rented Warehouse (RW). A model considering the effect of two warehouses was considered by Hartley (1976) in which he assumed that the holding cost in RW is greater than that in OW; therefore, items in RW are first transferred to OW to meet the demand until the stock level in RW drops to zero, and then items in OW are released.

In this direction, researchers have developed their inventory model for a single warehouse which has unlimited capacity. This assumption is not applicable in real-life situation. When an attractive price discount for bulk purchase is available, the management decides to purchase a huge quantity of items at a time. These goods cannot be stored in the existing storage (the owned warehouse with limited capacity). Another equally important aspect associated with inventory management is to decide where to stock the goods. There are many such situations requiring additional storage facility, for instance when one has to procure a larger stock that can’t be accommodated in one’s Own Warehouse (OW) because of its limited capacity. This additional storage capacity may be a Rented Warehouse (RW). A model considering the effect of two warehouses was considered by Hartley (1976) in which he assumed that the holding cost in RW is greater than that in OW, due to the non-availability of better preserving facility which results in higher deterioration rate; therefore, items in RW are first transferred to OW to meet the demand until the stock level in RW drops to zero, and then items in OW are released. Hence to reduce the holding cost, it is more economical to consume the goods of rented warehouse at the earliest.

Inventory model with double storage facility OW and RW was first developed by Hartley (1976). Palanivel M, Sundararajan R, Uthayakumar R (2016) Two warehouse inventory model with non-instantaneously deteriorating items, stock dependent demand, shortages and inflation, Sahu and Bishi (2017) extended the inventory deteriorating Items under permissible delay in payments. After his pioneering contribution, several other researchers have attempted to extend his work to various other realistic situations. In this connection, mention may be made of the studies undertaken by Sarma(1983, 1990), Murdeshwar and Sathe (1985), Pakkala and Achary (1992), Dave (1988), Bhunia and Maity (1997) Yang (2004), Singh and Sahu (2012), Lee (2006), Yang (2006), Dey et al.(2008) to name only a few.

In the classical time, the payment of the items was done exactly at the time of delivery or before it. But in the modern era, as the business is getting huge and complex, this practice is not possible. Nowadays, the retailer need not clear his dues at the time of delivery. Now Trade Credit is also known as permissible delay in payment, the practice followed by every business. In this, a grace period is provided by the supplier to his retailers to complete the payment. Trade credit is an essential tool for financing growth for many businesses. The number of days for which a credit is given is determined by the company allowing the credit and is agreed on by both the company allowing the credit and the company receiving it. By payment extension date, the company receiving the credit essentially could sell the goods and use the credited amount to pay back the debt. To encourage sales, such a credit is given. During this credit period, the retailer can accumulate and earn interest on the encouraged sales revenue. In case of an extension period, the supplier charges interest on the unpaid balance. Hence, the permissible delay period indirectly reduces the cost of holding cost. In addition, trade credit offered by the supplier encourages the retailer to buy more products. Hence, the trade credit plays a major role in inventory control for both the supplier as well as the retailer. Goyal (1985) developed an EOQ model under the condition of a permissible delay in payments. Aggarwal and Jaggi (1995) then extended Goyal's model to allow for deteriorating items under permissible delay in payments. Uthayakumar and Geetha (2009,2017) developed a replenishment policy for non-instantaneous deteriorating inventory system with partial backlogging and non-instantaneous deteriorating items with two levels of storage under trade credit policy.

This paper aims to develop a two-warehousing inventory model for non instantaneous deteriorating items with stock dependent demand and the supplier offers the retailer a trade credit period to settle the amount. It is also assumed that the inventory holding cost in RW is higher than that in OW but the deterioration rate in RW is less than that in OW because RW offers better preserving facilities. In addition, shortages are allowed in OW and are partially backlogged. The optimal replenishment schedule has also been proposed. Finally, the numerical examples elucidate the performance of this model.

Assumptions and notations

The following assumptions and notations have been used in the entire paper.

Assumptions

1. Demand rate is known and which is stock dependent. The Consumption rate D(t) at time t is assumed to be $D(t) = \begin{cases} a + b\, I(t), & I(t) > 0, \\ a, & I(t) \le 0 \end{cases}$ where a and b is a positive constant.
2. Demand is satisfied initially from goods stored in RW and continues with those in OW once inventory stored at RW is exhausted. This implies that $t_r < T$.
3. The replenishment rate is infinite and the lead time is zero. The time horizon is infinite. Shortages are allowed and

they are partially backlogged, the backlogged rate is defined to be $\frac{1}{1+\delta(T-t)}$ when inventory is negative. The backlogging parameter δ is a positive constant.

4. The owned warehouse OW has limited capacity of W units and the rented warehouse RW has unlimited capacity. For economic reasons, the items of RW are consumed first and next the items of OW.
5. The items deteriorate at a fixed rate α in OW and at β in RW, for the rented warehouse offers better facility, so $\alpha > \beta$, and $h_r - h_o > c(\alpha - \beta)$ (following Liang and Zhou (2011)). To guarantee that the optimal solution exists, we assume that $\alpha W < D$; that is, deteriorating quantity for items in OW is less than the demand rate.
6. When $T \geq M$, the account is settled at $T = M$. Beyond the fixed credit period, the retailer begins paying the interest charges on the items in stock at rate I_p. Before the settlement of the replenishment amount, the retailer can use the sales revenue to earn the interest at annual rate I_e, where $I_p \geq I_e$. When $T \leq M$, the account is settled at $T = M$ and the retailer does not pay any interest charge. Alternatively, the retailer can accumulate revenue and earn interest until the end of the trade credit period.

Notations

In addition, the following notations are used throughout this paper:

OW	-	The owned warehouse
RW	-	The rented warehouse
D	-	The demand per unit time
K	-	The replenishment cost per order ($/order)
P	-	The purchasing cost per unit item ($/unit)
p1	-	The selling price per unit item ($/unit)
S	-	The shortage cost per unit item ($/unit)
h_o	-	The holding cost per unit per unit time in RW
h_r	-	The holding cost per unit per unit time in OW
π	-	The opportunity cost per unit item ($/unit)
α	-	The deterioration rate in OW
β	-	The deterioration rate in RW
M	-	Permissible delay in settling the accounts
I_p	-	The interest charged per dollar in stocks per year
I_e	-	The interest earned per dollar per year
t_w	-	The length of time in which the product has no deterioration in Rented Warehouse
t_o	-	The length of time in which the product has no deterioration in Owned Warehouse
$I_o(t)$	-	The inventory level in OW at time t
$I_r(t)$	-	The inventory level in RW at time t
I_m	-	Maximum Inventory level.

I_b - Maximum amount of shortage demand to be backlogged
W - The storage capacity of OW
Q - The retailer's order quantity (a decision variable)
TC_i - The total relevant costs
t_r - The time at which the inventory level reaches zero in RW
t_1 - The time at which the inventory level reaches zero in OW
T - The length of replenishment cycle (a decision variable)

Model formulation (two-warehouse system)

There are certain circumstances, where the owned warehouse of the retailer is insufficient to store the goods. In that situation, the retailer may go for rented warehouse. To suit to this case, we develop the replenishment problem of a two-warehousing inventory model for a single non-instantaneous deteriorating item with trade credit period and partial backlogging has been considered. Initially a lot size of Q units enters the system. After meeting the backorders, I_m units enter the inventory system, out of which W units are kept in OW and the remaining $(I_m - W)$ units are kept in the RW. The items of OW are consumed only after consuming the goods kept in RW. As the deterioration of item is non-instantaneous, so initially, the units do not deteriorate for some period and after that, the deterioration begins. For the analysis of the inventory system, it is necessary to compare the value of the parameter t_o, t_w, t_r and M with the possible values that the decision variables T can take on.

During the time interval $(0, t_w)$, the inventory level at RW is decreasing only owing to demand rate. The inventory level is dropping to zero due to demand and deterioration during the time interval (t_w, t_r). The behaviour of the inventory system is depicted in Figure 1.

Hence, the change in the inventory level in RW at any time t in the interval $(0, t_r)$ is given by the following differential equations:

$$\frac{dI_{r1}(t)}{dt} = -\left(a + bI_{r1}(t)\right), \quad 0 \le t \le t_w$$

$$\frac{dI_{r2}(t)}{dt} = -\left(a + bI_{r2}(t)\right) - \beta I_{r2}(t), \quad t_w \le t \le t_r$$

With the initial and boundary condition $I_{r1}(0) = I_m - W, I_{r2}(t_r) = 0$

The solutions of the above differential equation are

$$I_{r1}(t) = \frac{-a}{b} + \frac{e^{-bt}}{b}(a + I_m b - Wb)$$

$$I_{r2}(t) = \frac{-a}{b+\beta}\left\{1 - e^{-(b+\beta)(t-t_r)}\right\}$$

Furthermore, the continuity of $I_r(t)$ at $t = t_w$ we get

$$I_m = W - \frac{a}{b} + \frac{ae^{bt_w}}{b(b+\beta)}\left(be^{-(b+\beta)(t_w-t_r)} + \beta\right)$$

During the interval $(0, t_o)$ there is no change in the inventory level in OW, as demand is met from RW.

