On Berinde and Upper Class Functions with Cyclic Admissible Contraction Maps in I-metric Space

Sunil Suman[1], Jayanta Biswas[2]

[1]P.G. Department of Mathematics, Magadh University, Bodhgaya, Bihar, India, sumansunil97@gmail.com
[2]Barasat Government College, Barasat, West Bengal, India, jbiswas37@gmail.com

Abstract- This work aims to generalization of (α,β)-Berinde-ϕ-contraction to(α,β)-Berinde-(ψ,ϕ)-rational contraction and to (α,β)-Berinde-(ψ,ϕ)-weak contraction in I-metric space and metric space and establish new fixed point results in I-metric space and its corresponding versions in metric space. Also, generalization of T-cyclic (α,β,H,F)-contraction to $\big((\alpha,\beta,\eta),(\gamma,\delta,\theta),H,F\big)$-$T$-cyclic contraction and T-cyclic (α,β,H,F)-rational contraction to $\big((\alpha,\beta,\eta),(\gamma,\delta,\theta),H,F\big)$-$T$-cyclic rational contraction took placeand establish new common fixed point theorems in I-metric spaceand its corresponding versions in metric space**.**

MSC: 47H10, 54H25

Keywords: $(\boldsymbol{\alpha},\boldsymbol{\beta})$**-Berinde-**$\boldsymbol{\phi}$**-contraction,** $(\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{H},\boldsymbol{F})$**-contraction, coincidence point, fixed point.**

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1. Introduction

Fixed point theory plays an important role to form mathematical models of several real life. Vasile Berinde [5] introduced a type of weak contraction in metric space, which extends fixed point theorems due to Banach [4], Kannan [10], Chatterjea [6] and many others, stated as: "Let (X,d) be a metric space. Amap $T: X \to X$ is called a weak contraction if $\exists\delta \in (0,1)$ and some $L \geq 0$ such that $d(Tx,Ty) \leq \delta d(x,y) + Ld(y,Tx), \forall x,y \in X$." and proved a fixed point theorem.
Mebawondu A.A. *et al.* [12] introduced(α,β)-cyclic admissible map, and(α,β)-Berinde-ϕ-contraction generalizing Berinde contraction and proved a fixed point theorem in metric space, stated as:.

"**Definition(1.1)[12]** For maps $T: X \to X, \alpha,\beta: X^2 \to \mathbb{R}^+$, T is said to be (α,β))-cyclic admissible, if $\forall x,y \in X$, (i) $\alpha(x,y) \geq 1 \Longrightarrow \beta(Tx,Ty) \geq 1$and (ii) $\beta(x,y) \geq 1 \Longrightarrow \alpha(Tx,Ty) \geq 1$ "

"**Definition(1.2)[12]**Let (X,d) be a metric space, $T: X \to X, \alpha,\beta: X^2 \to [0,\infty)$. $T$is said to be an (α,β)-Berinde-ϕ-contraction if $\exists L > 0$ such that $\forall x,y \in X$ with $Tx \neq Ty$, $\alpha(x,Tx)\beta(y,Ty) \geq 1 \Longrightarrow$
$d(Tx,Ty) \leq \phi\big(d(x,y)\big) + L \cdot min\{d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\}$, where
$\phi: [0,\infty) \to [0,\infty)$is continuous, $\phi(t) < t, \forall t > 0$ and $\phi(0) = 0$."

"**Theorem(1.3)[12]** Let (X,d) be a complete metric space and $T: X \to X$ be an (α,β)-Berinde-ϕ-contraction map and (i) T is an (α,β)-cyclic admissible map, (ii) $\exists x_\circ \in X$ such that $\alpha(x_\circ,Tx_\circ) \geq 1$ and $\beta(x_\circ,Tx_\circ) \geq 1$, (iii) T is continuous. Then T has a fixed point."

"**Theorem(1.4)[12]** Let (X,d) be a complete metric space and $T: X \to X$ be an (α,β)-Berinde-ϕ-contraction map and (i) T is an (α,β)-cyclic admissible map, (ii) $\exists x_\circ \in X$ such that $\alpha(x_\circ,Tx_\circ) \geq 1$ and $\beta(x_\circ,Tx_\circ) \geq 1$, (iii) for any sequence $\{x_n\}$ in X converging to x, $\alpha(x,Tx) \geq 1, \beta(x,Tx) \geq 1$. Then T has a fixed point. In addition, if $\alpha(x,Tx) \geq 1, \beta(y,Ty) \geq 1, \forall x,y \in$ Fix(T), then T has a unique fixed point."

Here we generalize $(\boldsymbol{\alpha},\boldsymbol{\beta})$-Berinde-$\boldsymbol{\phi}$-contraction in two ways and establish fixed point results under these generalized contractions in I-metric spaces and metric spaces. Also our results extend the results of Sattar Alizadeh *et al.* [1], stated as:

“**Definition(1.5)[1]**Let $T: X \to X, \alpha, \beta: X \to \mathbb{R}^+$. T is said to be a cyclic (α,β)-admissible if
(i) $\alpha(x) \geq 1$for some $x \in X \Longrightarrow \beta(Tx) \geq 1$, (ii) $\beta(x) \geq 1$ for some $x \in X \Longrightarrow \alpha(Tx) \geq 1$”

“**Definition(1.6)[1]** Let (X,d) be a metric space, $T: X \to X$ be cyclic (α,β)-admissible. T is said to be a $(\alpha,\beta)-(\psi,\phi)$-contractive map if $\alpha(x)\beta(y) \geq 1 \Longrightarrow \psi\big(d(Tx,Ty)\big) \leq \psi\big(d(x,y)\big) - \phi\big(d(x,y)\big)$, $\forall x, y \in X$, where $\psi, \phi: [0,\infty) \to [0,\infty)$ are continuous, non-decreasing and $\psi(t) = \phi(t) = 0$ iff $t = 0$.“

“**Theorem(1.7)[1]** Let (X,d) be a complete metric space and $T: X \to X$ is a $(\alpha,\beta)-(\psi,\phi)$-contractive map such that (a) $\exists x_o \in X$ for which $\alpha(x_o) \geq 1, \beta(x_o) \geq 1$. (b) T is continuous or (c) if $\{x_n\}$ is a sequence in X for which $x_n \to x$ and $\beta(x_n) \geq 1, \forall n \in \mathbb{N}$, then $\beta(x) \geq 1$. Then T has a fixed point. Moreover, if $\alpha(x) \geq 1, \beta(y) \geq 1, \forall x, y \in \mathrm{Fix}(T)$, then T has a unique fixed point. ”

Hussain N. *et al.* [8] have introduced a generalization of completeness and continuity of maps in metric spaces by means of $\alpha-\eta$-completeness and $\alpha-\eta$-continuity stated as:

''**Definition(1.8) [8]** Let (X,d) be metric space, $\alpha, \beta : X^2 \to [0,\infty)$.
(A)$(X,d)$is said to be $\alpha-\beta$-complete if every cauchy sequence $\{x_n\}$ with $\alpha(x_n, x_{n+1}) \geq \beta(x_n, x_{n+1}), \forall n \in \mathbb{N}$, converges in X.
If $\beta(x,y) = 1, \forall x, y \in X$, then (X,d) is called α-complete , and if $\alpha(x,y) = 1, \forall x, y \in X$, then $(X,d)$is called β-complete. "

''**(B)**$T : X \to X$ is said to be an $\alpha-\beta$-continuous map on (X,d) if for $x \in X$ and $\{x_n\}$ converging to x, $\alpha(x_n, x_{n+1}) \geq \beta(x_n, x_{n+1}), \forall n \in \mathbb{N} \Longrightarrow \{Tx_n\}$ converges to Tx.
If $\beta(x,y) = 1, \forall x, y \in X$, then T is called α-continuous on X ; and if $\alpha(x,y) = 1, \forall x, y \in X$, then T is called β-continuous on X."
.
Also Salimi P. *et al.* [13] placed a generalization of α-admissibility stated as:

''**Definition(1.9) [13]** Let $T : X \to X$, $\alpha, \beta : X^2 \to [0,\infty)$. $T$is said to be α-admissible map with respect to β if $\forall x, y \in X, \alpha(x,y) \geq \beta(x,y) \Longrightarrow \alpha(Tx,Ty) \geq \beta(Tx,Ty)$."

AgainIsik H. *et al*. [9] introduced T-cyclic (α,β)-admissible and T-cyclic(α,β)-sub admissible map stated as:
''**Definition(1.10)[9]** Let $S, T : X \to X$, $\alpha, \beta : X \to [0,\infty)$.
(A) $S$is said to be T-cyclic (α,β)-admissible map if
(a)$\alpha(Tx) \geq 1$for some $x \in X \Longrightarrow \beta(Sx) \geq 1$.*(b)*$\beta(Tx) \geq 1$for some $x \in X \Longrightarrow \alpha(Sx) \geq 1$."
''**(B)** S is said to be T-cyclic (α,β)-sub admissible map if
(a)$\alpha(Tx) \leq 1$for some $x \in X \Longrightarrow \beta(Sx) \leq 1$.*(b)*$\beta(Tx) \leq 1$ for some $x \in X \Longrightarrow \alpha(Sx) \leq 1$."

And Cho S.H. *et al*. [7] introduced a family $\mathcal{X}$ defined as:
''**Definition(1.11) [7]** We denote by $\mathcal{X}$ the family of all functions $\xi : [0,\infty)^4 \to [0,\infty)$ satisfying
(a)ξ is nondecreasing in each coordinate and continuous.
(b)$\xi(t,t,t,t) \leq t, \xi(t,0,0,t) \leq t, \xi\left(0,0,t,\frac{t}{2}\right) \leq t, \forall t > 0$.
(c)$\xi(t_1,t_2,t_3,t_4) = 0$iff $t_1 = t_2 = t_3 = t_4 = 0$."

Ansari A.H. *et al.* [3] have introduced pair of maps of upper classes stated as:
"**Definition(1.12) [3] (A)** Let $H: [0,\infty)^2 \to \mathbb{R}$ and $F: [0,\infty)^2 \to \mathbb{R}$. The pair (F,H) is said to be an upper class of type-I if *(a)*$x \geq 1 \Longrightarrow H(1,y) \leq H(x,y), \forall y \in [0,\infty)$.
(b)$0 \leq s \leq 1 \Longrightarrow F(s,t) \leq F(1,t), \forall t \in [0,\infty)$.*(c)*$H(1,y) \leq F(1,t) \Longrightarrow y \leq t, \forall y,t \in [0,\infty)$."
"**(B)** Let $H: [0,\infty)^3 \to \mathbb{R}$ and $F: [0,\infty)^2 \to \mathbb{R}$. The pair (F,H) is said to be an upper class of type-II if *(a)* for $x,y \geq 1, H(1,1,z) \leq H(x,y,z), \forall z \in [0,\infty)$.
(b)$0 \leq s \leq 1 \Longrightarrow F(s,t) \leq F(1,t), \forall t \in [0,\infty)$.*(c)*$H(1,1,z) \leq F(s,t) \Longrightarrow z \leq st, \forall z,s,t \in [0,\infty)$."

Ansari A. H. *et al.* [2] placed T-cyclic (α,β,H,F)-contractive map using a pair (F,H) of upper class functions [3], utilizing which a common fixed point theorem has been proved in metric spaces stated as:

"**Definition(1.13) [2]** Let (X,d) be a metric space and S be a T-cyclic (α,β)-admissible and T-cyclic(γ,δ)-subadmissible self map of X. S is said to be T-cyclic (α,β,H,F)-contractive map if $H(\alpha(Tx),\beta(Ty),\psi(d(Sx,Sy))) \leq F\left(\gamma(Tx)\delta(Ty),\phi\left(M(x,y)\right)\right), \forall x,y \in X$, where"
"$M(x,y) = \xi\left(d(Tx,Ty),d(Tx,Sx),d(Ty,Sy),\frac{1}{2}(d(Tx,Sy)+d(Ty,Sx))\right)$, for some $\xi \in \mathcal{X}$ (as defined in Definition(1.11)), the pair (F,H) is an upper class of type-II, $\psi: [0,\infty) \to [0,\infty)$ is an alternating distance function, and $\phi: [0,\infty) \to [0,\infty)$ is nondecreasing and right continuous such that $\phi(t) < \psi(t), \forall t > 0$."