Hence, at any epoch t, the inventory level at OW is

$I_{o1}(t) = W, 0 < t \leq t_o$

After the time t_o, the inventory level in OW decreases due to deterioration in the interval $(t_{o,} t_r)$ and decreases both by demand and by deterioration in the interval $(t_{r,} t_1)$. Hence the differential equation governing the inventory position is given by

$$\frac{dI_{o2}(t)}{dt} = -\alpha I_{o2}(t), \qquad t_o \leq t \leq t_r$$

$$\frac{dI_{o3}(t)}{dt} = -\big(a + bI_{o3}(t)\big) - \alpha I_{o3}(t), \qquad t_r \leq t \leq t_1$$

$$\frac{dI_{o4}(t)}{dt} = \frac{-a}{1 + \delta(T - t)}, \qquad t_1 \leq t \leq T$$

With the initial and boundary condition $I_{o3}(t_1) = 0,\ I_{o4}(t_1) = 0$ and from the continuity of $I_{o2}(t_r) = I_{o3}(t_r)$

The solutions of the above differential equation are

$$I_{o2}(t) = \frac{ae^{-\alpha(t - t_r)}}{(b + \alpha)}\left(e^{-(b+\alpha)(t_r - t_1)} - 1\right)$$

$$I_{o3}(t) = \frac{a}{b + \alpha}\left\{e^{-(b+\alpha)(t - t_1)} - 1\right\}$$

$$I_{o4}(t) = \frac{a}{\delta}\left(\log\big(1 + \delta(T - t)\big) - \log(1 + \delta(T - t_1))\right)$$

Furthermore, the continuity of $I_{o1}(t_o) = I_{o2}(t_o)$ we get

$$W = \frac{ae^{-\alpha(t_o - t_r)}}{(b + \alpha)}\left(e^{-(b+\alpha)(t_r - t_1)} - 1\right)$$

The maximum backlogging quantity is given by

$I_{o4}(T) = I_b$

$$I_b = \frac{a}{\delta}(-\log(1 + \delta(T - t_1)))$$

Hence the maximum order quantity is $Q = I_m + I_b$

The total inventory cost per cycle consists of the following elements

a) Cost of placing order is K

b) Inventory holding cost HC per cycle is given by

$$HC = h_r\left\{\int_0^{t_w} I_{r1}(t)dt + \int_{t_w}^{t_r} I_{r2}(t)dt\right\} + h_o\left\{\int_0^{t_o} I_{o1}(t)dt + \int_{t_o}^{t_r} I_{o2}(t)dt + \int_{t_r}^{t_1} I_{o3}(t)dt\right\}$$

$$HC = \frac{h_r}{b^2}\{(a + I_m b - Wb)(1 - e^{-bt_w}) - abt_w\} + \frac{h_r a}{(b+\beta)^2}\left\{(b+\beta)(t_w - t_r) + e^{-(b+\beta)(t_w - t_r)} - 1\right\} + h_o W t_o + \frac{h_o a}{\alpha(b+\alpha)}\left\{\left(e^{(b+\alpha)(t_1 - t_r)} - 1\right)\left(e^{-\alpha(t_o - t_r)} - 1\right)\right\} + \frac{h_o a}{(b+\alpha)^2}\left\{(b+\alpha)(t_r - t_1) + e^{(b+\alpha)(t_1 - t_r)} - 1\right\}$$

c) Deterioration cost per cycle is given by

$$DC = p\beta\left\{\int_{t_w}^{t_r} I_{r2}(t)dt\right\} + p\alpha\left\{\int_{t_o}^{t_r} I_{o2}(t)dt + \int_{t_r}^{t_1} I_{o3}(t)dt\right\}$$

$$DC = \frac{p\beta a}{(b+\beta)^2}\left\{(b+\beta)(t_w - t_r) + e^{-(b+\beta)(t_w - t_r)} - 1\right\} + \frac{pa}{(b+\alpha)}\left\{\left(e^{(b+\alpha)(t_1 - t_r)} - 1\right)\left(e^{-\alpha(t_o - t_r)} - 1\right)\right\} + \frac{p\alpha a}{(b+\alpha)^2}\left\{(b+\alpha)(t_r - t_1)e^{(b+\alpha)(t_1 - t_r)} - 1\right\}$$

d) Shortage cost per cycle SC is given by

$$SC = s\int_{t_1}^{T} -I_{o4}(t)dt$$

$$SC = -\frac{s.a}{\delta^2}\left\{\log\left(1 + \delta(T - t_1)\right) - \delta(T - t_1)\right\}$$

e) Opportunity cost per cycle due to lost sales OC is given by

$$OC = \pi\int_{t_1}^{T}\left(a - \frac{a}{1 + \delta(T - t)}\right)dt$$

$$OC = \pi a\left\{(T - t_1) - \frac{\log\left(1 + \delta(T - t_1)\right)}{\delta}\right\}$$

Based on the assumptions and description of the model, the total annual cost which is a function of $t_r, t_1,$ and T is given by

$$TC(t_1, T) = \begin{cases} TC_1(t_1, T), & 0 < M \le t_o \\ TC_2(t_1, T), & t_o < M \le t_w \\ TC_3(t_1, T), & t_w < M \le t_r \\ TC_4(t_1, T), & t_r < M \le t_1 \\ TC_5(t_1, T), & M > t_1 \end{cases}$$

Figure 1 depicts the following 5 cases.

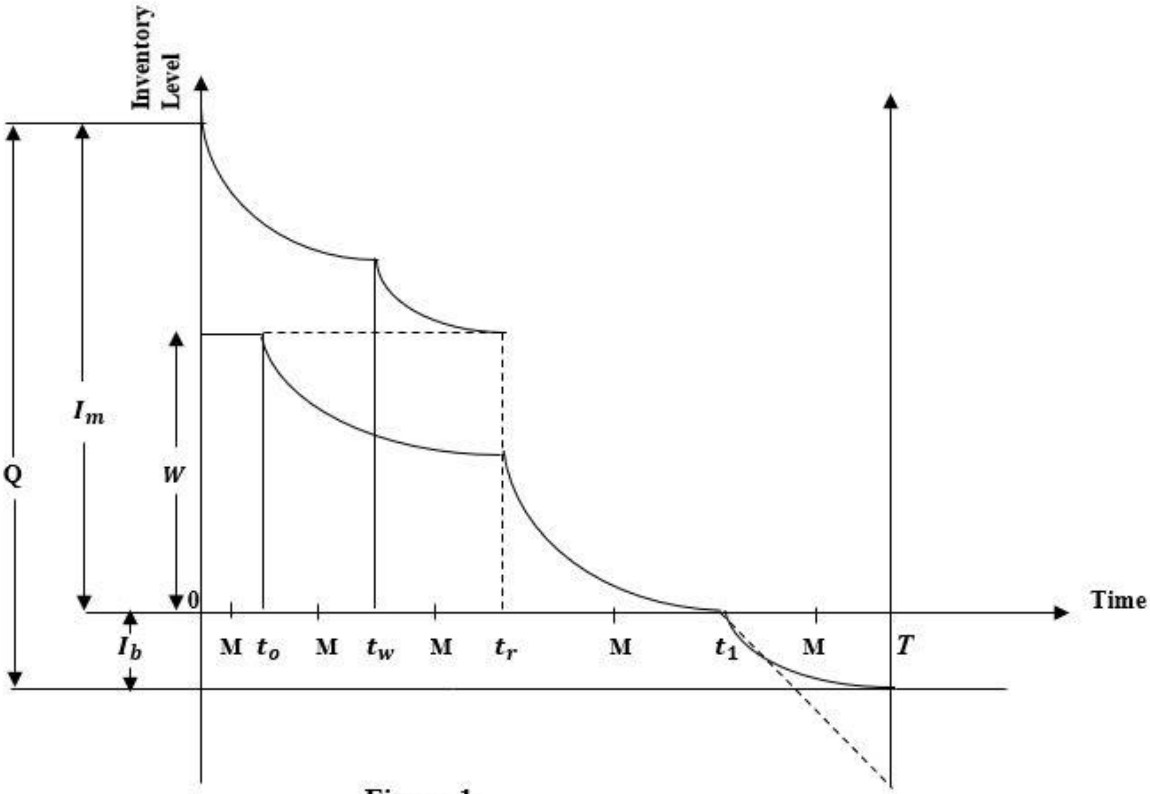

Figure 1:

Case 1: $0 < M \le t_o$

When the end point of the credit period is shorter than or equal to the length of period with positive inventory stock of items ($M \le t_1$), payment for goods is settled and retailer starts paying the interest for the goods still in stocks with annual rate I_p Thus the interest payable denoted by IP_1 and it is given by

$$IP_1 = pI_p\left\{\int_M^{t_w} I_{r1}(t)dt + \int_{t_w}^{t_r} I_{r2}(t)dt + \int_M^{t_o} I_{o1}(t)dt + \int_{t_o}^{t_r} I_{o2}(t)dt + \int_{t_r}^{t_1} I_{o3}(t)dt\right\}$$

$$IP_1 = pI_p\left\{\frac{(a+I_m b-Wb)\left(e^{-bM}-e^{-bt_w}\right)+ab(M-t_w)}{b^2} + \frac{a}{(b+\beta)^2}\left((b+\beta)(t_w-t_r) + e^{-(b+\beta)(t_w-t_r)} - 1\right) + W(t_o-M) + \frac{a}{\alpha(b+\alpha)}\left\{\left(e^{(b+\alpha)(t_1-t_r)} - 1\right)\left(e^{-\alpha(t_o-t_r)} - 1\right)\right\} + \frac{a}{(b+\alpha)^2}\left\{(b+\alpha)(t_r-t_1) + e^{(b+\alpha)(t_1-t_r)} - 1\right\}\right\}$$

We assume that during the time when the account is not settled, the retailer sells the goods and continues to accumulate sales revenue and earns the interest with a rate I_e thus the interest earned per cycle is given by IE_1

$$IE_1 = p_1 I_e \left\{ \int_0^M \left(a + bI_{r1}(t)\right) t dt + \int_0^M \left(a + bI_{o1}(t)\right) t dt \right\}$$

$$IE_1 = p_1 I_e \left\{ \frac{(a + I_m b - Wb)\left(1 - e^{-bM} - bMe^{-bM}\right)}{b^2} + \frac{(a + Wb)M^2}{2} \right\}$$

Thus, the total annual cost which is a function of t_1 and T is given by

$$TC_1(t_1, T) = \frac{K + HC + DC + SC + OC + IP_1 - IE_1}{T}$$

$$TC_1(t_1, T) = \frac{1}{T} \Bigg\{ K + \frac{h_r}{b^2}\{(a + I_m b - Wb)(1 - e^{-bt_w}) - abt_w\} + \frac{h_r a}{(b+\beta)^2}\{(b + \beta)(t_w - t_r) + e^{-(b+\beta)(t_w - t_r)} - 1\} + h_o W t_o + \frac{h_o a}{\alpha(b+\alpha)}\left\{\left(e^{(b+\alpha)(t_1 - t_r)} - 1\right)\left(e^{-\alpha(t_o - t_r)} - 1\right)\right\} + \frac{h_o a}{(b+\alpha)^2}\{(b + \alpha)(t_r - t_1) + e^{(b+\alpha)(t_1 - t_r)} - 1\} + \frac{p\beta a}{(b+\beta)^2}\{(b + \beta)(t_w - t_r) + e^{-(b+\beta)(t_w - t_r)} - 1\} + \frac{pa}{(b+\alpha)}\left\{\left(e^{(b+\alpha)(t_1 - t_r)} - 1\right)\left(e^{-\alpha(t_o - t_r)} - 1\right)\right\} + \frac{p\alpha a}{(b+\alpha)^2}\{(b + \alpha)(t_r - t_1) + e^{(b+\alpha)(t_1 - t_r)} - 1\} - \frac{s.a}{\delta^2}\{\log\left(1 + \delta(T - t_1)\right) - \delta(T - t_1)\} + \pi a\left\{(T - t_1) - \frac{\log\left(1 + \delta(T - t_1)\right)}{\delta}\right\} + pI_p\Bigg\{\frac{(a + I_m b - Wb)\left(e^{-bM} - e^{-bt_w}\right) + ab(M - t_w)}{b^2} + \frac{a}{(b+\beta)^2}\left((b + \beta)(t_w - t_r) + e^{-(b+\beta)(t_w - t_r)} - 1\right) + W(t_o - M) + \frac{a}{\alpha(b+\alpha)}\left\{\left(e^{(b+\alpha)(t_1 - t_r)} - 1\right)\left(e^{-\alpha(t_o - t_r)} - 1\right)\right\} + \frac{a}{(b+\alpha)^2}\{(b + \alpha)(t_r - t_1) + e^{(b+\alpha)(t_1 - t_r)} - 1\}\Bigg\} - p_1 I_e\left\{\frac{(a + I_m b - Wb)\left(1 - e^{-bM} - bMe^{-bM}\right)}{b^2} + \frac{(a + Wb)M^2}{2}\right\}\Bigg\} \quad \text{..........................(1)}$$

The necessary conditions for the total annual cost $\partial TC_1(t_1, T)$ is convex with respect to t_1 and T are $\frac{\partial TC_1(t_1,T)}{\partial t_1} = 0$ and $\frac{\partial TC_1(t_1,T)}{\partial T} = 0$ (2)

Provided they satisfy the sufficient conditions $\left.\frac{\partial^2 TC_1(t_1,T))}{\partial t_1{}^2}\right|_{(t_1{}^*,T^*)} > 0$, $\left.\frac{\partial^2 TC_1(t_1,T))}{\partial T^2}\right|_{(t_1{}^*,T^*)} > 0$ and $\left.\left\{\left(\frac{\partial^2 TC_1(t_1,T)}{\partial t_1{}^2}\right)\left(\frac{\partial^2 TC_1(t_1,T))}{\partial T^2}\right) - \left(\frac{\partial^2 TC_1(t_1,T)}{\partial t_1 \partial T}\right)^2\right\}\right|_{(t_1{}^*,T^*)} > 0$ (3)

To acquire the optimal values of t_1 and T that minimize $TC_1(t_1, T)$, we develop the following algorithm to find the optimal values of t_1 and T (say, $t_1{}^*$ and T^*).

Algorithm – 1 [case(1)]

Step 1: Start

Step 2: Evaluate $\frac{\partial TC_1(t_1,T)}{\partial t_1}$ and $\frac{\partial TC_1(t_1,T)}{\partial T}$

Step 3: Solve the simultaneous equation $\frac{\partial TC_1(t_1,T)}{\partial t_1} = 0$ and $\frac{\partial TC_1(t_1,T)}{\partial T} = 0$ by fixing M, t_r and initializing the values of $K, a, b, \alpha, \beta, \delta, s, \pi, C, p, p_1, h_r, h_o, t_o, t_w, I_p, I_e$

Step 4: Choose one set of solution from step 3.

Step 5: If the values in equation (3) are greater than zero, then this set of solution is optimal and go to step 6. Otherwise go to step 4 with next set of solution obtained from step 3.

Step 6: Evaluate $TC_1({t_1}^*, T^*)$

Step 7: End

Case 2: $t_o < M \le t_w$

Thus, interest payable denoted by IP_2 is given by

$$IP_2 = pI_p\left\{\int_M^{t_w} I_{r1}(t)dt + \int_{t_w}^{t_r} I_{r2}(t)dt + \int_M^{t_r} I_{o2}(t)dt + \int_{t_r}^{t_1} I_{o3}(t)dt\right\}$$

$$IP_2 = pI_p\left\{\frac{(a+I_mb-Wb)\left(e^{-bM}-e^{-bt_w}\right)+ab(M-t_w)}{b^2} + \frac{a}{(b+\beta)^2}\left((b+\beta)(t_w-t_r) + e^{-(b+\beta)(t_w-t_r)} - 1\right) + \frac{a}{\alpha(b+\alpha)}\left(e^{(b+\alpha)(t_1-t_r)}-1\right)\left(e^{\alpha(t_r-M)}-1\right) + \frac{a}{(b+\alpha)^2}\left(e^{(b+\alpha)(t_1-t_r)} - 1 - (b+\alpha)(t_1-t_r)\right)\right\}$$

Thus, interest earned from the accumulated sales during this period is

$$IE_2 = p_1I_e\left\{\int_0^M\left(a+bI_{r1}(t)\right)t\,dt + \int_0^{t_o}\left(a+bI_{o1}(t)\right)t\,dt + \int_{t_o}^M\left(a+bI_{o2}(t)\right)t\,dt\right\}$$

$$IE_2 = \frac{p_1I_e}{b^2}\{(a+I_mb-Wb)(1-e^{-bM}-bMe^{-bt_w})\} + \frac{p_1I_e}{2}(a+Wb){t_o}^2 + \frac{p_1I_ea}{2\alpha^2(b+\alpha)}\left\{\alpha^2\left(M^2-{t_o}^2\right)(b+\alpha) + 2b(\alpha t_o+1)(e^{b(t_1-t_r)+\alpha(t_r-t_o)} - e^{\alpha(t_r-t_o)}) - 2b(\alpha M+1)(e^{\alpha(t_1-M)+b(t_1-t_r)} - e^{\alpha(t_r-M)})\right\}$$

Thus, the total annual cost which is a function of t_1 and T is given by

$$TC_2(t_1,T) = \frac{K+HC+DC+SC+OC+IP_2-IE_2}{T}$$

$$TC_2(t_1,T) = \frac{1}{T}\Bigg\{K + \frac{h_r}{b^2}\{(a+I_mb-Wb)(1-e^{-bt_w}) - abt_w\} + \frac{h_ra}{(b+\beta)^2}\{(b+\beta)(t_w-t_r) + e^{-(b+\beta)(t_w-t_r)} - 1\} + h_oWt_o + \frac{h_oa}{\alpha(b+\alpha)}\{\left(e^{(b+\alpha)(t_1-t_r)}-1\right)\left(e^{-\alpha(t_o-t_r)}-1\right)\} + \frac{h_oa}{(b+\alpha)^2}\{(b+\alpha)(t_r-t_1) + e^{(b+\alpha)(t_1-t_r)} - 1\} + \frac{p\beta a}{(b+\beta)^2}\{(b+\beta)(t_w-t_r) + e^{-(b+\beta)(t_w-t_r)} - 1\} + \frac{pa}{(b+\alpha)}\{\left(e^{(b+\alpha)(t_1-t_r)}-1\right)\left(e^{-\alpha(t_o-t_r)}-1\right)\} + \frac{p\alpha a}{(b+\alpha)^2}\{(b+\alpha)(t_r-t_1) + e^{(b+\alpha)(t_1-t_r)} - 1\} - \frac{s.a}{\delta^2}\{\log\left(1+\delta(T-t_1)\right) - \delta(T-t_1)\} + \pi a\left\{(T-t_1) - \frac{\log\left(1+\delta(T-t_1)\right)}{\delta}\right\} + pI_p\left\{\frac{(a+I_mb-Wb)\left(e^{-bM}-e^{-bt_w}\right)+ab(M-t_w)}{b^2} + \frac{a}{(b+\beta)^2}\left((b+\beta)(t_w-t_r) + e^{-(b+\beta)(t_w-t_r)} - 1\right) + \frac{a}{\alpha(b+\alpha)}\left(e^{(b+\alpha)(t_1-t_r)}-1\right)\left(e^{\alpha(t_r-M)}-1\right) + \frac{a}{(b+\alpha)^2}\left(e^{(b+\alpha)(t_1-t_r)} - 1 - (b+\alpha)(t_1-t_r)\right)\right\} - \left\{\frac{p_1I_e}{b^2}\{(a+I_mb-Wb)(1-e^{-bM}-bMe^{-bt_w})\} + \frac{p_1I_e}{2}(a+Wb){t_o}^2 + \frac{p_1I_ea}{2\alpha^2(b+\alpha)}\left\{\alpha^2\left(M^2-{t_o}^2\right)(b+\alpha) + 2b(\alpha t_o+1)(e^{b(t_1-t_r)+\alpha(t_r-t_o)} - e^{\alpha(t_r-t_o)}) - 2b(\alpha M+1)(e^{\alpha(t_1-M)+b(t_1-t_r)} - e^{\alpha(t_r-M)})\right\}\right\}\Bigg\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4)$$

The necessary conditions for the total annual cost $\partial TC_2(t_1,T)$ is convex with respect to t_1 and T are $\frac{\partial TC_2(t_1,T)}{\partial t_1} = 0$ and $\frac{\partial TC_2(t_1,T)}{\partial T} = 0$ (5)

Provided they satisfy the sufficient conditions $\frac{\partial^2 TC_2(t_1,T))}{\partial t_1{}^2}\Big|_{(t_1{}^*,T^*)} > 0$, $\frac{\partial^2 TC_2(t_1,T))}{\partial T^2}\Big|_{(t_1{}^*,T^*)} > 0$ and $\left\{\left(\frac{\partial^2 TC_2(t_1,T)}{\partial t_1{}^2}\right)\left(\frac{\partial^2 TC_2(t_1,T))}{\partial T^2}\right) - \left(\frac{\partial^2 TC_2(t_1,T)}{\partial t_1 \partial T}\right)^2\right\}\Big|_{(t_1{}^*,T^*)} > 0$ (6)

To acquire the optimal values of t_1 and T that minimize $TC_2(t_1,T)$, we develop the following algorithm to find the optimal values of t_1 and T (say, $t_1{}^*$ and T^*).