Here we shall generalize T-cyclic (α,β,H,F)-contraction [2] considering generalized α-admissibility [13], and prove a common fixed point result in the environment of generalized completeness [8] in I-metric spaces and metric spaces replacing the function ξ by an alternating function so that the contraction condition be more weak, and by modifying Definition(1.12)(B) so that this definition becomes more general, and by generalization of some definitions (see from Definition(3.7)).

2. Preliminaries

We have generalized metric space by introducing an idempotent map, called I-metric space.

Definition(2.1) Let X be a nonempty set, $f: X \to X$ be an idempotent map, *i.e.*$f^2 = f$. A map $d: X^2 \to \mathbb{R}$is said to be an I-metric on X iff
I_1 $d(x,y) \geq 0, \forall x \text{ and } y \in X$.
I_2 $\forall x \text{ and } y \in X, d(x,f(y)) = 0$ iff $f(x) = f(y)$&$d(f(x),y) = 0$ iff $f(x) = f(y)$.
I_3 $d(x,f(y)) = d(y,f(x))$&$d(f(x),y) = d(f(y),x), \forall x \text{ and } y \in X$.
I_4 $d(x,z) \leq d(f(x),y) + d(y,f(z)), \forall x,y \text{ and } z \in X$.
The order triple (X,d,f) is called an I-metric space.

Example(2.2) (i) Consider the set $\mathbb{R}$ of reals and the idempotent map $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = [x], \forall x \in \mathbb{R}$, the largest integer function. Then $(\mathbb{R},d,f)$ is an I-metric space, where $d: \mathbb{R}^2 \to \mathbb{R}$ is defined by $d(x,y) = |[x]-[y]|, \forall x,y \in \mathbb{R}$.
(ii) It is clear that every metric space (X,d) is the I-metric space (X,d,I_X).

Theorem(2.3) Let (X,d,f) be an I-metric space. Then(i) $d(x,x) = 0, \forall x \in X$.
(ii) $d(x,f(y)) = d(y,f(x)) = d(f(x),f(y)) = d(f(y),f(x)) = d(f(x),y) = d(f(y),x) \geq d(x,y), d(y,x), \forall x \text{ and } y \in X$.(iii) $d(x,f(x)) = 0, \forall x \in X$.
Proof: Follows from Definition(2.1).

Definition(2.4)[Convergence of a sequence] A sequence $\{x_n\}$ in an I-metric space (X,d,f) is said to I-converge to a point *x* of X, if for any $\epsilon > 0, \exists m \in \mathbb{N}$ such that $x_n \in S_f(x,\epsilon), \forall n \geq m$. In this case *x* is called I-limit of the sequence $\{x_n\}$.

Definition(2.5) [I-uniqueness] Let *X* be a nonempty set and $f : X \to X$ be an idempotent map. Two elements $x, y \in X$ are said to be I-unique with respect to *f*, or simply I-unique, if $f(x) = f(y)$; otherwise they are called I-distinct elements in *X*.

Definition(2.6)[Cauchy sequence] A sequence $\{x_n\}$ in an I-metric space (X, d, f) is said to be an I-cauchy sequence in *X* if corresponding to every $\epsilon > 0, \exists n_\circ \in \mathbb{N}$ such that $d(f(x_m), x_n) < \epsilon$, $\forall m, n \geq n_\circ$, *i.e.*, $d\big(f(x_{n+p}), x_n\big) < \epsilon$, $\forall n \geq n_\circ, \forall p \geq 1$.

Definition(2.7)[Complete I-metric space] An I-metric space (X, d, f) is said to be I-complete if every I-cauchy sequence in *X* I-converges to some point of *X*; otherwise (X, d, f) is called I-incomplete.

Definition(2.8) [I-continuity] Let (X, d_1, f) and (Y, d_2, g) be two I-metric spaces. Then a function $h : X \to Y$ is said to be I-continuous at a point $a \in X$, if corresponding to every $\epsilon > 0, \exists \delta > 0$ such that $d_1(f(x), a) < \delta \Longrightarrow d_2\big((gh)(x), h(a)\big) < \epsilon$. h is said to be I-continuous on *X* if it is I-continuous at every point of *X*.

Theorem(2.9) Let (X, d_1, f) and (Y, d_2, g) be I-metric spaces and let $h : (X, d_1, f) \to (Y, d_2, g)$ be a function. Then h is I-continuous at a point $a \in X$ iff the sequence $\{h(x_n)\}$ in *Y* I-converges to $h(a)$ for each sequence $\{x_n\}$ in *X* I-converging to the point *a* in (X, d_1, f).

Theorem(2.10) Let (X, d, f) be an I-metric space and $h : (X, d, f) \to (X, d, f)$ be an operator, Then *h* is said to have an I-fixed point *x* in *X* if $(fh)(x) = f(x)$.

3. Main results

Definition(3.1) Let (X, d, f) be an I-metric space, $T : X \to X, \alpha, \beta : X^2 \to [0, \infty)$. $T$is said to be an (α, β)-Berinde-(ψ, ϕ)-rational I-contraction map if $\exists L \geq 0, M \geq 0$ and $\forall x, y \in X$ with $(fT)x \neq (fT)y$,

(***A***) $\alpha(x, Tx)\beta(y, Ty) \geq 1 \Longrightarrow \psi\Big(d\big((fT)x, Ty\big)\Big) \leq \psi\big(N(x, y)\big) - \phi\big(d(fx, y)\big)$

$+L \cdot min\{d(fx, Tx), d(fy, Ty), d(fx, Ty), d(fy, Tx)\}(1 + Md(fx, y))$, where

$N(x, y) = max\left\{d(fx, y), \frac{p_1(p_2 + d(fx, Tx))d(fy, Ty)}{p_3 + d(fx, y)}\right\}$, $\psi, \phi : [0, \infty) \to [0, \infty)$ are continuous,

$\psi(t) = 0$ iff $t = 0$, $\phi(t) = 0$ iff $t = 0$, ψ is strictly increasing, and $1 \geq p_1 \geq 0, p_3 > p_2 > 0$.

Replacing f by the identity map on *X*, we shall get the definition of similar type generalized contraction in a metric space (X, d), called (α, β)-Berinde-(ψ, ϕ)-rational contraction.

Theorem(3.2) Let (X, d, f) be an I-complete I-metric space, and $T : X \to X$ be an (α, β)-Berinde-(ψ, ϕ)-rational I-contraction map such that(i) T is (α, β)-cyclic admissible.

(ii) $\exists x_\circ \in X$such that $\alpha(x_\circ, Tx_\circ) \geq 1, \beta(x_\circ, Tx_\circ) \geq 1$.

(iii) T is I-continuous or for any sequence $\{x_n\}$ generated from $x_\circ$ by T, I-converging to x, $\alpha(x, Tx) \geq 1, \beta(x, Tx) \geq 1$.

Then T has an I-fixed point in *X*.

In addition, if $\alpha(x, Tx) \geq 1$ and $\beta(x, Tx) \geq 1, \forall x \in$ IFix(T), then T has an I-unique I-fixed point in *X*, whereIFix(T) is the set of all I-fixed points of T in *X*.

Proof: Given that $\exists x_\circ \in X$ such that $\alpha(x_\circ, Tx_\circ) \geq 1, \beta(x_\circ, Tx_\circ) \geq 1$. Define $x_n = Tx_{n-1}, \forall n \in \mathbb{N}$. If $fx_m = fx_{m-1}$ for some $m \in \mathbb{N}$, then x_{m-1} is an I-fixed point of T. Let $fx_n \neq fx_{n-1}, \forall n \in \mathbb{N}$.

By (i) and (ii), since $\alpha(x_\circ, x_1) \geq 1$, hence $\beta(x_1, x_2) \geq 1$, and this implies that $\alpha(x_2, x_3) \geq 1$ and so on. Thus, in general, $\alpha(x_{2n}, x_{2n+1}) \geq 1, \beta(x_{2n+1}, x_{2n+2}) \geq 1, \forall n \geq 0$.
Again by (i) and (ii), since $\beta(x_\circ, x_1) \geq 1$, similarly we get
$\beta(x_{2n}, x_{2n+1}) \geq 1, \alpha(x_{2n+1}, x_{2n+2}) \geq 1, \forall n \geq 0$.
Therefore $\alpha(x_n, x_{n+1}) \geq 1, \beta(x_n, x_{n+1}) \geq 1, \forall n \geq 0$. (1)
This implies that $\alpha(x_n, x_{n+1})\beta(x_{n+1}, x_{n+2}) = \alpha(x_n, Tx_n)\beta(x_{n+1}, Tx_{n+1}) \geq 1$.
Therefore from (A) of Definition(3.1) we get (taking $x = x_n, y = x_{n+1}$)

$$\psi\big(d(fx_{n+1}, x_{n+2})\big) = \psi\Big(d\big((fT)x_n, Tx_{n+1}\big)\Big) \leq \psi\big(N(x_n, x_{n+1})\big) - \phi(d(fx_n, x_{n+1})) \quad (2)$$

$< \psi\big(N(x_n, x_{n+1})\big)$ (since $\phi\big(d(fx_n, x_{n+1})\big) > 0$). $\Rightarrow d(fx_{n+1}, x_{n+2}) < N(x_n, x_{n+1})$.

$$\Rightarrow d(fx_{n+1}, x_{n+2}) < max\left\{d(fx_n, x_{n+1}), \frac{p_1(p_2 + d(fx_n, x_{n+1}))d(fx_{n+1}, x_{n+2})}{p_3 + d(fx_n, x_{n+1})}\right\}$$

$$\leq max\{d(fx_n, x_{n+1}), p_1 d(fx_{n+1}, x_{n+2})\} \leq max\{d(fx_n, x_{n+1}), d(fx_{n+1}, x_{n+2})\} \quad (3)$$

(since $0 \leq p_1 \leq 1, 0 < p_2 < p_3$).
Let $d(fx_n, x_{n+1}) < d(fx_{n+1}, x_{n+2})$ for some $n \geq 0$. Then from (3) we get
$d(fx_{n+1}, x_{n+2}) < d(fx_{n+1}, x_{n+2})$, a contradiction. Therefore $d(fx_{n+1}, x_{n+2}) \leq d(fx_n, x_{n+1}), \forall n \geq 0$.
Therefore $N(x_n, x_{n+1}) \leq d(fx_n, x_{n+1}), \forall n \geq 0$ (4)
Thus the sequence $\{d(fx_n, x_{n+1})\}$ is a decreasing sequence of nonnegative real numbers so that it converges to some nonnegative real number r. Since ψ is increasing, from (2) and (4) we get

$$\psi\big(d(fx_{n+1}, x_{n+2})\big) \leq \psi\big(d(fx_n, x_{n+1})\big) - \phi(d(fx_n, x_{n+1})) \quad (5)$$

Let $r > 0$. Taking limit as $n \to \infty$ in (5) and using continuity of ψ, ϕ, we get
$\psi(r) \leq \psi(r) - \phi(r) < \psi(r)$ (since $\phi(r) > 0$), a contradiction. Therefore $r = 0$.
Therefore $\lim_{n\to\infty} d(fx_n, x_{n+1}) = 0$. (6)
We claim that $\{x_n\}$ is I-cauchy in X. If not, then $\exists \epsilon > 0$, two subsequences $\{x_{m_k}\}$ and $\{x_{n_k}\}$ of $\{x_n\}$
Such that for every $k \in \mathbb{N}, m_k$ is the smallest positive integer for which $m_k > n_k > k$ and $d\big(fx_{n_k}, x_{m_k}\big) \geq \epsilon$ (7). Then $d\big(fx_{n_k}, x_{m_k-1}\big) < \epsilon$ (8)
From (1) we have $\alpha\big(x_{n_k}, x_{n_k+1}\big) \geq 1$ and $\beta\big(x_{m_k}, x_{m_k+1}\big) \geq 1, \forall k \in \mathbb{N}$.
Therefore $\alpha\big(x_{n_k}, x_{n_k+1}\big)\beta\big(x_{m_k}, x_{m_k+1}\big) \geq 1, \forall k \in \mathbb{N}$.
Therefore by (A) of Definition(3.1) we get (taking $x = x_{n_k}$, $y = x_{m_k}$)

$$\psi\big(d(fx_{n_k+1}, x_{m_k+1})\big) = \psi\big(d((fT)x_{n_k}, Tx_{m_k})\big) \leq \psi\Big(N\big(x_{n_k}, x_{m_k}\big)\Big) - \phi(d(fx_{n_k}, x_{m_k}))$$

$$+L \cdot min\big\{d\big(fx_{n_k}, x_{n_k+1}\big), d\big(fx_{m_k}, x_{m_k+1}\big), d\big(fx_{n_k}, x_{m_k+1}\big), d\big(fx_{n_k+1}, x_{m_k}\big)\big\}(1 + Md(fx_{n_k}, x_{m_k})) \quad (9)$$

$$\text{where} N\big(x_{n_k}, x_{m_k}\big) = max\left\{d\big(fx_{n_k}, x_{m_k}\big), \frac{p_1\Big(p_2+d\big(fx_{n_k}, x_{n_k+1}\big)\Big)d\big(fx_{m_k}, x_{m_k+1}\big)}{p_3+d\big(fx_{n_k}, x_{m_k}\big)}\right\} \quad (10)$$

Using (6), (7), (8), triangle inequality we shall get

$\lim_{k\to\infty} d\big(fx_{n_k}, x_{m_k}\big) = \epsilon$ (11) $\lim_{k\to\infty} d\big(fx_{n_k+1}, x_{m_k+1}\big) = \epsilon$ (12)

$\lim_{k\to\infty} d\big(fx_{n_k}, x_{m_k+1}\big) = \epsilon$ (13) $\lim_{k\to\infty} d\big(fx_{n_k+1}, x_{m_k}\big) = \epsilon$ (14)

Therefore from (10) we get $\lim_{k\to\infty} N\big(x_{n_k}, x_{m_k}\big) = \epsilon$ (By (6), (11)) (15)
Taking the limit as $k \to \infty$ in (9), and by (6), (11), (12), (13), (14), (15) and continuity of ψ, ϕ we get $\psi(\epsilon) \leq \psi(\epsilon) - \phi(\epsilon) < \psi(\epsilon)$, a contradiction.
Therefore $\{x_n\}$ is I-cauchy in X so that it I-converges to some point $u \in X$.
Let T is I-continuous. Then $\{Tx_n\}$ I-converges to Tu. But $\{Tx_n\} = \{x_{n+1}\}$ I-converges to u.
Therefore $(fT)u = fu$ so that u is an I-fixed point of T.

Let for any sequence $\{x_n\}$ generated from $x_\circ$by T, I-converging to x, $\alpha(x,Tx) \geq 1, \beta(x,Tx) \geq 1$.
Therefore $\alpha(u,Tu) \geq 1, \beta(u,Tu) \geq 1$ (16)
Therefore $\alpha(x_n,x_{n+1})\beta(u,Tu) \geq 1$ (By (1), (16)).
Therefore from (*A*) of Definition(3.1) we get (taking $x = x_n, y = u$)
$\psi\big(d(fx_{n+1},Tu)\big) = \psi\left(d\big((fT)x_n,Tu\big)\right) \leq \psi\big(N(x_n,u)\big) - \phi(d(fx_n,u))$
$L \cdot min\{d(fx_n,x_{n+1}), d(fu,Tu), d(fx_n,Tu), d(fu,x_{n+1})\}(1 + Md(fx_n,u))$ (17)
where $N(x_n,u) = max\left\{d(fx_n,u), \frac{p_1(p_2+d(fx_n,x_{n+1}))d(fu,Tu)}{p_3+d(fx_n,u)}\right\}$.
Therefore $\lim_{n\to\infty} N(x_n,u) = \frac{p_1p_2d(fu,Tu)}{p_3}$ (since $\{x_n\}$ I-converges to u). (18)
Taking the limit as $n \to \infty$ in (17) and using continuity of ψ, ϕ and by (18), I-convergence of $\{x_n\}$ to$u$ we get$\psi\big(d(fu,Tu)\big) \leq \psi\left(\frac{p_1p_2d(fu,Tu)}{p_3}\right) < \psi(d(fu,Tu))$ if $d(fu,Tu) > 0$.
(since $0 \leq p_1 \leq 1, 0 < p_2 < p_3$ and ψ is strictly increasing), a contradiction.
Therefore $d(fu,Tu) = 0$. Thus $(fT)u = fu$. Therefore u is an I-fixed point of T.

Now let for all $x \in$ IFix(T), $\alpha(x,Tx) \geq 1, \beta(x,Tx) \geq 1$. Let v be an I-fixed point of T such that $fv \neq fu$. Then $(fT)v = fv, d(fu,v) > 0$. Also $\alpha(u,Tu) \geq 1, \beta(u,Tu) \geq 1, \alpha(v,Tv) \geq 1, \beta(v,Tv) \geq 1$. Therefore $\alpha(u,Tu)\,\beta(v,Tv) \geq 1$.Therefore from (*A*) of Definition(3.1) we get (taking $x = u, y = v$)
$\psi\big(d(fu,v)\big) = \psi\big(d(fu,fv)\big) = \psi\left(d\big((fT)u,(fT)v\big)\right) = \psi\left(d\big((fT)u,Tv\big)\right)$
$\leq \psi\big(N(u,v)\big) - \phi(d(fu,v))$ (19), where $N(u,v) = max\{d(fu,v), 0\} = d(fu,v)$.
Therefore (19) becomes $\psi\big(d(fu,v)\big) \leq \psi\big(d(fu,v)\big) - \phi\big(d(fu,v)\big) < \psi\big(d(fu,v)\big)$
(since $\phi\big(d(fu,v)\big) > 0$), a contradiction. Therefore T has an I-unique I-fixed point.

Corollary(3.3) Let (X,d) be a complete metric space, and $T : X \to X$ be an (α,β)-Berinde-(ψ,ϕ)-rational contraction map such that(i) T is (α,β)-cyclic admissible.
(ii) $\exists x_\circ \in X$ such that $\alpha(x_\circ,Tx_\circ) \geq 1, \beta(x_\circ,Tx_\circ) \geq 1$.
(iii) T is continuous or for any sequence $\{x_n\}$ generated from $x_\circ$ by T, converging to x, $\alpha(x,Tx) \geq 1, \beta(x,Tx) \geq 1$.
Then T has a fixed point in *X*.
In addition, if $\alpha(x,Tx) \geq 1$ and $\beta(x,Tx) \geq 1, \forall x \in$ Fix(T), then T has a unique fixed point in *X*, where Fix(T) is the set of all fixed points of T in *X*.
Proof: Replacing f by the identity map of *X* in Theorem(3.2), we shall get the result.

Now we have another generalization of $(\boldsymbol{\alpha},\boldsymbol{\beta})$-Berinde-$\boldsymbol{\phi}$-contraction, named $(\boldsymbol{\alpha},\boldsymbol{\beta})$-Berinde-$(\boldsymbol{\psi},\boldsymbol{\phi})$-weak contraction map and established fixed point results under this contraction in I-metric spacesand metric spaces.

Definition(3.4) Let (X,d,f) be an I-metric space, $\alpha,\beta : X^2 \to [0,\infty)$ and $T : X \to X$. $T$is said to be an (α,β)-Berinde-(ψ,ϕ)-weak I-contraction map if $\exists L \geq 0, M \geq 0$ and $\forall x,y \in X$ with $(fT)x \neq (fT)y$,
(***A***) $\alpha(x,Tx)\beta(y,Ty) \geq 1 \Longrightarrow \psi\left(d\big((fT)x,Ty\big)\right) \leq \psi\big(N(x,y)\big) - \phi\big(N(x,y)\big)$
$+L \cdot min\{d(fx,Tx), d(fy,Ty), d(fx,Ty), d(fy,Tx)\}(1 + Md(fx,y))$, where
$N(x,y) = max\{d(fx,y), d(fx,Tx), d(fy,Ty), d(fx,Ty), d(fy,Tx)\}$, $\psi,\phi : [0,\infty) \to [0,\infty)$ are continuous, $\psi(t) = 0$ iff $t = 0$, $\phi(0) = 0$ and ψ is increasing.
If f be the identity map on *X*, then this contraction is called an (α,β)-Berinde-(ψ,ϕ)-weak contraction in the metric space (X,d).

Theorem(3.5) Let (X,d,f) be an I-complete I-metric space and $T:X\to X$ be an (α,β)-Berinde-(ψ,ϕ)-weak I-contraction map such that (i) T is (α,β)-cyclic admissible.
(ii) $\exists x_\circ\in X$ such that $\alpha(x_\circ,Tx_\circ)\geq 1,\beta(x_\circ,Tx_\circ)\geq 1$.
(iii) T is I-continuous or for any sequence $\{x_n\}$ generated from $x_\circ$ by T, I-converging to x, $\alpha(x,Tx)\geq 1,\beta(x,Tx)\geq 1$.
Then T has an I-fixed point in X.
In addition, if $\alpha(x,Tx)\geq 1$ and $\beta(x,Tx)\geq 1,\forall x\in \text{IFix}(T)$, then T has an I-unique I-fixed point in X.
Proof: Similar to Theorem(3.2).

Corollary(3.6) Let (X,d) be a complete metric space, and $T:X\to X$ be an (α,β)-Berinde-(ψ,ϕ)-weak contraction map such that(i) T is (α,β)-cyclic admissible.
(ii) $\exists x_\circ\in X$such that $\alpha(x_\circ,Tx_\circ)\geq 1,\beta(x_\circ,Tx_\circ)\geq 1$.
(iii) T is continuous or for any sequence $\{x_n\}$ generated from $x_\circ$ by T, converging to x, $\alpha(x,Tx)\geq 1,\beta(x,Tx)\geq 1$.
Then T has a fixed point in X.
In addition, if $\alpha(x,Tx)\geq 1$ and $\beta(x,Tx)\geq 1,\forall x\in \text{Fix}(T)$, then T has a unique fixed point in X,
Proof: Replacing f by the identity map of X in Theorem(3.5), we shall get the result.

Now we have the following generalized definitions and common fixed point results.

Definition(3.7)[Generalization of Definition(1.10)] Let $S,T:X\to X$and $\alpha,\beta,\eta:\ X^2\to[0,\infty)$. $S$is said to be**(A)**(α,β)- η-T-cyclic admissible map if $\forall x,y\in X$
(a)$\alpha(Tx,Ty)\geq\eta(Tx,Ty)\Longrightarrow\beta(Sx,Sy)\geq\eta(Sx,Sy)$.
(b)$\beta(Tx,Ty)\geq\eta(Tx,Ty)\Longrightarrow\alpha(Sx,Sy)\geq\eta(Sx,Sy)$.
In addition, if *(c)*$\alpha(x,y)\geq\eta(x,y),\alpha(y,z)\geq\eta(y,z)\Longrightarrow\alpha(x,z)\geq\eta(x,z)$ and $\beta(x,y)\geq\eta(x,y),\beta(y,z)\geq\eta(y,z)\Longrightarrow\beta(x,z)\geq\eta(x,z),\forall x,y,z\in X$, then S is called triangular(α,β)- η-T-cyclic admissible map.

(B)(α,β)- η-T-cyclic subadmissible map if $\forall x,y\in X$
(a)$\alpha(Tx,Ty)\leq\eta(Tx,Ty)\Longrightarrow\beta(Sx,Sy)\leq\eta(Sx,Sy)$.
(b)$\beta(Tx,Ty)\leq\eta(Tx,Ty)\Longrightarrow\alpha(Sx,Sy)\leq\eta(Sx,Sy)$.
In addition, if *(c)*$\alpha(x,y)\leq\eta(x,y),\alpha(y,z)\leq\eta(y,z)\Longrightarrow\alpha(x,z)\leq\eta(x,z)$ and $\beta(x,y)\leq\eta(x,y),\beta(y,z)\leq\eta(y,z)\Longrightarrow\beta(x,z)\leq\eta(x,z),\forall x,y,z\in X$, then S is called triangular(α,β)- η-T-cyclic subadmissible map.