Algorithm – 2 [case(2)]

Step 1: Start

Step 2: Evaluate $\frac{\partial TC_2(t_1,T)}{\partial t_1}$ and $\frac{\partial TC_2(t_1,T)}{\partial T}$

Step 3: Solve the simultaneous equation $\frac{\partial TC_2(t_1,T)}{\partial t_1} = 0$ and $\frac{\partial TC_2(t_1,T)}{\partial T} = 0$ by fixing M, t_r and initializing the values of $K, a, b, \alpha, \beta, \delta, s, \pi, C, p, p_1, h_r, h_o, t_o, t_w, I_p, I_e$

Step 4: Choose one set of solution from step 3.

Step 5: If the values in equation (6) are greater than zero, then this set of solution is optimal and go to step 6. Otherwise go to step 4 with next set of solution obtained from step 3.

Step 6: Evaluate $TC_2(t_1{}^*, T^*)$

Step 7: End

Case 3: $t_w < M \le t_r$

The interest payable denoted by IP_2 is given by

$$IP_3 = pI_p\left\{\int_M^{t_r} I_{r2}(t)dt + \int_M^{t_r} I_{o2}(t)dt + \int_{t_r}^{t_1} I_{o3}(t)dt\right\}$$

$$IP_3 = pI_p\left\{\frac{a}{(b+\beta)^2}\left((M-t_r)(b+\beta) + e^{-(b+\beta)(M-t_r)} - 1\right) + \frac{a}{\alpha(b+\alpha)}\left(e^{(b+\alpha)(t_1-t_r)} - 1\right)\left(e^{\alpha(t_r-M)} - 1\right) + \frac{a}{(b+\alpha)^2}\left(e^{(b+\alpha)(t_1-t_r)} - 1 - (b+\alpha)(t_1-t_r)\right)\right\}$$

Thus, interest earned from the accumulated sales during this period is

$$IE_3 = p_1 I_e\left\{\int_0^{t_w}\left(a + bI_{r1}(t)\right)t\,dt + \int_{t_w}^{M}\left(a + bI_{r2}(t)\right)t\,dt + \int_0^{t_o}\left(a + bI_{o1}(t)\right)t\,dt + \int_{t_0}^{M}\left(a + bI_{o2}(t)\right)t\,dt\right\}$$

$$IE_3 = \frac{p_1 I_e}{b^2}\{(a + I_m b - Wb)(1 - e^{-bt_w} - bt_d e^{-bt_w})\} + \frac{p_1 I_e a}{2(b+\beta)^3}\{2b(1 + bt_w + \beta t_w)e^{(b+\beta)(t_r-t_w)} - 2b(1 + bM + \beta M)e^{(b+\beta)(t_r-M)} + \beta(M^2 - t_w{}^2)(\beta^2 + 2b\beta + b^2)\} + \frac{p_1 I_e}{2}(a + Wb)t_o{}^2 + \frac{p_1 I_e a}{2\alpha^2(b+\alpha)}\left\{\alpha^2\left(M^2 - t_o{}^2\right)(b+\alpha) + 2b(\alpha t_o + 1)(e^{b(t_1-t_r)+\alpha(t_r-t_o)} - e^{\alpha(t_r-t_o)}) - 2b(\alpha M + 1)(e^{\alpha(t_1-M)+b(t_1-t_r)} - e^{\alpha(t_r-M)})\right\}$$

Thus, the total annual cost which is a function of t_1 and T is given by

$$TC_3(t_1,T) = \frac{K + HC + DC + SC + OC + IP_3 - IE_3}{T}$$

$$TC_3(t_1,T) = \frac{1}{T}\left\{K + \frac{h_r}{b^2}\{(a + I_m b - Wb)(1 - e^{-bt_w}) - abt_w\} + \frac{h_r a}{(b+\beta)^2}\{(b+\beta)(t_w - t_r) + e^{-(b+\beta)(t_w-t_r)} - 1\} + h_o W t_o + \frac{h_o a}{\alpha(b+\alpha)}\{\left(e^{(b+\alpha)(t_1-t_r)} - 1\right)\left(e^{-\alpha(t_o-t_r)} - 1\right)\} + \right.$$

$\frac{h_o a}{(b+\alpha)^2}\left\{(b+\alpha)(t_r-t_1)+e^{(b+\alpha)(t_1-t_r)}-1\right\}+\frac{p\beta a}{(b+\beta)^2}\left\{(b+\beta)(t_w-t_r)+e^{-(b+\beta)(t_w-t_r)}-1\right\}+\frac{pa}{(b+\alpha)}\left\{\left(e^{(b+\alpha)(t_1-t_r)}-1\right)\left(e^{-\alpha(t_o-t_r)}-1\right)\right\}+\frac{p\alpha a}{(b+\alpha)^2}\left\{(b+\alpha)(t_r-t_1)+e^{(b+\alpha)(t_1-t_r)}-1\right\}-\frac{s.a}{\delta^2}\left\{\log\left(1+\delta(T-t_1)\right)-\delta(T-t_1)\right\}+\pi a\left\{(T-t_1)-\frac{\log(1+\delta(T-t_1))}{\delta}\right\}+pI_p\left\{\frac{a}{(b+\beta)^2}\left((M-t_r)(b+\beta)+e^{-(b+\beta)(M-t_r)}-1\right)+\frac{a}{\alpha(b+\alpha)}\left(e^{(b+\alpha)(t_1-t_r)}-1\right)\left(e^{\alpha(t_r-M)}-1\right)+\frac{a}{(b+\alpha)^2}\left(e^{(b+\alpha)(t_1-t_r)}-1-(b+\alpha)(t_1-t_r)\right)\right\}-\frac{p_1 I_e}{b^2}\{(a+I_m b-Wb)(1-e^{-bt_w}-bt_d e^{-bt_w})\}+\frac{p_1 I_e a}{2(b+\beta)^3}\left\{2b(1+bt_w+\beta t_w)e^{(b+\beta)(t_r-t_w)}-2b(1+bM+\beta M)e^{(b+\beta)(t_r-M)}+\beta(M^2-{t_w}^2)(\beta^2+2b\beta+b^2)\right\}+\frac{p_1 I_e}{2}(a+Wb){t_o}^2+\frac{p_1 I_e a}{2\alpha^2(b+\alpha)}\left\{\alpha^2\left(M^2-{t_o}^2\right)(b+\alpha)+2b(\alpha t_o+1)(e^{b(t_1-t_r)+\alpha(t_r-t_o)}-e^{\alpha(t_r-t_o)})-2b(\alpha M+1)(e^{\alpha(t_1-M)+b(t_1-t_r)}-e^{\alpha(t_r-M)})\right\}\Bigg\}$ (7)

The necessary conditions for the total annual cost $\partial TC_3(t_1,T)$ is convex with respect to t_1 and T are $\frac{\partial TC_3(t_1,T)}{\partial t_1}=0$ and $\frac{\partial TC_3(t_1,T)}{\partial T}=0$.(8)

Provided they satisfy the sufficient conditions $\left.\frac{\partial^2 TC_3(t_1,T))}{\partial {t_1}^2}\right|_{({t_1}^*,T^*)}>0$, $\left.\frac{\partial^2 TC_3(t_1,T))}{\partial T^2}\right|_{({t_1}^*,T^*)}>0$

and $\left.\left\{\left(\frac{\partial^2 TC_3(t_1,T)}{\partial {t_1}^2}\right)\left(\frac{\partial^2 TC_3(t_1,T))}{\partial T^2}\right)-\left(\frac{\partial^2 TC_3(t_1,T)}{\partial t_1 \partial T}\right)^2\right\}\right|_{({t_1}^*,T^*)}>0$ (9)

To acquire the optimal values of t_1 and T that minimize $TC_3(t_1,T)$, we develop the following algorithm to find the optimal values of t_1 and T (say, ${t_1}^*$ and T^*).

Algorithm – 3 [case(3)]

Step 1: Start

Step 2: Evaluate $\frac{\partial TC_3(t_1,T)}{\partial t_1}$ and $\frac{\partial TC_3(t_1,T)}{\partial T}$

Step 3: Solve the simultaneous equation $\frac{\partial TC_3(t_1,T)}{\partial t_1}=0$ and $\frac{\partial TC_3(t_1,T)}{\partial T}=0$ by fixing M, t_r and initializing the values of $K, a, b, \alpha, \beta, \delta, s, \pi, C, p, p_1, h_r, h_o, t_o, t_w, I_p, I_e$

Step 4: Choose one set of solution from step 3.

Step 5: If the values in equation (9) are greater than zero, then this set of solution is optimal and go to step 6. Otherwise go to step 4 with next set of solution obtained from step 3.