Example(3.8) Let $X=[0,\infty)$, $\alpha,\beta,\gamma,\delta,\eta,\theta:\ X^2\to[0,\infty)$ given by $\alpha(x,y)=60(x+y)$, $\beta(x,y)=55(x+y),\eta(x,y)=50(x+y),\theta(x,y)=45(x+y),\gamma(x,y)=40(x+y)$, $\delta(x,y)=35(x+y),\forall x,y\in X.S,T:X\to X$ are given by $Sx=\ |cosx|,Tx=|sinx|,\forall x\in X.$
Then it is obvious that S is a triangular (α,β)- η-T-cyclic admissible map, and a triangular (γ,δ)-θ-T-cyclic subadmissible map.

Note(3.9) Following Definition(1.11) of the function ξ, we can say that, if we replace it by an alternating distance function $\xi_1:[0,\infty)\to[0,\infty)$ which is nondecreasing, continuous, $\xi_1(t)<t,\forall t>0$ and $\xi_1(t)=0$ iff $t=0$ in Definition(1.11), then the contraction become more weak, since it is clear that $\xi(t_1,t_2,t_3,t_4)\leq max\{t_1,t_2,t_3,t_4\}$ and ϕ is non-decreasing in Definition(1.11).

Definition(3.10)[Modification of Definition(1.12)(B)] Let H: $[0,\infty)^3\to\mathbb{R}$ and $F:\ [0,\infty)^2\to\mathbb{R}$. The pair (F,H) is said to be a modified upper class of type-II if

*(a)*for $x \geq x_1, y \geq y_1, H(x_1, y_1, z) \leq H(x, y, z), \forall z \in [0, \infty)$.
(b)$0 \leq s \leq s_1 \Longrightarrow F(s, t) \leq F(s_1, t), \forall t \in [0, \infty)$.
(c)$H(x, y, z) \leq F(s, t) \Longrightarrow xyz \leq st, \forall x, y, z, s, t \in [0, \infty)$.

Example(3.11) Let $H: [0, \infty)^3 \to \mathbb{R}$ and $F: [0, \infty)^2 \to \mathbb{R}$ are given by
$H(x, y, z) = (xyz)^p, \forall x, y, z \in [0, \infty)$and $F(s, t) = (st)^p, \forall s, t \in [0, \infty)$, for some given $p \in \mathbb{N}$.
Then clearly, (F, H) is a modified upper class of type-II.

Definition(3.12) Let (X, d, f) be an I-metric space, $S, T: X \to X$.
(A) If $\exists x \in X$ such that $(fS)x = (fT)x = fy$ (say), then x is called an I-coincidence point of S and T, and y is called a point of I-coincidence of S and T (See Example(3.17)).

(B)$S$and$T$ are said to be weakly I-compatible, if fS and fT commute at I-coincidence point(s) of S and T (See Example(3.17)).

Definition(3.13)[Generalization of Definition(1.13)] Let (X, d, f) be an I-metric space, $S, T : X \to X$ such that S is a triangular (α, β)- η-T-cyclic admissible and triangular (γ, δ)-θ-T-cyclic subadmissible map for some $\alpha, \beta, \gamma, \delta, \eta, \theta : X^2 \to [0, \infty)$. $S$is said to be $\big((\alpha, \beta, \eta), (\gamma, \delta, \theta), H, F\big)$-$T$-cyclic I-contractive map if
(i) $H(\alpha(Tx, Ty), \beta(Tx, Ty), \psi(d((fS)x, Sy))) \leq F\Big(\gamma(Tx, Ty)\delta(Tx, Ty), \phi\big(M(x, y)\big)\Big), \forall x, y \in X$,
where$M(x, y) = \xi\left(max\left\{d\big((fT)x, Ty\big), d\big((fT)x, Sx\big), d\big((fT)y, Sy\big), \frac{1}{2}d\big((fT)x, Sy\big), d\big((fT)y, Sx\big)\right\}\right)$
for some $\xi : [0, \infty) \to [0, \infty)$ which is nondecreasing, continuous, $\xi(t) < t, \forall t > 0$&$\xi(t) = 0$ iff $t = 0$.
(ii) the pair (F, H) is a modified upper class of type-II.
(iii) $\psi : [0, \infty) \to [0, \infty)$ is nondecreasing, continuous and $\psi(t) = 0$ iff $t = 0$.
(iv) $\phi : [0, \infty) \to [0, \infty)$ is nondecreasing, right continuous and $\phi(t) < \psi(t), \forall t > 0$.
If $f = I_X$, the identity map on X, then this contraction in a metric space (X, d) is called an $\big((\alpha, \beta, \eta), (\gamma, \delta, \theta), H, F\big)$-$T$-cyclic contractive map (See Example(3.17)).

Definition(3.14)[Generalization of Definition(1.8)(A)] Let (X, d, f) be an I-metric space. $\alpha, \beta, \eta : X^2 \to [0, \infty)$. Then (X, d, f) is called
(A)admissibly(α, β)-η I-complete if every I-cauchy sequence $\{x_n\}$ with
$\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1}), \beta(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1}), \forall n \in \mathbb{N}$, I-converges in X.

(B)sub admissibly(α, β)-η I-complete if every I-cauchy sequence $\{x_n\}$ with $\alpha(x_n, x_{n+1}) \leq \eta(x_n, x_{n+1})$, $\beta(x_n, x_{n+1}) \leq \eta(x_n, x_{n+1}), \forall n \in \mathbb{N}$, I-converges in X. (See Example(3.17)).
If f be the identity map on X, then we have admissibly (α, β)-η-complete metric space (X, d) and subadmissibly (α, β)-η-complete metric space (X, d) respectively.

Theorem(3.15) Let (X, d, f) be an I-metric space, $S, T : X \to X$ satisfy(i) $S(X) \subseteq T(X)$.
(ii) $S$is$\big((\alpha, \beta, \eta), (\gamma, \delta, \theta), H, F\big)$-$T$-cyclic I-contractive map with $\theta(x, y) \leq \eta(x, y), \forall x, y \in X$; $\theta(x, y) > 0$ whenever $fx \neq fy$.(iii) $S(X)$or$T(X)$ is I-closed in X.
(iv) $\exists x_1 \in X$such that $\alpha(Tx_1, Tx_2) \geq \eta(Tx_1, Tx_2), \beta(Tx_1, Tx_2) \geq \eta(Tx_1, Tx_2)$,
$\gamma(Tx_1, Tx_2) \leq \theta(Tx_1, Tx_2), \delta(Tx_1, Tx_2) \leq \theta(Tx_1, Tx_2)$, where $x_2 \in X$ such that $Sx_1 = Tx_2$.
(since$Sx_1 \in S(X) \subseteq T(X), Sx_1 = Tx_2$ for some $x_2 \in X$).
(v) $(X, d, f)$is admissibly (α, β)-η I-complete or subadmissibly (γ, δ)-θ I-complete.
(vi) if$\{y_n\}$ be a sequence I-converging to u in X, and $\alpha(y_n, y_{n+1}) \geq \eta(y_n, y_{n+1})$ and
$\beta(y_n, y_{n+1}) \geq \eta(y_n, y_{n+1}), \forall n \in \mathbb{N}$, then $\alpha(y_n, u) \geq \eta(y_n, u), \beta(y_n, u) \geq \eta(y_n, u)$;

and if $\gamma(y_n, y_{n+1}) \leq \theta(y_n, y_{n+1}), \delta(y_n, y_{n+1}) \leq \theta(y_n, y_{n+1}), \forall n \in \mathbb{N}$, then $\gamma(y_n, u) \leq \theta(y_n, u)$, $\delta(y_n, u) \leq \theta(y_n, u)$.
(vii) $\alpha(Tp, Tq) \geq \eta(Tp, Tq), \beta(Tp, Tq) \geq \eta(Tp, Tq), \gamma(Tp, Tq) \leq \theta(Tp, Tq)$, $\delta(Tp, Tq) \leq \theta(Tp, Tq)$, whenever $(fS)p = (fT)p, (fS)q = (fT)q$.
Then S and T have an I-unique point of I-coincidence in X.
In addition, if S and T are weakly I-compatible, then S and T have an I-unique common I-fixed point in X.

Proof: Let $y_1 = Sx_1 = Tx_2, y_2 = Sx_2 = Tx_3$ (since $Sx_2 \in S(X) \subseteq T(X), Sx_2 = Tx_3$ for some $x_3 \in X$), and in general, $y_n = Sx_n = Tx_{n+1}, \forall n \in \mathbb{N}$. If $fy_m = fy_{m+1}$ for some $m \in \mathbb{N}$, then y_{m+1} is a point of I-coincidence of $S$and T.Let $fy_n \neq fy_{n+1}, \forall n \in \mathbb{N}$. Since S is triangular (α, β)- η-T-cyclic admissible and $\alpha(Tx_1, Tx_2) \geq \eta(Tx_1, Tx_2)$, hence $\beta(Sx_1, Sx_2) = \beta(Tx_2, Tx_3) \geq \eta(Sx_1, Sx_2) = \eta(Tx_2, Tx_3)$.
This implies that $\alpha(Sx_2, Sx_3) = \alpha(Tx_3, Tx_4) \geq \eta(Sx_2, Sx_3) = \eta(Tx_3, Tx_4)$.
Proceeding in this way, we have, in general, $\alpha(Tx_{2n-1}, Tx_{2n}) \geq \eta(Tx_{2n-1}, Tx_{2n})$, $\beta(Tx_{2n}, Tx_{2n+1}) \geq \eta(Tx_{2n}, Tx_{2n+1}), \forall n \in \mathbb{N}$.
Similarly, S is triangular (α, β)- η-T-cyclic admissible and $\beta(Tx_1, Tx_2) \geq \eta(Tx_1, Tx_2)$ implies that $\alpha(Tx_{2n}, Tx_{2n+1}) \geq \eta(Tx_{2n}, Tx_{2n+1}),\ \beta(Tx_{2n-1}, Tx_{2n}) \geq \eta(Tx_{2n-1}, Tx_{2n}), \forall n \in \mathbb{N}$.
Therefore $\alpha(Tx_n, Tx_{n+1}) \geq \eta(Tx_n, Tx_{n+1}),\ \beta(Tx_n, Tx_{n+1}) \geq \eta(Tx_n, Tx_{n+1}), \forall n \in \mathbb{N}$. (1)

Again since S is triangular (γ, δ)-θ-T-cyclic subadmissible and
$\gamma(Tx_1, Tx_2) \leq \theta(Tx_1, Tx_2)$, $\delta(Tx_1, Tx_2) \leq \theta(Tx_1, Tx_2)$, similarly, we shall get
$\gamma(Tx_n, Tx_{n+1}) \leq \theta(Tx_n, Tx_{n+1}),\ \delta(Tx_n, Tx_{n+1}) \leq \theta(Tx_n, Tx_{n+1}), \forall n \in \mathbb{N}$. (2)
Therefore $\gamma(Tx_n, Tx_{n+1})\delta(Tx_n, Tx_{n+1}) \leq \theta^2(Tx_n, Tx_{n+1}), \forall n \in \mathbb{N}$. (3)
By (ii) we get (taking $x = x_{n+1}, y = x_{n+2}$)$H\left(\eta(y_n, y_{n+1}), \eta(y_n, y_{n+1}), \psi\big(d(fy_{n+1}, y_{n+2})\big)\right)$
$= H(\eta(Tx_{n+1}, Tx_{n+2}), \eta(Tx_{n+1}, Tx_{n+2}), \psi(d((fS)x_{n+1}, Sx_{n+2})))$
$\leq H(\alpha(Tx_{n+1}, Tx_{n+2}), \beta(Tx_{n+1}, Tx_{n+2}), \psi(d((fS)x_{n+1}, Sx_{n+2})))$ (By property of H).
$\leq F(\gamma(Tx_{n+1}, Tx_{n+2})\delta(Tx_{n+1}, Tx_{n+2}), \phi(M(x_{n+1}, x_{n+2})))$
$\leq F\left(\theta^2(Tx_{n+1}, Tx_{n+2}), \phi\big(M(x_{n+1}, x_{n+2})\big)\right) = F(\theta^2(y_n, y_{n+1}), \phi(M(x_{n+1}, x_{n+2})))$.
$\Longrightarrow \eta^2(y_n, y_{n+1})\psi\big(d(fy_{n+1}, y_{n+2})\big) \leq \theta^2(y_n, y_{n+1})\phi(M(x_{n+1}, x_{n+2}))$. (4)
$\Longrightarrow \eta^2(y_n, y_{n+1})\psi\big(d(fy_{n+1}, y_{n+2})\big) \leq \eta^2(y_n, y_{n+1})\phi(M(x_{n+1}, x_{n+2}))$. (5)
(since $\theta(x, y) \leq \eta(x, y), \forall x, y \in X$). $\Longrightarrow \psi\big(d(fy_{n+1}, y_{n+2})\big) \leq \phi(M(x_{n+1}, x_{n+2}))$. (6)
(since$0 < \theta(y_n, y_{n+1}) \leq \eta(y_n, y_{n+1})$, as $fy_n \neq fy_{n+1}, \forall n \in \mathbb{N}$).
$\Longrightarrow \psi\big(d(fy_{n+1}, y_{n+2})\big) < \psi(M(x_{n+1}, x_{n+2}))$ (since $\phi(t) < \psi(t), \forall t > 0$). (7)
$\Longrightarrow d(fy_{n+1}, y_{n+2}) < M(x_{n+1}, x_{n+2}), \forall n \in \mathbb{N}$ (since ψ is nondecreasing). (8)
where$M(x_{n+1}, x_{n+2}) = \xi\left(max\left\{d(fy_n, y_{n+1}), d(fy_{n+1}, y_{n+2}), \frac{1}{2}d(fy_n, y_{n+2})\right\}\right)$
$\leq \xi\left(max\left\{d(fy_n, y_{n+1}), d(fy_{n+1}, y_{n+2}), \frac{1}{2}\big(d(fy_n, y_{n+1}) + d(fy_{n+1}, y_{n+2})\big)\right\}\right)$. (9)
(sinceξ is nondecreasing)
If $d(fy_n, y_{n+1}) < d(fy_{n+1}, y_{n+2})$ for some $n \in \mathbb{N}$, then from (9) we get
$M(x_{n+1}, x_{n+2}) \leq \xi\big(d(fy_{n+1}, y_{n+2})\big) < d(fy_{n+1}, y_{n+2})$. (10)
(since ξ is nondecreasing, $fy_{n+1} \neq y_{n+2}$ and $\xi(t) < t, \forall t > 0$).
From (8) and (10) we get $d(fy_{n+1}, y_{n+2}) < d(fy_{n+1}, y_{n+2})$, a contradiction.
Therefore $d(fy_{n+1}, y_{n+2}) \leq d(fy_n, y_{n+1}), \forall n \in \mathbb{N}$. (11)
Then $M(x_{n+1}, x_{n+2}) \leq \xi\big(d(fy_n, y_{n+1})\big)$. (12)
Therefore $\{d(fy_n, y_{n+1})\}$ is a decreasing sequence of nonnegative real numbers so that it converges to some nonnegative real number r. Let $r > 0$. From (12) we get

$\lim_{n\to\infty} M(x_{n+1}, x_{n+2}) \leq \xi(r)$ (By continuity of ξ). (13)
Taking limit as $n \to \infty$ in (6), using continuity of ψ, ϕ we get
$\psi(r) \leq \phi(\xi(r))$ (since ϕ is nondecreasing) $< \psi(\xi(r))$ (since $\phi(t) < \psi(t), \forall t > 0$).
$\Rightarrow r < \xi(r)$ (since ψ is nondecreasing) $< r$ (since $\xi(t) < t, \forall t > 0$), a contradiction.
Therefore $r = 0$. Thus $\lim_{n\to\infty} d(fy_n, y_{n+1}) = 0$. (14)
We claim that $\{y_n\}$ is I-cauchy in X. If not, then $\exists \epsilon > 0$, two strictly increasing sequences of positive integers $\{m_k\}$ and $\{n_k\}$ such that for every $k \in \mathbb{N}$, m_k is the smallest positive integer for which $m_k > n_k > k$ and $d(fy_{n_k}, y_{m_k}) \geq \epsilon$. (15) Then $d(fy_{n_k}, y_{m_k-1}) < \epsilon$. (16)
By (14), (15), (16) and triangle inequality, subsequently we can easily prove that
$\lim_{k\to\infty} d(fy_{n_k}, y_{m_k}) = \epsilon$ (17) $\lim_{k\to\infty} d(fy_{n_k}, y_{m_k-1}) = \epsilon$ (18)
$\lim_{k\to\infty} d(fy_{n_k-1}, y_{m_k-1}) = \epsilon$ (19) $\lim_{k\to\infty} d(fy_{n_k-1}, y_{m_k}) = \epsilon$ (20)
Now we shall prove that for all $n, p \in \mathbb{N}$, $\alpha(y_n, y_{n+p}) \geq \eta(y_n, y_{n+p})$, $\beta(y_n, y_{n+p}) \geq \eta(y_n, y_{n+p})$, $\gamma(y_n, y_{n+p}) \leq \theta(y_n, y_{n+p})$, $\delta(y_n, y_{n+p}) \leq \theta(y_n, y_{n+p})$ by induction on p.
For $p = 1$, the result holds by (1) and (2).
Let $p = 2$. By (1), we have $\alpha(y_n, y_{n+1}) \geq \eta(y_n, y_{n+1})$ and $\alpha(y_{n+1}, y_{n+2}) \geq \eta(y_{n+1}, y_{n+2})$.
This implies that $\alpha(y_n, y_{n+2}) \geq \eta(y_n, y_{n+2})$ (since S is a triangular (α,β)-η-T-cyclic admissible).
Similarly, we shall get $\beta(y_n, y_{n+2}) \geq \eta(y_n, y_{n+2}), \gamma(y_n, y_{n+2}) \leq \theta(y_n, y_{n+2})$ and $\delta(y_n, y_{n+2}) \leq \theta(y_n, y_{n+2})$. Therefore the result holds for $p = 2$.
Let the result hold for any positive integer $p \geq 2$.
Then $\alpha(y_n, y_{n+p}) \geq \eta(y_n, y_{n+p})$, $\beta(y_n, y_{n+p}) \geq \eta(y_n, y_{n+p})$, $\gamma(y_n, y_{n+p}) \leq \theta(y_n, y_{n+p})$, $\delta(y_n, y_{n+p}) \leq \theta(y_n, y_{n+p})$. Now by (1) we have $\alpha(y_{n+p}, y_{n+p+1}) \geq \eta(y_{n+p}, y_{n+p+1})$.
Also $\alpha(y_n, y_{n+p}) \geq \eta(y_n, y_{n+p})$.Since S is a triangular (α,β)-η-T-cyclic admissible, hence $\alpha(y_n, y_{n+p+1}) \geq \eta(y_n, y_{n+p+1})$.Similarly, we shall get $\beta(y_n, y_{n+p+1}) \geq \eta(y_n, y_{n+p+1})$, $\gamma(y_n, y_{n+p+1}) \leq \theta(y_n, y_{n+p+1})$ and $\delta(y_n, y_{n+p+1}) \leq \theta(y_n, y_{n+p+1})$.
Therefore by mathematical induction, we have the result.
Therefore we have $\alpha(y_{n_k-1}, y_{m_k-1}) \geq \eta(y_{n_k-1}, y_{m_k-1})$ (21)
$\beta(y_{n_k-1}, y_{m_k-1}) \geq \eta(y_{n_k-1}, y_{m_k-1})$ (22) $\gamma(y_{n_k-1}, y_{m_k-1}) \leq \theta(y_{n_k-1}, y_{m_k-1})$ (23)
$\delta(y_{n_k-1}, y_{m_k-1}) \leq \theta(y_{n_k-1}, y_{m_k-1})$ (24)
From (ii) we get (taking $x = x_{n_k}, y = x_{m_k}$)$H(\eta(y_{n_k-1}, y_{m_k-1}), \eta(y_{n_k-1}, y_{m_k-1}), \psi(d(fy_{n_k}, y_{m_k})))$
$= H(\eta(Tx_{n_k}, Tx_{m_k}), \eta(Tx_{n_k}, Tx_{m_k}), \psi(d((fS)x_{n_k}, Sx_{m_k})))$
$\leq H(\alpha(Tx_{n_k}, Tx_{m_k}), \beta(Tx_{n_k}, Tx_{m_k}), \psi(d((fS)x_{n_k}, Sx_{m_k})))$ (By (21), (22) and property of H).
$\leq F(\gamma(Tx_{n_k}, Tx_{m_k})\delta(Tx_{n_k}, Tx_{m_k}), \phi(M(x_{n_k}, x_{m_k})))$
$\leq F(\theta^2(Tx_{n_k}, Tx_{m_k}), \phi(M(x_{n_k}, x_{m_k})))$ (By (23), (24) and property of F).
$= F(\theta^2(y_{n_k-1}, y_{m_k-1}), \phi(M(x_{n_k}, x_{m_k})))$.
$\Rightarrow \eta^2(y_{n_k-1}, y_{m_k-1})\, \psi\left(d(fy_{n_k}, y_{m_k})\right) \leq \theta^2(y_{n_k-1}, y_{m_k-1})\phi(M(x_{n_k}, x_{m_k}))$ (25)
$\Rightarrow \eta^2(y_{n_k-1}, y_{m_k-1})\, \psi\left(d(fy_{n_k}, y_{m_k})\right) \leq \eta^2(y_{n_k-1}, y_{m_k-1})\phi(M(x_{n_k}, x_{m_k}))$ (26)
(since $\theta(x, y) \leq \eta(x, y)$).$\Rightarrow \psi\left(d(fy_{n_k}, y_{m_k})\right) \leq \phi(M(x_{n_k}, x_{m_k}))$. (27)
$\Rightarrow \psi\left(d(fy_{n_k}, y_{m_k})\right) < \psi(M(x_{n_k}, x_{m_k}))$ (since $\phi(t) < \psi(t), \forall t > 0$) (28)
$\Rightarrow d(fy_{n_k}, y_{m_k}) < M(x_{n_k}, x_{m_k})$ (since ψ is nodecreasing) (29)
where$M(x_{n_k}, x_{m_k}) =$

$\xi\left(max\left\{d\left(fy_{n_k-1},y_{m_k-1}\right),d\left(fy_{n_k-1},y_{n_k}\right),d\left(fy_{m_k-1},y_{m_k}\right),\frac{1}{2}d\left(fy_{n_k-1},y_{m_k}\right),d\left(fy_{m_k-1},y_{n_k}\right)\right\}\right)$.

Therefore $\lim_{k\to\infty} M\left(x_{n_k},x_{m_k}\right)=\xi\left(max\left\{\epsilon,0,0,\frac{\epsilon}{2},\epsilon\right\}\right)=\xi(\epsilon)$ (30)

(By (14), (18), (19), (20) and continuity of ξ).

Taking the limit as $k\to\infty$ in (27), and using continuity of ψ,ϕ and by (17), (30) we get

$\psi(\epsilon)\le\phi\left(\xi(\epsilon)\right)<\psi(\xi(\epsilon))$ (since$\epsilon>0\Longrightarrow\xi(\epsilon)>0$ and $\phi(t)<\psi(t),\forall t>0$).

$\Longrightarrow\epsilon<\xi(\epsilon)<\epsilon$ (since ψ is nondecreasing and $\xi(\epsilon)<\epsilon$, as $\epsilon>0$), a contradiction.

Therefore $\{y_n\}$ is I-Cauchy in X. Now since (X,d,f) is either admissibly (α,β)-η I-complete or subadmissibly (γ,δ)-θ I-complete, hence $\{y_n\}$ I-converges to some point $u\in X$,

i.e., $\{Sx_n\}=\{Tx_{n+1}\}=\{y_n\}$ I-converges to u.

Since $S(X)$ or $T(X)$ is I-closed, hence $u\in S(X)\cup T(X)=T(X)$ (since $S(X)\subseteq T(X)$).