Step 6: Evaluate $TC_3({t_1}^*,T^*)$

Step 7: End

Case 4: $t_r < M \le t_1$

Thus, interest payable denoted by IP_4 is given by

$IP_4 = pI_p\int_M^{t_1} I_{o3}(t)dt$

$$IP_4=\frac{pI_p a}{(b+\alpha)^2}\left(e^{-(b+\alpha)(M-t_1)}-1+(b+\alpha)(M-t_1)\right)$$

Thus, interest earned from the accumulated sales during this period is

$IE_4 = p_1 I_e \left\{\int_0^{t_w}\left(a + bI_{r1}(t)\right)t\,dt + \int_{t_w}^{t_r}\left(a + bI_{r2}(t)\right)t\,dt + \int_0^{t_o}\left(a + bI_{o1}(t)\right)t\,dt + \int_{t_o}^{t_r}\left(a + bI_{o2}(t)\right)t\,dt + \int_{t_r}^{M}\left(a + bI_{o3}(t)\right)t\,dt\right\}$

$IE_4 = \frac{p_1 I_e}{b^2}\{(a + I_m b - Wb)(1 - e^{-bt_w} - bt_d e^{-bt_w})\} + \frac{p_1 I_e a}{2(b+\beta)^3}\left\{2b(bt_w + \beta t_w + 1)e^{(b+\beta)(t_r - t_w)} + \beta(t_r{}^2 - t_w{}^2)(\beta^2 + 2b\beta + 1) - 2b(\beta t_r + bt_r + 1)\right\} + \frac{p_1 I_e}{2}\{(a + Wb)t_o{}^2\} - \frac{p_1 I_e a}{2\alpha^2(b+\alpha)}\left\{2b(\alpha t_o + 1)\left(e^{\alpha(t_r - t_o)} - e^{b(t_1 - t_r) + \alpha(t_1 - t_o)}\right) + 2b(1 + \alpha t_r)e^{(b+\alpha)(t_1 - t_r)} + (\alpha^2(\alpha + b)(t_o{}^2 - t_r{}^2) - 2b(\alpha t_r + 1)\right\} + \frac{p_1 I_e a}{2(b+\alpha)^3}\left\{2b(bt_r + \alpha t_r + 1)e^{(b+\alpha)(t_1 - t_r)} - 2b(1 + bM + \alpha M)e^{(b+\alpha)(t_1 - M)} + \alpha(M^2 - t_r{}^2)(\alpha + b)^2\right\}$

Thus, the total annual cost which is a function of t_1 and T is given by

$$TC_4(t_1, T) = \frac{K + HC + DC + SC + OC + IP_4 - IE_4}{T}$$

$TC_4(t_1, T) = \frac{1}{T}\Bigg\{K + \frac{h_r}{b^2}\{(a + I_m b - Wb)(1 - e^{-bt_w}) - abt_w\} + \frac{h_r a}{(b+\beta)^2}\left\{(b + \beta)(t_w - t_r) + e^{-(b+\beta)(t_w - t_r)} - 1\right\} + h_o W t_o + \frac{h_o a}{\alpha(b+\alpha)}\left\{\left(e^{(b+\alpha)(t_1 - t_r)} - 1\right)\left(e^{-\alpha(t_o - t_r)} - 1\right)\right\} + \frac{h_o a}{(b+\alpha)^2}\left\{(b + \alpha)(t_r - t_1) + e^{(b+\alpha)(t_1 - t_r)} - 1\right\} + \frac{p\beta a}{(b+\beta)^2}\left\{(b + \beta)(t_w - t_r) + e^{-(b+\beta)(t_w - t_r)} - 1\right\} + \frac{pa}{(b+\alpha)}\left\{\left(e^{(b+\alpha)(t_1 - t_r)} - 1\right)\left(e^{-\alpha(t_o - t_r)} - 1\right)\right\} + \frac{p\alpha a}{(b+\alpha)^2}\left\{(b + \alpha)(t_r - t_1) + e^{(b+\alpha)(t_1 - t_r)} - 1\right\} - \frac{s.a}{\delta^2}\left\{\log\left(1 + \delta(T - t_1)\right) - \delta(T - t_1)\right\} + \pi a\left\{(T - t_1) - \frac{\log\left(1 + \delta(T - t_1)\right)}{\delta}\right\} + \frac{pI_p a}{(b+\alpha)^2}\left(e^{-(b+\alpha)(M - t_1)} - 1 + (b + \alpha)(M - t_1)\right) - \Bigg\{\frac{p_1 I_e}{b^2}\{(a + I_m b - Wb)(1 - e^{-bt_w} - bt_d e^{-bt_w})\} + \frac{p_1 I_e a}{2(b+\beta)^3}\left\{2b(bt_w + \beta t_w + 1)e^{(b+\beta)(t_r - t_w)} + \beta(t_r{}^2 - t_w{}^2)(\beta^2 + 2b\beta + 1) - 2b(\beta t_r + bt_r + 1)\right\} + \frac{p_1 I_e}{2}\{(a + Wb)t_o{}^2\} - \frac{p_1 I_e a}{2\alpha^2(b+\alpha)}\left\{2b(\alpha t_o + 1)\left(e^{\alpha(t_r - t_o)} - e^{b(t_1 - t_r) + \alpha(t_1 - t_o)}\right) + 2b(1 + \alpha t_r)e^{(b+\alpha)(t_1 - t_r)} + (\alpha^2(\alpha + b)(t_o{}^2 - t_r{}^2) - 2b(\alpha t_r + 1)\right\} + \frac{p_1 I_e a}{2(b+\alpha)^3}\left\{2b(bt_r + \alpha t_r + 1)e^{(b+\alpha)(t_1 - t_r)} - 2b(1 + bM + \alpha M)e^{(b+\alpha)(t_1 - M)} + \alpha(M^2 - t_r{}^2)(\alpha + b)^2\right\}\Bigg\}\Bigg\}$ …………..(10)

The necessary conditions for the total annual cost $\partial TC_4(t_1, T)$ is convex with respect to t_1 and T are $\frac{\partial TC_4(t_1,T)}{\partial t_1} = 0$ and $\frac{\partial TC_4(t_1,T)}{\partial T} = 0$ (11)

Provided they satisfy the sufficient conditions $\left.\frac{\partial^2 TC_4(t_1,T))}{\partial t_1{}^2}\right|_{(t_1{}^*,T^*)} > 0$, $\left.\frac{\partial^2 TC_4(t_1,T))}{\partial T^2}\right|_{(t_1{}^*,T^*)} > 0$ and $\left.\left\{\left(\frac{\partial^2 TC_4(t_1,T)}{\partial t_1{}^2}\right)\left(\frac{\partial^2 TC_4(t_1,T))}{\partial T^2}\right) - \left(\frac{\partial^2 TC_4(t_1,T)}{\partial t_1 \partial T}\right)^2\right\}\right|_{(t_1{}^*,T^*)} > 0$ (12)

To acquire the optimal values of t_1 and T that minimize $TC_4(t_1, T)$, we develop the following algorithm to find the optimal values of t_1 and T (say, $t_1{}^*$ and T^*).

Algorithm – 4 [case(4)]

Step 1: Start

Step 2: Evaluate $\frac{\partial TC_4(t_1,T)}{\partial t_1}$ and $\frac{\partial TC_4(t_1,T)}{\partial T}$

Step 3: Solve the simultaneous equation $\frac{\partial TC_4(t_1,T)}{\partial t_1}=0$ and $\frac{\partial TC_4(t_1,T)}{\partial T}=0$ by fixing M, t_r and initializing the values of $K, a, b, \alpha, \beta, \delta, s, \pi, C, p, p_1, h_r, h_o, t_o, t_w, I_p, I_e$

Step 4: Choose one set of solution from step 3.

Step 5: If the values in equation (12) are greater than zero, then this set of solution is optimal and go to step 6. Otherwise go to step 4 with next set of solution obtained from step 3.

Step 6: Evaluate $TC_4(t_1^*, T^*)$

Step 7: End

Case 5: $M \geq t_1$

In this case there is no interest payable but the interest earned from the accumulated sales during this period is given by

$$IE_5 = p_1 I_e \Big\{\Big\{\int_0^{t_w}\big(a+bI_{r1}(t)\big)t\,dt + \int_{t_w}^{t_r}\big(a+bI_{r2}(t)\big)t\,dt + \int_0^{t_o}\big(a+bI_{o1}(t)\big)t\,dt + \int_{t_o}^{t_r}\big(a+bI_{o2}(t)\big)t\,dt + \int_{t_r}^{t_1}\big(a+bI_{o3}(t)\big)t dt\Big\} + (M-t_1)\Big\{\int_0^{t_w}\big(a+bI_{r1}(t)\big)\,dt + \int_{t_w}^{t_r}\big(a+bI_{r2}(t)\big)\,dt + \int_0^{t_o}\big(a+bI_{o1}(t)\big)\,dt + \int_{t_o}^{t_r}\big(a+bI_{o2}(t)\big)\,dt + \int_{t_r}^{t_1}\big(a+bI_{o3}(t)\big)dt\Big\}\Big\}$$

$$IE_5 = \frac{p_1I_e}{b^2}\{(a+I_mb-Wb)(1-e^{-bt_w}-bt_we^{-bt_w})\} + \frac{p_1I_ea}{2(b+\beta)^3}\big\{2b(bt_w+\beta t_w+1)e^{(b+\beta)(t_r-t_w)} + \beta(t_r^2-t_w^2)(\beta+b)^2 - 2b(\beta t_r+bt_r+1)\big\} + \frac{p_1I_e}{2}\{(a+Wb)t_o^2\} - \frac{p_1I_ea}{2\alpha^2(b+\alpha)}\big\{(\alpha^2(\alpha+b)(t_d^2-t_r^2) - 2b(\alpha t_r+1) - 2b(\alpha t_o+1)\big(e^{\alpha(t_1-t_o)+b(t_1-t_r)} - e^{\alpha(t_r-t_o)}\big) + 2b(1+\alpha t_r)e^{(b+\alpha)(t_1-t_r)} + \big\} + \frac{p_1I_ea}{2(b+\alpha)^3}\big\{2b(bt_r+\alpha t_r+1)e^{(b+\alpha)(t_1-t_r)} + \alpha(t_1^2-t_r^2)(b+\alpha)^2 - 2b(\alpha(t_1+b)+1)\big\} + (M-t_1)\Big\{\frac{p_1I_e}{b}\{(a+I_mb-Wb)(1-e^{-bt_w})\} + \frac{p_1I_ea}{(b+\beta)^2}\big\{be^{(b+\beta)(t_r-t_w)} + \beta(\beta+b)(t_r-t_w) - b\big\} + p_1I_e\{(a+Wb)t_o\} - \frac{p_1I_ea}{\alpha(b+\alpha)}\big\{b\big(e^{(b+\alpha)(t_1-t_r)} - e^{\alpha(t_1-t_o)+b(t_1-t_r)} + e^{-\alpha(t_o-t_r)}\big) + b\alpha(t_o-t_r) + \alpha^2(t_o-t_r) - b\big\} + \frac{p_1I_ea}{(b+\alpha)^2}\big\{(\alpha^2+b\alpha)(t_1-t_r) - b + be^{(b+\alpha)(t_1-t_r)}\big\}\Big\}$$