Therefore $u=Tv$ for some $v\in X$.(31). Now by (vi), (1) and (2) we get

$\alpha(y_n,u)\ge\eta(y_n,u),\beta(y_n,u)\ge\eta(y_n,u),\gamma(y_n,u)\le\theta(y_n,u),\delta(y_n,u)\le\theta(y_n,u)$. (32)

i.e., $\alpha(Tx_{n+1},Tv)\ge\eta(Tx_{n+1},Tv),\beta(Tx_{n+1},Tv)\ge\eta(Tx_{n+1},Tv),\gamma(Tx_{n+1},Tv)\le\theta(Tx_{n+1},Tv)$, $\delta(Tx_{n+1},Tv)\le\theta(Tx_{n+1},Tv),\forall\, n\in\mathbb{N}$. (33)

Now by (ii) we get (taking $=x_{n+1},y=v$)$H\left(\eta(y_n,u),\eta(y_n,u),\psi\left(d(fy_{n+1},Sv)\right)\right)\le H(\alpha(y_n,u),\beta(y_n,u),\psi(d(fy_{n+1},Sv)))$

(By property of H and (32)).$=H(\alpha(Tx_{n+1},Tv),\beta(Tx_{n+1},Tv),\psi(d((fS)x_{n+1},Sv)))$.

$$\le F\left(\gamma(Tx_{n+1},Tv)\delta(Tx_{n+1},Tv),\phi\left(M(x_{n+1},v)\right)\right)\le F\left(\theta^2(Tx_{n+1},Tv),\phi\left(M(x_{n+1},v)\right)\right)$$

(By property of F and by (33)) $=F\left(\theta^2(y_n,u),\phi\left(M(x_{n+1},v)\right)\right)$.

$\Longrightarrow\eta^2(y_n,u)\psi\left(d(fy_{n+1},Sv)\right)\le\theta^2(y_n,u)\,\phi\left(M(x_{n+1},v)\right)\le\eta^2(y_n,u)\,\phi\left(M(x_{n+1},v)\right)$

(since $\theta(x,y)\le\eta(x,y),\forall x,y\in X$).$\Longrightarrow\psi\left(d(fy_{n+1},Sv)\right)\le\phi\left(M(x_{n+1},v)\right)$. (34)

$\Longrightarrow\psi\left(d(fy_{n+1},Sv)\right)<\psi\left(M(x_{n+1},v)\right)$ (since $\phi(t)<\psi(t),\forall t>0$). (35)

$\Longrightarrow d(fy_{n+1},Sv)<M(x_{n+1},v)$ (sinceψ is nondecreasing). (36)

where $M(x_{n+1},v)=\xi\left(max\left\{d(fy_n,u),d(fy_n,y_{n+1}),d(fu,Sv),\frac{1}{2}d(fy_n,Sv),d(fu,y_{n+1})\right\}\right)$.

Therefore $\lim_{n\to\infty}M(x_{n+1},v)=\xi\left(max\left\{0,0,d(fu,Sv),\frac{1}{2}d(fu,Sv),0\right\}\right)$ (since $\{y_n\}$ I-converges to u).

$\Longrightarrow\lim_{n\to\infty}M(x_{n+1},v)=\xi(d(fu,Sv))$. (37)

Taking the limit as $n\to\infty$ in (34), using continuity of ψ,ϕ, by (37) and I-convergence of $\{y_n\}$ to u we get $\psi\left(d(fu,Sv)\right)\le\phi(\xi(d(fu,Sv)))$. (38)

Let $d(fu,Sv)>0$, *i.e.*, $(fS)v\ne fu$. Then $\xi\left(d(fu,Sv)\right)>0$ and from (38) we get

$\psi\left(d(fu,Sv)\right)\le\phi\left(\xi\left(d(fu,Sv)\right)\right)<\psi\left(\xi\left(d(fu,Sv)\right)\right)\Longrightarrow d(fu,Sv)<\xi\left(d(fu,Sv)\right)<d(fu,Sv)$,

a contradiction. Therefore $(fS)v=fu=(fT)v$ (39)

Therefore u is a point of I-coincidence of S and T.

Let w be a point of I-coincidence of S and T such that $fw\ne fu$, *i.e.*, $d(fu,w)>0$. Then $\exists z\in X$ such that $(fS)z=(fT)z=fw$. (40). By (vii), (39), (40) we get

$\alpha(Tv,Tz)\ge\eta(Tv,Tz),\beta(Tv,Tz)\ge\eta(Tv,Tz),\gamma(Tv,Tz)\le\theta(Tv,Tz),\delta(Tv,Tz)\le\theta(Tv,Tz)$.

Therefore from (ii) we get (taking $x=v,y=z$) $H(\eta(Tv,Tz),\eta(Tv,Tz),\psi(d((fS)v,Sz)))\le H(\alpha(Tv,Tz),\beta(Tv,Tz),\psi(d((fS)v,Sz)))\le F\left(\gamma(Tv,Tz)\delta(Tv,Tz),\phi\left(M(v,z)\right)\right)$

$\le F(\theta^2(Tv,Tz),\phi(M(v,z)))$

$\Longrightarrow\eta^2(Tv,Tz)\psi\left(d\left((fS)v,Sz\right)\right)\le\theta^2(Tv,Tz)\phi\left(M(v,z)\right)\le\eta^2(Tv,Tz)\phi(M(v,z))$

$\Rightarrow \psi\left(d\left((fS)v, Sz\right)\right) \leq \phi(M(v,z)).$ (41)

$\Rightarrow \psi\left(d\left((fS)v, Sz\right)\right) < \psi(M(v,z)).$ (42) $\Rightarrow d\left((fS)v, Sz\right) < M(v,z).$ (43)

where $M(v,z) = \xi\left(max\left\{d(fu,w), 0, 0, \frac{1}{2}d(fu,w), d(fw,u)\right\}\right) = \xi(d(fu,w))$ (44)

Using (44) in (43) we get $d(fu,w) < \xi\left(d(fu,w)\right) < d(fu,w)$, a contradiction.

Therefore $fw = fu$. Therefore S and T have an I-unique point of I-coincidence.

Let S and T are weakly I-compatible. Then from (39) we get $(fSf)u = (fTf)u = a$ (say).
$\Rightarrow (fS)(fu) = (fT)(fu) = fa$. $\Rightarrow a$is a point of I-coincidence of S and T, so that $fa = fu$.
Therefore $(fS)(fu) = (fT)(fu) = fu = f(fu)$. $\Rightarrow fu$is a common I-fixed point of S and T. Since a common I-fixed point of S and T is a point of I-coincidence of S and T also, and S and T have an I-unique point of I-coincidence, hence S and T have an I-unique common I-fixed point in X.

Corollary(3.16) Let (X,d) be a metric space, and $S,T : X \to X$ satisfy(i) $S(X) \subseteq T(X)$.
(ii) $S$is$\left((\alpha,\beta,\eta),(\gamma,\delta,\theta),H,F\right)$-$T$-cyclic contractive map with $\theta(x,y) \leq \eta(x,y), \forall x,y \in X$ and $\theta(x,y) > 0$ whenever $x \neq y$.(iii) $S(X)$or$T(X)$ is closed in X.
(iv) $\exists x_1 \in X$ such that $\alpha(Tx_1,Tx_2) \geq \eta(Tx_1,Tx_2), \beta(Tx_1,Tx_2) \geq \eta(Tx_1,Tx_2)$,
$\gamma(Tx_1,Tx_2) \leq \theta(Tx_1,Tx_2), \delta(Tx_1,Tx_2) \leq \theta(Tx_1,Tx_2)$, where $x_2 \in X$ such that $Sx_1 = Tx_2$.
(since $Sx_1 \in S(X) \subseteq T(X), Sx_1 = Tx_2$ for some $x_2 \in X$).
(v) (X,d) is admissibly (α,β)-η-complete or subadmissibly (γ,δ)-θ-complete.
(vi) if$\{y_n\}$ be a sequence converging to u in X, and $\alpha(y_n,y_{n+1}) \geq \eta(y_n,y_{n+1})$ and $\beta(y_n,y_{n+1}) \geq \eta(y_n,y_{n+1}), \forall n \in \mathbb{N}$, then $\alpha(y_n,u) \geq \eta(y_n,u), \beta(y_n,u) \geq \eta(y_n,u)$;
and if $\gamma(y_n,y_{n+1}) \leq \theta(y_n,y_{n+1})$, $\delta(y_n,y_{n+1}) \leq \theta(y_n,y_{n+1}), \forall n \in \mathbb{N}$, then $\gamma(y_n,u) \leq \theta(y_n,u)$, $\delta(y_n,u) \leq \theta(y_n,u)$.
(vii) $\alpha(Tp,Tq) \geq \eta(Tp,Tq), \beta(Tp,Tq) \geq \eta(Tp,Tq), \gamma(Tp,Tq) \leq \theta(Tp,Tq)$,
$\delta(Tp,Tq) \leq \theta(Tp,Tq)$, whenever $Sp = Tp$, $Sq = Tq$.
Then S and T have a unique point of coincidence in X.
In addition, if S and T are weakly compatible, then S and T have a unique common fixed point in X.
Proof: Replacing f by the identity map of X in Theorem(3.15), we shall get the result.

Example(3.17) Let $X = \mathbb{R}$. Consider the idempotent map $f: X \to X$, given by $f(x) = |x|, \forall x \in X$.
Consider $d : X^2 \to \mathbb{R}$, given by $d(x,y) = \left||x| - |y|\right|, \forall x,y \in X$. Then (X,d,f) is an I-metric space.
Let $H : [0,\infty)^3 \to \mathbb{R}$ be given by $H(x,y,z) = xyz, \forall x,y,z \in [0,\infty)$. And $F : [0,\infty)^2 \to \mathbb{R}$ is given by $F(s,t) = st, \forall s,t \in [0,\infty)$.$\psi,\phi,\xi : [0,\infty) \to [0,\infty)$are given by $\psi(t) = 5t, \phi(t) = 4.9t$, $\xi(t) = 0.9t, \forall t \in [0,\infty)$.$S,T : X \to X$are given by $Sx = \frac{x}{81}, Tx = \frac{x}{9}$, $\forall x \in X$.
And $\alpha,\beta,\eta,\gamma,\delta,\theta : X^2 \to [0,\infty)$ are given by $\alpha(x,y) = |x| + |y| = \beta(x,y), \forall x,y \in X$,
$\eta(x,y) = 0.9(|x| + |y|), \gamma(x,y) = 0.88(|x| + |y|) = \delta(x,y), \theta(x,y) = 0.89(|x| + |y|), \forall x,y \in X$.
Let for $x,y \in X, \alpha(Tx,Ty) \geq \eta(Tx,Ty)$. Then $\frac{1}{9}(|x| + |y|) \geq \frac{0.9}{9}(|x| + |y|)$ (1)
This implies that $\frac{1}{81}(|x| + |y|) \geq \frac{0.9}{81}(|x| + |y|)$. $\Rightarrow \beta(Sx,Sy) \geq \eta(Sx,Sy)$.
Similarly, we can prove that, for all $x,y \in X, \beta(Tx,Ty) \geq \eta(Tx,Ty) \Rightarrow \alpha(Sx,Sy) \geq \eta(Sx,Sy)$.
Again for $x,y,z \in X$, let $\alpha(x,y) \geq \eta(x,y)$ and $\alpha(y,z) \geq \eta(y,z)$. Then $\alpha(x,z) \geq \eta(x,z)$.
Similarly, we can prove, for $x,y,z \in X$, $\beta(x,y) \geq \eta(x,y)$ and $\beta(y,z) \geq \eta(y,z) \Rightarrow \beta(x,z) \geq \eta(x,z)$.
Therefore S is a triangular (α,β)-η-T-cyclic admissible map.
Similarly, we can prove that S is a triangular (γ,δ)-θ-T-cyclic subadmissible map.
Clearly ξ is nondecreasing, continuous, $\xi(t) < t, \forall t > 0$ and $\xi(t) = 0$ iff $t = 0$.

Again ψ is nondecreasing, continuous and $\psi(t) = 0$iff $t = 0$.
Also, clearly ϕ is nondecreasing and right continuous (since it is continuous), and $\phi(t) < \psi(t), \forall t > 0$.
Also, obviously (F, H) is a modified upper class of type-II.
Clearly, $\theta(x, y) \leq \eta(x, y), \forall x, y \in X$, and $\theta(x, y) > 0$ when ever $fx = |x| \neq |y| = fy$.
Again it is easy to show that $S(X) \subseteq T(X)$. Here $S(X) = X = T(X)$ so that $S(X)$ or $T(X)$ is I-closed in X.
Let $x_1 \in X$. Then $Sx_1 = \frac{x_1}{81} = T(x_2)$, where $x_2 = \frac{x_1}{9}$, and obviously, $\alpha(Tx_1, Tx_2) \geq \eta(Tx_1, Tx_2)$, $\beta(Tx_1, Tx_2) \geq \eta(Tx_1, Tx_2)$, $\gamma(Tx_1, Tx_2) \leq \theta(Tx_1, Tx_2)$, $\delta(Tx_1, Tx_2) \leq \theta(Tx_1, Tx_2)$.
Now $H\big(\alpha(Tx, Ty), \beta(Tx, Ty), \psi(d((fS)x, Sy))\big) = \alpha\left(\frac{x}{9}, \frac{y}{9}\right)\beta\left(\frac{x}{9}, \frac{y}{9}\right)5d(f\left(\frac{x}{81}\right), \frac{y}{81})$

$$= \frac{(|x|+|y|)^2}{81} \cdot \frac{5}{81} \cdot \big||x| - |y|\big| = \frac{5}{6561}(|x| + |y|)^2\big||x| - |y|\big| \quad (2)$$

and $F(\gamma(Tx, Ty)\delta(Tx, Ty), \phi(M(x, y))) = \gamma\left(\frac{x}{9}, \frac{y}{9}\right)\delta\left(\frac{x}{9}, \frac{y}{9}\right)4.9M(x, y)$ (3)

where $M(x, y) = \xi\left(max\left\{\frac{1}{9}\big||x| - |y|\big|, \frac{1}{9}\left||x| - \frac{1}{9}|x|\right|, \frac{1}{9}\left||y| - \frac{1}{9}|y|\right|, \frac{1}{18}\left||x| - \frac{1}{9}|y|\right|, \frac{1}{9}\left||y| - \frac{1}{9}|x|\right|\right\}\right)$

$$= 0.9\, max\left\{\frac{1}{9}\big||x| - |y|\big|, \frac{8|x|}{81}, \frac{8|y|}{81}, \frac{1}{18}\left||x| - \frac{1}{9}|y|\right|, \frac{1}{9}\left||y| - \frac{1}{9}|x|\right|\right\}$$

$$= 0.1\, max\left\{\big||x| - |y|\big|, \frac{8|x|}{9}, \frac{8|y|}{9}, \frac{1}{2}\left||x| - \frac{1}{9}|y|\right|, \left||y| - \frac{1}{9}|x|\right|\right\}$$

$$= 0.1\, max\left\{\big||x| - |y|\big|, \frac{8|x|}{9}, \frac{8|y|}{9}, \left||y| - \frac{1}{9}|x|\right|\right\}$$

$$\left(\text{since } \frac{1}{2}\left||x| - \frac{1}{9}|y|\right| \leq \frac{1}{2}\left(\big||x| - |y|\big| + \frac{8|y|}{9}\right) \leq max\left\{\big||x| - |y|\big|, \frac{8|y|}{9}\right\}\right)$$

$= 0.1\, max\left\{\frac{8|x|}{9}, \frac{8|y|}{9}, \left||y| - \frac{1}{9}|x|\right|\right\}$.
Therefore (3) becomes $F(\gamma(Tx, Ty)\delta(Tx, Ty), \phi(M(x, y)))$

$$= \frac{(0.88)^2}{81} \times 0.49\,(|x| + |y|)^2 \cdot max\left\{\frac{8|x|}{9}, \frac{8|y|}{9}, \left||y| - \frac{1}{9}|x|\right|\right\}$$

$$= \frac{0.379456}{81}(|x| + |y|)^2 \cdot max\left\{\frac{8|x|}{9}, \frac{8|y|}{9}, \left||y| - \frac{1}{9}|x|\right|\right\} \quad (4)$$

Since $\frac{5}{6561} < \frac{0.379456}{81}$and $\big||x| - |y|\big| \leq \left||y| - \frac{1}{9}|x|\right| \leq max\left\{\frac{8|x|}{9}, \frac{8|y|}{9}, \left||y| - \frac{1}{9}|x|\right|\right\}$, from (2) and (4) we get

$$H\big(\alpha(Tx, Ty), \beta(Tx, Ty), \psi(d((fS)x, Sy))\big) \leq F\left(\gamma(Tx, Ty)\delta(Tx, Ty), \phi\big(M(x, y)\big)\right), \forall x, y \in X.$$

Therefore S is an $\big((\alpha, \beta, \eta), (\gamma, \delta, \theta), H, F\big)$-$T$-cyclic I-contractive map.
Let $\{x_n\}$ be any I-cauchy sequence in (X, d, f) such that $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$, $\beta(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1}), \forall n \in \mathbb{N}$ which obviously holds in this example. Then for any $\epsilon > 0$, $\exists n_\circ \in \mathbb{N}$such that $d(fx_n, x_m) < \epsilon, \forall m, n \geq n_\circ. \Longrightarrow \big||x_n| - |x_m|\big| < \epsilon, \forall m, n \geq n_\circ \Longrightarrow \{|x_n|\}$ I-converges in (X, d, f).(since$d$ can be considered as usual metric in $f(X) = [0, \infty)$ and $[0, \infty)$ is complete with respect to the usual metric).
$\Longrightarrow \{fx_n\}$ I-converges in $(X, d, f) \Longrightarrow \{x_n\}$ I-converges in (X, d, f).
Therefore (X, d, f) is admissibly (α, β)-η I-complete. Let $\{y_n\}$ be a sequence I-converging to u in (X, d, f) such that $\alpha(y_n, y_{n+1}) \geq \eta(y_n, y_{n+1})$,$\beta(y_n, y_{n+1}) \geq \eta(y_n, y_{n+1}), \forall n \in \mathbb{N}$ which holds in this example.Now $\alpha(y_n, u) = |y_n| + |u| = \beta(y_n, u) \geq 0.9\,(|y_n| + |u|) = \eta(y_n, u)$.
And assuming $\gamma(y_n, y_{n+1}) \leq \theta(y_n, y_{n+1})$, $\delta(y_n, y_{n+1}) \leq \theta(y_n, y_{n+1}), \forall n \in \mathbb{N}$ which holds in this example, we have $\gamma(y_n, u) = 0.88\,(|y_n| + |u|) = \delta(y_n, u) \leq 0.89\,(|y_n| + |u|) = \theta(y_n, u)$.
Condition (vii) obviously holds.Now for $x \in X$, x is an I-coincidence point of S and T iff

$(fS)x = (fT)x$ iff $\frac{|x|}{81} = \frac{|x|}{9}$ iff $x = 0$.Therefore $(fS)(0) = (fT)(0) = 0 = fy = |y|$iff $y = 0$.
Therefore $y = 0$ is the only point of I-coincidence of S and T, *i.e.*, here S and T have I-unique point of I-coincidence in X.
Now let x be an I-coincidence point of S and T in X. Then $(fS)x = (fT)x \Longrightarrow x = 0$.
Therefore $(fS)(0) = (fT)(0) = 0. \Longrightarrow (fT)(fS)(0) = (fT)(0) = 0 = (fS)(0) = (fS)(fT)(0)$.
$\Longrightarrow (fT)(fS)(0) = (fS)(fT)(0) \Longrightarrow S$ and T are weakly I-compatible.
Here 0 is the only common I-fixed point of S and T so that 0 is I-unique common I-fixed point of S and T. Therefore Theorem(3.15) is verified.
Also by this example, Definition(3.12), Definition(3.13), Definition(3.14) are exampled.

Note(3.18) Let $\eta(x, y) = 1 = \theta(x, y), \forall x \ and \ y \in X$. In this case
(i)Swill be called (triangular) (α, β)-T-cyclic admissible map in Definition(3.7)(A).
Swill be called (triangular) (α, β)-T-cyclic subadmissible map in Definition(3.7)(B).
(ii)Swill be called $\big((\alpha, \beta), (\gamma, \delta), H, F\big)$-$T$-cyclic I-contractive map in Definition(3.13); and if $f$be the identity map on X in Definition(3.13), then S will be called $\big((\alpha, \beta), (\gamma, \delta), H, F\big)$-$T$-cyclic contractivemap.
(iii) In Definition(3.14)(A), (X, d, f) will be called admissibly (α, β)-I-complete; and in Definition(3.14)(B), (X, d, f) will be called subadmissibly (α, β)-I-complete.
Again if f be the identity map in these definitions, then the metric space (X, d) will be called admissibly(α, β)-complete and subadmissibly (α, β)-complete respectively.

Now we have the following corollaries immediately come from Theorem(3.15).

Corollary(3.19) Let (X, d, f) be an I-metric space, and $S, T : X \to X$ satisfy(i) $S(X) \subseteq T(X)$.
(ii) $S$is$\big((\alpha, \beta), (\gamma, \delta), H, F\big)$-$T$-cyclic I-contractive map.(iii) $S(X)$or$T(X)$ is I-closed in X.
(iv) $\exists x_1 \in X$such that $\alpha(Tx_1, Tx_2) \geq 1, \beta(Tx_1, Tx_2) \geq 1, \gamma(Tx_1, Tx_2) \leq 1, \delta(Tx_1, Tx_2) \leq 1$,
where$x_2 \in X$ such that $Sx_1 = Tx_2$. (since $Sx_1 \in S(X) \subseteq T(X), Sx_1 = Tx_2$ for some $x_2 \in X$).
(v) (X, d, f) is admissibly (α, β) I-complete or subadmissibly (γ, δ) I-complete.
(vi) if$\{y_n\}$ be a sequence I-converging to u in X, and $\alpha(y_n, y_{n+1}) \geq 1$, $\beta(y_n, y_{n+1}) \geq 1, \forall n \in \mathbb{N}$,
then $\alpha(y_n, u) \geq 1, \beta(y_n, u) \geq 1$; and if $\gamma(y_n, y_{n+1}) \leq 1$, $\delta(y_n, y_{n+1}) \leq 1, \forall n \in \mathbb{N}$, then
$\gamma(y_n, u) \leq 1, \delta(y_n, u) \leq 1$.
(vii) $(Tp, Tq) \geq 1, \beta(Tp, Tq) \geq 1, \gamma(Tp, Tq) \leq 1, \delta(Tp, Tq) \leq 1$, whenever $(fS)p = (fT)p$,
$(fS)q = (fT)q$.
Then S and T have an I-unique point of I-coincidence in X.In addition, if S and T are weakly I-compatible, then S and T have an I-unique common I-fixed point in X.

Corollary(3.20) Let (X, d) be a metric space, and $S, T : X \to X$ satisfy(i) $S(X) \subseteq T(X)$.
(ii) $S$is$\big((\alpha, \beta), (\gamma, \delta), H, F\big)$-$T$-cyclic contractive map.(iii) $S(X)$or$T(X)$ is closed in X.
(iv) $\exists x_1 \in X$such that $\alpha(Tx_1, Tx_2) \geq 1, \beta(Tx_1, Tx_2) \geq 1, \gamma(Tx_1, Tx_2) \leq 1, \delta(Tx_1, Tx_2) \leq 1$,
where$x_2 \in X$ such that $Sx_1 = Tx_2$. (since $Sx_1 \in S(X) \subseteq T(X), Sx_1 = Tx_2$ for some $x_2 \in X$).
(v) (X, d) is admissibly (α, β) complete or subadmissibly (γ, δ) complete.
(vi) if $\{y_n\}$ be a sequence converging to u in X, and $\alpha(y_n, y_{n+1}) \geq 1$, $\beta(y_n, y_{n+1}) \geq 1, \forall n \in \mathbb{N}$,
then $\alpha(y_n, u) \geq 1, \beta(y_n, u) \geq 1$; if $\gamma(y_n, y_{n+1}) \leq 1$, $\delta(y_n, y_{n+1}) \leq 1, \forall n \in \mathbb{N}$, then
$\gamma(y_n, u) \leq 1, \delta(y_n, u) \leq 1$.
(vii) $(Tp, Tq) \geq 1, \beta(Tp, Tq) \geq 1, \gamma(Tp, Tq) \leq 1, \delta(Tp, Tq) \leq 1$, whenever $Sp = Tp$, $Sq = Tq$.
Then S and T have a unique point of coincidence in X.
In addition, if S and T are weakly compatible, then S and T have a unique common fixed point in X.