Thus, the total annual cost which is a function of t_1 and T is given by

$$TC_5(t_1,T) = \frac{K+HC+DC+SC+OC-IE_5}{T}$$

$$TC_5(t_1,T) = \frac{1}{T}\Big\{K + \frac{h_r}{b^2}\{(a+I_mb-Wb)(1-e^{-bt_w}) - abt_w\} + \frac{h_ra}{(b+\beta)^2}\big\{(b+\beta)(t_w-t_r) + e^{-(b+\beta)(t_w-t_r)} - 1\big\} + h_oWt_o + \frac{h_oa}{\alpha(b+\alpha)}\big\{\big(e^{(b+\alpha)(t_1-t_r)}-1\big)\big(e^{-\alpha(t_o-t_r)}-1\big)\big\} + \frac{h_oa}{(b+\alpha)^2}\big\{(b+\alpha)(t_r-t_1) + e^{(b+\alpha)(t_1-t_r)} - 1\big\} + \frac{p\beta a}{(b+\beta)^2}\big\{(b+\beta)(t_w-t_r) + e^{-(b+\beta)(t_w-t_r)} - 1\big\} + \frac{pa}{(b+\alpha)}\big\{\big(e^{(b+\alpha)(t_1-t_r)}-1\big)\big(e^{-\alpha(t_o-t_r)}-1\big)\big\} + \frac{p\alpha a}{(b+\alpha)^2}\big\{(b+\alpha)(t_r-t_1) +$$

$e^{(b+\alpha)(t_1-t_r)}-1\Big\}-\frac{s.a}{\delta^2}\big\{\log\big(1+\delta(T-t_1)\big)-\delta(T-t_1)\big\}+\pi a\Big\{(T-t_1)-\frac{\log(1+\delta(T-t_1))}{\delta}\Big\}-\Bigg\{\frac{p_1I_e}{b^2}\{(a+I_mb-Wb)(1-e^{-bt_w}-bt_we^{-bt_w})\}+\frac{p_1I_ea}{2(b+\beta)^3}\big\{2b(bt_w+\beta t_w+1)e^{(b+\beta)(t_r-t_w)}+\beta({t_r}^2-{t_w}^2)(\beta+b)^2-2b(\beta t_r+bt_r+1)\big\}+\frac{p_1I_e}{2}\{(a+Wb){t_o}^2\}-\frac{p_1I_ea}{2\alpha^2(b+\alpha)}\big\{(\alpha^2(\alpha+b)({t_d}^2-{t_r}^2)-2b(\alpha t_r+1)-2b(\alpha t_o+1)\big(e^{\alpha(t_1-t_o)+b(t_1-t_r)}-e^{\alpha(t_r-t_o)}\big)+2b(1+\alpha t_r)e^{(b+\alpha)(t_1-t_r)}+\big\}+\frac{p_1I_ea}{2(b+\alpha)^3}\big\{2b(bt_r+\alpha t_r+1)e^{(b+\alpha)(t_1-t_r)}+\alpha({t_1}^2-{t_r}^2)(b+\alpha)^2-2b(\alpha(t_1+b)+1)\big\}+(M-t_1)\Bigg\{\frac{p_1I_e}{b}\{(a+I_mb-Wb)(1-e^{-bt_w})\}+\frac{p_1I_ea}{(b+\beta)^2}\big\{be^{(b+\beta)(t_r-t_w)}+\beta(\beta+b)(t_r-t_w)-b\big\}+p_1I_e\{(a+Wb)t_o\}-\frac{p_1I_ea}{\alpha(b+\alpha)}\big\{b\big(e^{(b+\alpha)(t_1-t_r)}-e^{\alpha(t_1-t_o)+b(t_1-t_r)}+e^{-\alpha(t_o-t_r)}\big)+b\alpha(t_o-t_r)+\alpha^2(t_o-t_r)-b\big\}+\frac{p_1I_ea}{(b+\alpha)^2}\big\{(\alpha^2+b\alpha)(t_1-t_r)-b+be^{(b+\alpha)(t_1-t_r)}\big\}\Bigg\}\Bigg\}\Bigg\}$ ….(13)

The necessary conditions for the total annual cost $\partial TC_5(t_1,T)$ is convex with respect to t_1 and T are $\frac{\partial TC_5(t_1,T)}{\partial t_1}=0$ and $\frac{\partial TC_5(t_1,T)}{\partial T}=0$ ………………..…….(14)

Provided they satisfy the sufficient conditions $\frac{\partial^2 TC_5(t_1,T))}{\partial {t_1}^2}\Big|_{({t_1}^*,T^*)}>0$, $\frac{\partial^2 TC_5(t_1,T))}{\partial T^2}\Big|_{({t_1}^*,T^*)}>0$ and $\left\{\left(\frac{\partial^2 TC_5(t_1,T)}{\partial {t_1}^2}\right)\left(\frac{\partial^2 TC_5(t_1,T))}{\partial T^2}\right)-\left(\frac{\partial^2 TC_5(t_1,T)}{\partial t_1\partial T}\right)^2\right\}\Big|_{({t_1}^*,T^*)}>0$ ………………….(15)

To acquire the optimal values of t_1 and T that minimize $TC_5(t_1,T)$, we develop the following algorithm to find the optimal values of t_1 and T (say, ${t_1}^*$ and T^*).

Algorithm – 5 [case(5)]

Step 1: Start

Step 2: Evaluate $\frac{\partial TC_5(t_1,T)}{\partial t_1}$ and $\frac{\partial TC_5(t_1,T)}{\partial T}$

Step 3: Solve the simultaneous equation $\frac{\partial TC_5(t_1,T)}{\partial t_1}=0$ and $\frac{\partial TC_5(t_1,T)}{\partial T}=0$ by fixing M, t_r and initializing the values of $K, a, b, \alpha, \beta, \delta, s, \pi, C, p, p_1, h_r, h_o, t_o, t_w, I_p, I_e$

Step 4: Choose one set of solution from step 3.

Step 5: If the values in equation (15) are greater than zero, then this set of solution is optimal and go to step 6. Otherwise go to step 4 with next set of solution obtained from step 3.

Step 6: Evaluate $TC_5({t_1}^*,T^*)$

Step 7: End

Our aim is to find the optimal values of t_1 and T which minimize $TC({t_1}^*,T^*)$

$TC({t_1}^*,T^*)=Min\{TC_1({t_1}^*,T^*),TC_2({t_1}^*,T^*),TC_3({t_1}^*,T^*),TC_4({t_1}^*,T^*),TC_5({t_1}^*,T^*)\}$

Numerical Examples:

The following examples illustrate our solution procedure:

Example 1: Consider an inventory system with the following data: $K = 1000, a = 1000,$ $b = 0.5, t_o = 0.041, t_w = 0.082, t_r = 0.247, \alpha = 0.05, \beta = 0.02, h_r = 3, h_o = 1.5, p = 15,$

$s = 12, \pi = 10, p1 = 25, M = 0.027, \delta = 0.3, I_r = 0.12, I_e = 0.13, W = 200$ in appropriate units. In this case, we see that $M < t_o$. Therefore, applying algorithm 1 of Case 1, we get the optimal solutions, $t_1 = 0.6103, T = 0.7819$ the corresponding total cost $TC = 2553.36$, Maximum inventory level, $I_M = 463.20$ and the ordering quantity $Q = 630.55$ units.

Example 2: The data are the same as in Example 1 except $M = 0.055$ in appropriate units. In this case, we see that $t_o < M < t_w$. Therefore, applying algorithm 2 of Case 2, we get the optimal solutions, $t_1 = 0.6079, T = 0.7763$ the corresponding total cost $TC = 2509.02$, Maximum inventory level, $I_M = 463.20$ and the ordering quantity $Q = 627.48$ units.

Example 3: The data are the same as in Example 1 except $M = 0.205$ in appropriate units. In this case, we see that $t_w < M < t_r$. Therefore, applying algorithm 3 of Case 3, we get the optimal solutions, $t_1 = 0.6111, T = 0.7685$ the corresponding total cost $TC = 2362.63$, Maximum inventory level, $I_M = 463.20$ and the ordering quantity $Q = 617.06$ units.

Example 4: The data are the same as in Example 1 except $M = 0.274$ in appropriate units. In this case, we see that $t_r < M < t_1$. Therefore, applying algorithm 4 of Case 4, we get the optimal solutions, $t_1 = 0.6057, T = 0.7329$ the corresponding total cost $TC = 1945.76$, Maximum inventory level, $I_M = 463.20$ and the ordering quantity $Q = 588.01$ units.

Example 5: The data are the same as in Example 1 except $M = 0.493$ in appropriate units. In this case, we see that $M > t_1$. Therefore, applying algorithm 5 of Case 5, we get the optimal solutions, $t_1 = 0.4148, T = 0.5041$ the corresponding total cost $TC = 1401.97$, Maximum inventory level, $I_M = 463.20$ and the ordering quantity $Q = 551.32$ units.

Effect of change in various parameters of the inventory in the two-warehouse model

Changing parameter	% Change in parameter	Change in parameter	t_1	T	TC	I_M	Q
a	-20%	800	0.4878	0.6013	1490.22	410.56	499.84
	-10%	900	0.4490	0.5500	1454.45	436.88	526.37
	0%	1000	0.4148	0.5041	1401.97	463.20	551.32
	+10%	1100	0.3839	0.4622	1333.17	489.52	574.75
	+20%	1200	0.3554	0.4235	1248.04	515.84	596.71
b	-20%	0.40	0.4174	0.5086	1411.17	459.91	549.86
	-10%	0.45	0.4160	0.5062	1406.58	461.55	550.58
	0%	0.50	0.4148	0.5041	1401.97	463.20	551.32
	+10%	0.55	0.4138	0.5021	1397.32	464.87	552.07
	+20%	0.60	0.4129	0.5003	1392.63	466.55	552.85

α	-20%	0.040	0.4226	0.5106	1386.47	463.20	550.11
	-10%	0.045	0.4187	0.5073	1394.30	463.20	550.72
	0%	0.050	0.4148	0.5041	1401.97	463.20	551.32
	+10%	0.055	0.4110	0.5009	1409.47	463.20	551.90
	+20%	0.060	0.4073	0.4978	1416.82	463.20	552.47
β	-20%	0.016	0.4145	0.5037	1400.28	463.14	551.15
	-10%	0.018	0.4147	0.5039	1401.12	463.17	551.23
	0%	0.020	0.4148	0.5041	1401.97	463.20	551.32
	+10%	0.022	0.4149	0.5043	1402.82	463.23	551.40
	+20%	0.024	0.4151	0.5045	1403.66	463.26	551.48
h_r	-20%	2.4	0.4087	0.4955	1364.20	463.20	548.91
	-10%	2.7	0.4118	0.4998	1383.16	463.20	550.12
	0%	3.0	0.4148	0.5041	1401.97	463.20	551.32
	+10%	3.3	0.4178	0.5083	1420.62	463.20	552.50
	+20%	3.6	0.4208	0.5125	1439.12	463.20	553.68
h_o	-20%	1.20	0.4296	0.5160	1366.14	463.20	548.54
	-10%	1.35	0.4220	0.5099	1384.37	463.20	549.95
	0%	1.50	0.4148	0.5041	1401.97	463.20	551.32
	+10%	1.65	0.4079	0.4985	1418.96	463.20	552.63
	+20%	1.80	0.4012	0.4932	1435.37	463.20	553.89
δ	-20%	0.24	0.4109	0.5033	1397.02	463.20	554.60
	-10%	0.27	0.4128	0.5036	1399.52	463.20	552.93
	0%	0.30	0.4148	0.5041	1401.97	463.20	551.32
	+10%	0.33	0.4170	0.5048	1404.39	463.20	549.75
	+20%	0.36	0.4194	0.5057	1406.78	463.20	548.23
t_o	-20%	0.0328	0.4121	0.5016	1403.57	463.20	551.49
	-10%	0.0369	0.4134	0.5028	1402.78	463.20	551.40
	0%	0.0410	0.4148	0.5041	1401.97	463.20	551.32
	+10%	0.0451	0.4162	0.5054	1401.14	463.20	551.23
	+20%	0.0492	0.4175	0.5066	1400.29	463.20	551.13
t_w	-20%	0.0656	0.4151	0.5045	1403.74	463.26	551.49
	-10%	0.0738	0.4149	0.5043	1402.83	463.23	551.40
	0%	0.0820	0.4148	0.5041	1401.97	463.20	551.32
	+10%	0.0902	0.4147	0.5039	1401.15	463.17	551.23
	+20%	0.0984	0.4145	0.5037	1400.36	463.14	551.16
t_r	-20%	0.1976	0.4484	0.5405	1461.12	407.84	498.71

	-10%	0.2223	0.4321	0.5230	1434.31	435.34	525.02
	0%	0.2470	0.4148	0.5041	1401.97	463.20	551.32
	+10%	0.2717	0.3963	0.4835	1363.52	491.42	577.55
	+20%	0.2964	0.3764	0.4611	1318.19	520.01	603.69
M	-20%	0.394	0.4226	0.5427	1834.09	463.20	581.17
	-10%	0.444	0.4192	0.5239	1619.06	463.20	566.25
	0%	0.493	0.4148	0.5041	1401.97	463.20	551.32
	+10%	0.543	0.4090	0.4823	1173.22	463.20	535.70
	+20%	0.592	0.4018	0.4590	941.06	463.20	519.99

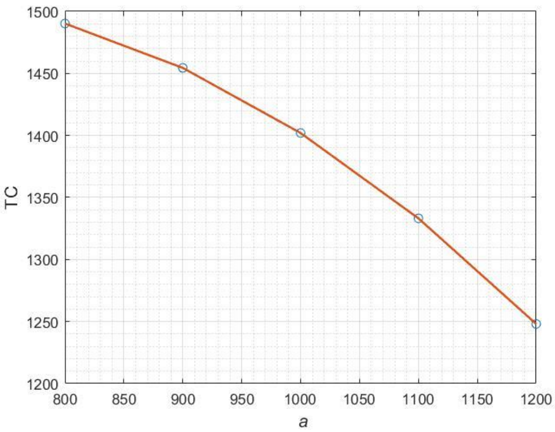

Fig 2. Effect of change of a on Total cost

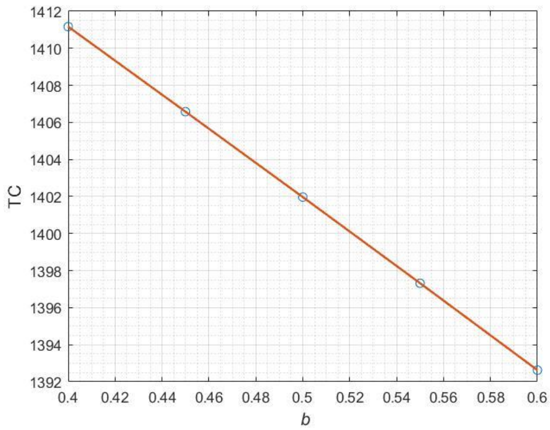

Fig 3. Effect of change of b on Total cost

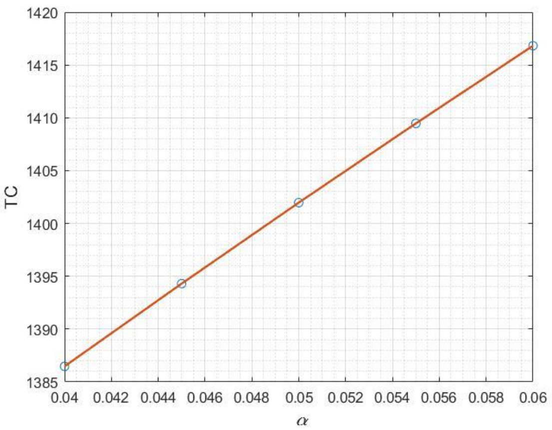

Fig 4. Effect of change of α on Total cost

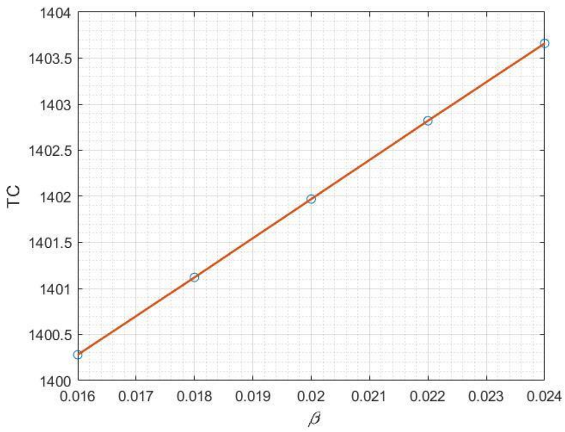

Fig 5. Effect of change of β on Total cost

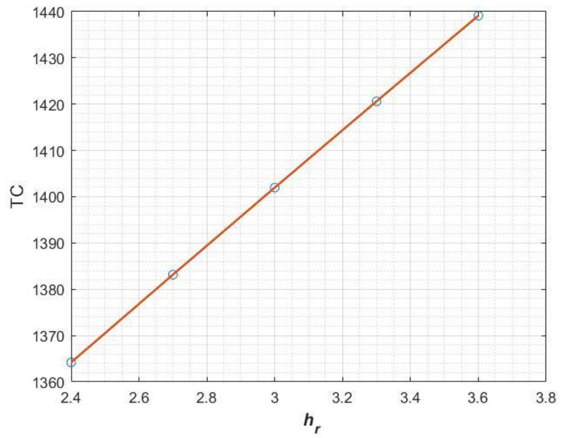

Fig 6. Effect of change of h_r on Total cost

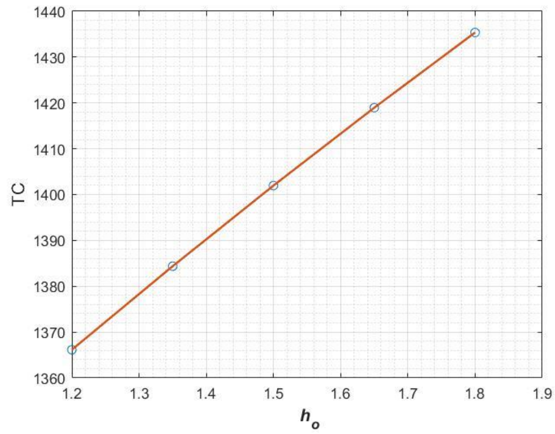

Fig 7. Effect of change of h_o on Total cost

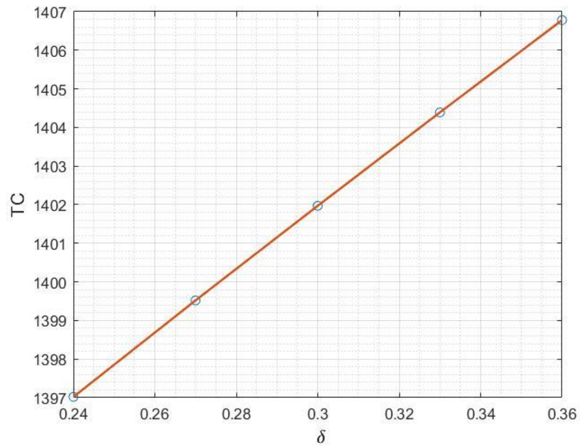

Fig 8. Effect of change of δ on Total cost

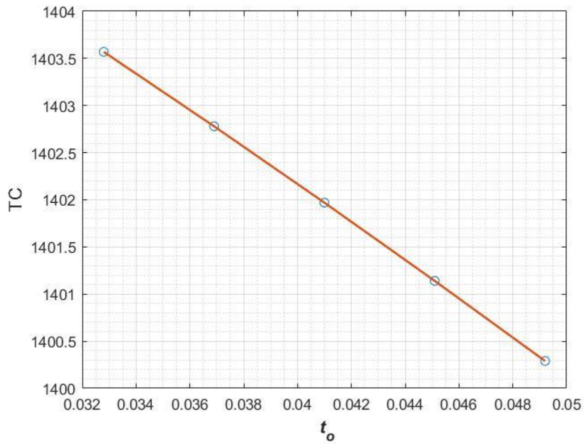

Fig 9. Effect of change of t_o on Total cost

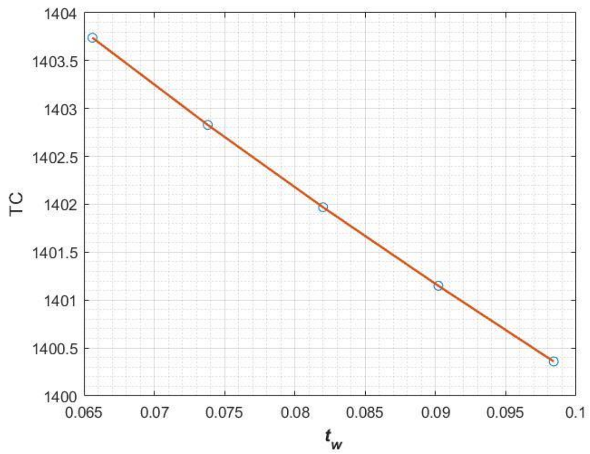

Fig 10. Effect of change of t_w on Total cost

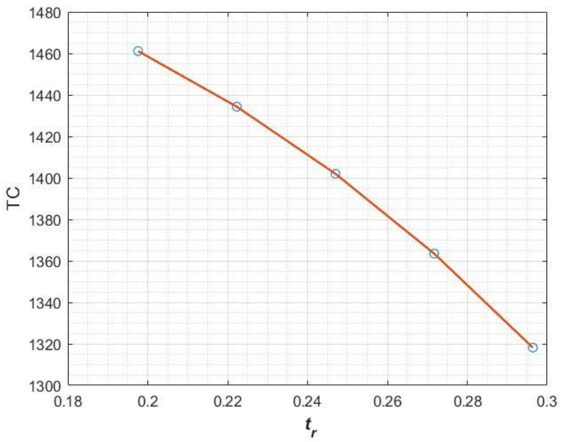

Fig 11. Effect of change of t_r on Total cost

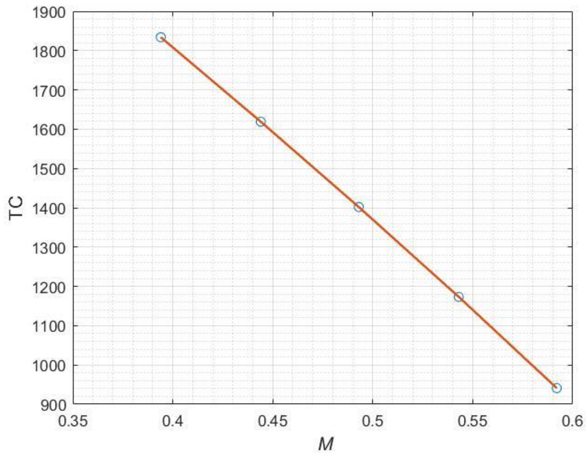

Fig 12. Effect of change of M on Total cost

Sensitivity analysis

A change in the values of parameters may happen due to uncertainties in any decision making situation. In order to examine the implications of these changes, a sensitivity analysis will be of great help in decision-making. We now study the effects of changes in the values of the system parameters $a, b, \alpha, \beta, h_r, h_o, \delta$ and M on the optimal replenishment policy of Example 5. We change one parameter at a time, keeping the other parameters unchanged. The results are summarized in Table 1. Based on our numerical results, we obtain the following managerial implications:

(1) When the parameter a is increasing, the total optimal cost (TC) is highly decreasing. The time at which the inventory level becomes zero in OW and the cycle length (T) are decreasing. But the order quantity (Q) is increasing. That is, increasing of the parameter a will decrease the total cost of the retailer.

(2) When the parameter b is increasing, the total optimal cost (TC), the time at which the inventory level becomes zero in OW and the cycle length (T) are decreasing. But the order quantity (Q) is increasing. That is, increasing of the parameter b will decrease the total cost of the retailer.

(3) When the deterioration rate α is increasing, the total optimal cost (TC) is increasing, the time at which the inventory level becomes zero in OW and the cycle length (T) are decreasing. But the order quantity (Q) is increasing. That is, increasing of the deterioration rate α will increase the total cost of the retailer.

(4) When the deterioration rate β is increasing, the total optimal cost (TC), the time at which the inventory level becomes zero in OW, the cycle length (T) and the order quantity (Q) are increasing. That is, increasing of the deterioration rate β will increase the total cost of the retailer.

(5) When the holding costs h_r in RW is increasing, the total optimal cost (TC), the time at which the inventory level becomes zero in OW, the cycle length (T) and the order quantity (Q) is increasing. That is, increasing of the holding costs h_r in RW will increase the total cost of the retailer.

(6) When the holding costs h_o in OW is increasing, the total optimal cost (TC), and the order quantity (Q) are increasing. But the time at which the inventory level becomes zero in OW and the cycle length (T) are decreasing. That is, increasing of the holding costs h_o in OW will increase the total cost of the retailer.

(7) If the backlogging parameter δ increases, the total optimal cost (TC), the time at which the inventory level becomes zero in OW and the cycle length (T) are increasing. But the order quantity (Q) is decreasing. That is, in order to minimize the

cost, the retailers should decrease the backlogging parameter.

(8) If the length of time in which the product in OW has no deterioration increases, the total optimal cost (TC) and the order quantity (Q) are decreasing. But the time at which the inventory level becomes zero in OW and the cycle length (T) are increasing. That is, in order to minimize the cost, the retailers should increase the length of time in which the product in OW has no deterioration.

(9) If the length of time in which the product in RW has no deterioration increases, the total optimal cost (TC), the time at which the inventory level becomes zero in OW and the cycle length (T) and the order quantity (Q) are decreasing. That is, in order to minimize the cost, the retailers should increase the length of time in which the product in RW has no deterioration.

(10) If the time at which the inventory level becomes zero in RW increases, the total optimal cost (TC), the time at which the inventory level becomes zero in OW and the cycle length (T) and are decreasing. But the order quantity (Q) is increasing. That is, in order to minimize the cost, the retailers should increase time at which the inventory level becomes zero in RW.

(11) If the Credit period M increases, the total optimal cost (TC), the time at which the inventory level becomes zero in OW, the cycle length (T) and order quantity (Q) are decreasing. That is, in order to minimize the cost, the retailers should increase the Credit period M.

Conclusion:

In this paper, a two-warehouse inventory model for non-instantaneously deteriorating items with stock-dependent demand under trade credit period is developed. Shortages are allowed and are partially backlogged. In marketing management, display stock level plays a very important role in different sectors. Thus, it is very clear that the demand rate increases rapidly if the stored amount is high, and vice versa. Since the capacity of any warehouse is limited, the company has to rent a warehouse for storing the excess units over the fixed capacity W of their own warehouse in practice. Our model suits well for the retailer in situations involving unlimited storage space. Thus, the decision maker can easily determine whether it will be financially advantageous to rent a warehouse to hold much more items to avail a trade credit period. For the case of perishable product, the retailer may need to backlog demand to avoid costs due to deterioration. Therefore, shortage is allowed and can be partially backlogged, where the backlogging rate is dependent on the time of waiting for the next replenishment. The aim of this paper is to obtain the optimal solution of cycle length, time intervals and order quantity simultaneously with the objective of minimizing the total cost of the retailer. We presented an analytical closed-form solution for the identified problem, and a computational algorithm has been framed. The necessary and sufficient conditions for the existence and uniqueness of the optimal solutions are also provided. Numerical examples and a sensitivity analysis are given to illustrate the application and the performance of the proposed methodology. From the managerial insights we could see that the rate of change of the parameters $a, b, \alpha, \beta, h_r, h_0, t_o, t_w, t_r$, and M affects the total annual inventory cost and ordering quantity. From the results obtained, we see that the retailer can reduce total annual inventory cost by ordering lower quantity when the supplier provides a permissible delay in payments by improving storage conditions for non-instantaneous deteriorating items. Therefore, this model provides a new managerial insight that helps the industry to reduce the total inventory cost.

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