Note(3.21) Ansari A.H. *et al.* [2] introduced$T$-cyclic (α,β,H,F)-rational contraction, utilizing which a common fixed point result has been proved in metric spaces, stated as:
"**Definition: [2]** Let (X,d) be a metric space and let S be a T-cyclic (α,β)-admissible map and a cyclic(λ,γ)-subadmissible map. $S$is said to be a T-cyclic (α,β,H,F)-rational contractive map if
$H(\alpha(Tx),\beta(Ty),\phi(d(Sx,Sy)))\leq F\left(\gamma(Tx)\lambda(Ty),\eta\big(N(x,y)\big)\right),\forall x,y\in X$, where
$N(x,y)=\psi\left(d(Tx,Ty),\frac{1}{2}d(Tx,Sy),d(Ty,Sx),\frac{[1+d(Tx,Sx)]d(Ty,Sy)}{1+d(Tx,Ty)}\right)$, for $\psi\in\mathcal{X}$ (as per Definition(1.11)), ϕ is an alternating distance function, pair (F,H) is an upper class of type-II, $\eta:[0,\infty)\to[0,\infty)$is nondecreasing and right continuous with $\phi(t)>\eta(t),\forall t>0$" and
"**Theorem: [2]** Let (X,d) be a complete metric space and let $S,T:X\to X$ such that $S(X)\subset T(X)$. Let S be a T-cyclic (α,β,H,F)-rational contractive map, $T(X)$ is closed in X and
(i) $\exists x_\circ\in X$ such that $\alpha(Tx_\circ)\geq 1,\beta(Tx_\circ)\geq 1,\lambda(Tx_\circ)\leq 1,\gamma(Tx_\circ)\leq 1$.
(ii) if$\{x_n\}$ converges to x in X and $\beta(x_n)\geq 1,\forall n\in\mathbb{N}$, then $\beta(x)\geq 1$, and $\gamma(x_n)\leq 1,\forall n\in\mathbb{N}$, then $\gamma(x)\leq 1$.
(iii) $\alpha(Tu)\geq 1,\beta(Tv)\geq 1,\lambda(Tu)\leq 1,\gamma(Tv)\leq 1$ when ever $Su=Tu$ and $Sv=Tv$. Then S and T have a unique point of coincidence in X. In addition, if S and T are weakly compatible, then S and T have a unique common fixed point in X."

Generalizing this contraction, here we shall prove a common fixed point result in I-metric Spaces with an example, and then get an analogous result in metric spaces as a corollary.

Definition(3.22) Let (X,d,f) be an I-metric space, $S,T:X\to X$, $\alpha,\beta,\eta,\gamma,\delta,\theta:\ X^2\to[0,\infty)$ such that S is triangular (α,β)-η-T-cyclic admissible and triangular (γ,δ)-θ-T-cyclic subadmissible. S is said to be $\big((\alpha,\beta,\eta),(\gamma,\delta,\theta),H,F\big)$-T-cyclic rational I-contractive map if
(i) $H(\alpha(Tx,Ty),\beta(Tx,Ty),\psi(d((fS)x,Sy)))\leq F\left(\gamma(Tx,Ty)\delta(Tx,Ty),\phi\big(M(x,y)\big)\right),\forall x,y\in X$,
Where for some constants a,b,p,q with $0\leq a,b,p\leq 1,q>0$, $M(x,y)=\xi\left(max\left\{d\big((fT)x,Ty\big),\frac{1}{2}d\big((fT)x,Sy\big),d\big((fT)y,Sx\big),ad\big((fT)x,Sx\big),bd\big((fT)y,Sy\big),\frac{p[q+d((fT)x,Sx)]d((fT)y,Sy)}{q+d((fT)x,Ty)}\right\}\right)$
(ii) $\psi:[0,\infty)\to[0,\infty)$ is nondecreasing, continuous and $\psi(t)=0$ iff $t=0$.
(iii) $\phi:[0,\infty)\to[0,\infty)$ is nondecreasing, right continuous and $\phi(t)<\psi(t),\forall t>0$.
(iv) $\xi:[0,\infty)\to[0,\infty)$ is nondecreasing, continuous, $\xi(t)<t$ and $\forall t>0$; and $\xi(t)=0$ iff $t=0$.
(v) the pair (F,H) is a modified upper class of type-II.
If f be the identity map on X, then S is called $\big((\alpha,\beta,\eta),(\gamma,\delta,\theta),H,F\big)$-T-cyclic rational contractive map.

Theorem(3.23) Let (X,d,f) be an I-metric space, and $S,T:X\to X$ satisfy(i) $S(X)\subseteq T(X)$.
(ii) $S$is$\big((\alpha,\beta,\eta),(\gamma,\delta,\theta),H,F\big)$-T-cyclic rational I-contractive map with $\theta(x,y)\leq\eta(x,y),\forall x,y\in X$, and $\theta(x,y)>0$ whenever $fx\neq fy$.(iii) $S(X)$or$T(X)$ is I-closed in X.
(iv) $\exists x_\circ\in X$such that $\alpha(Tx_\circ,Sx_\circ)\geq\eta(Tx_\circ,Sx_\circ),\beta(Tx_\circ,Sx_\circ)\geq\eta(Tx_\circ,Sx_\circ)$, $\gamma(Tx_\circ,Sx_\circ)\leq\theta(Tx_\circ,Sx_\circ),\delta(Tx_\circ,Sx_\circ)\leq\theta(Tx_\circ,Sx_\circ)$.
(v) $(X,d,f)$is admissibly (α,β)-η I-complete or subadmissibly (γ,δ)-θ I-complete.
(vi) If $\{y_n\}_{n\geq 0}$ be a sequence I-converging to u in X, and $\alpha(y_n,y_{n+1})\geq\eta(y_n,y_{n+1})$, $\beta(y_n,y_{n+1})\geq\eta(y_n,y_{n+1}),\forall n\in\mathbb{N}$, then $\alpha(y_n,u)\geq\eta(y_n,u)$, $\beta(y_n,u)\geq\eta(y_n,u)$; and if $\gamma(y_n,y_{n+1})\leq\theta(y_n,y_{n+1})$, $\delta(y_n,y_{n+1})\leq\theta(y_n,y_{n+1}),\forall n\in\mathbb{N}$, then $\gamma(y_n,u)\leq\theta(y_n,u)$, $\delta(y_n,u)\leq\theta(y_n,u)$.
(vii) $\alpha(Tu,Tv)\geq\eta(Tu,Tv),\beta(Tu,Tv)\geq\eta(Tu,Tv),\gamma(Tu,Tv)\leq\theta(Tu,Tv)$,

$\delta(Tu,Tv) \le \theta(Tu,Tv)$when ever $(fS)u = (fT)u, (fS)v = (fT)v$.
Then S and T have an I-unique point of I-coincidence in X.In addition, if S and T are weakly I-compatible, then S and T have an I-unique common I-fixed point in X.

Proof: Let $y_\circ = Sx_\circ = Tx_1$ (since $Sx_\circ \in S(X) \subseteq T(X), Sx_\circ = Tx_1$ for some $x_1 \in X$), $y_1 = Sx_1 = Tx_2$ (since $Sx_1 \in S(X) \subseteq T(X), Sx_1 = Tx_2$ for some $x_2 \in X$), and in general, $y_n = Sx_n = Tx_{n+1}$, $\forall n \in \mathbb{N} \cup \{0\}$. If $fy_m = fy_{m-1}$ for some $m \in \mathbb{N}$, then y_m is a point of I-coincidence of S and T.
Let $fy_n \ne fy_{n-1}, \forall n \in \mathbb{N}$. Since S is triangular (α,β)- η-T-cyclic admissible and $\alpha(Tx_\circ,Tx_1) \ge \eta(Tx_\circ,Tx_1)$, hence $\beta(Sx_\circ,Sx_1) = \beta(Tx_1,Tx_2) \ge \eta(Sx_\circ,Sx_1) = \eta(Tx_1,Tx_2)$.
This implies that $\alpha(Sx_1,Sx_2) = \alpha(Tx_2,Tx_3) \ge \eta(Sx_1,Sx_2) = \eta(Tx_2,Tx_3)$.
Proceeding in this way, we have, in general, $\alpha(Tx_{2n},Tx_{2n+1}) \ge \eta(Tx_{2n},Tx_{2n+1})$, $\beta(Tx_{2n+1},Tx_{2n+2}) \ge \eta(Tx_{2n+1},Tx_{2n+2}), \forall n \in \mathbb{N} \cup \{0\}$.
Similarly, S is triangular (α,β)- η-T-cyclic admissible and $\beta(Tx_\circ,Tx_1) \ge \eta(Tx_\circ,Tx_1)$ implies that $\alpha(Tx_{2n+1},Tx_{2n+2}) \ge \eta(Tx_{2n+1},Tx_{2n+2}),\ \beta(Tx_{2n},Tx_{2n+1}) \ge \eta(Tx_{2n},Tx_{2n+1}), \forall n \in \mathbb{N} \cup \{0\}$.
Therefore $\alpha(Tx_n,Tx_{n+1}) \ge \eta(Tx_n,Tx_{n+1}),\ \beta(Tx_n,Tx_{n+1}) \ge \eta(Tx_n,Tx_{n+1}), \forall n \in \mathbb{N} \cup \{0\}$. (1)
Again since S is triangular (γ,δ)-θ-T-cyclic subadmissible and $\gamma(Tx_\circ,Tx_1) \le \theta(Tx_\circ,Tx_1)$, $\delta(Tx_\circ,Tx_1) \le \theta(Tx_\circ,Tx_1)$, similarly, we shall get
$\gamma(Tx_n,Tx_{n+1}) \le \theta(Tx_n,Tx_{n+1}),\ \delta(Tx_n,Tx_{n+1}) \le \theta(Tx_n,Tx_{n+1}), \forall n \in \mathbb{N} \cup \{0\}$. (2)
Therefore $\gamma(Tx_n,Tx_{n+1})\delta(Tx_n,Tx_{n+1}) \le \theta^2(Tx_n,Tx_{n+1}), \forall n \in \mathbb{N} \cup \{0\}$. (3)
By (ii) we get (taking $x = x_n, y = x_{n+1}$)$H\left(\eta(y_{n-1},y_n),\eta(y_{n-1},y_n),\psi\big(d(fy_n,y_{n+1})\big)\right)$
$= H(\eta(Tx_n,Tx_{n+1}),\eta(Tx_n,Tx_{n+1}),\psi(d((fS)x_n,Sx_{n+1})))$
$\le H(\alpha(Tx_n,Tx_{n+1}),\beta(Tx_n,Tx_{n+1}),\psi(d((fS)x_n,Sx_{n+1})))$ (By property of H).
$\le F(\gamma(Tx_n,Tx_{n+1})\delta(Tx_n,Tx_{n+1}),\phi(M(x_n,x_{n+1})))$
$\le F\left(\theta^2(Tx_n,Tx_{n+1}),\phi\big(M(x_n,x_{n+1})\big)\right) = F(\theta^2(y_{n-1},y_n),\phi(M(x_n,x_{n+1})))$.
$\Rightarrow \eta^2(y_{n-1},y_n)\psi\big(d(fy_n,y_{n+1})\big) \le \theta^2(y_{n-1},y_n)\phi(M(x_n,x_{n+1}))$. (4)
$\Rightarrow \eta^2(y_{n-1},y_n)\psi\big(d(fy_n,y_{n+1})\big) \le \eta^2(y_{n-1},y_n)\phi(M(x_n,x_{n+1}))$. (5)
(since $\theta(x,y) \le \eta(x,y), \forall x,y \in X$).
$\Rightarrow \psi\big(d(fy_n,y_{n+1})\big) \le \phi(M(x_n,x_{n+1}))$. (6)
(since $0 < \theta(y_{n-1},y_n) \le \eta(y_{n-1},y_n)$, as $fy_{n-1} \ne fy_n, \forall n \in \mathbb{N}$).
$\Rightarrow \psi\big(d(fy_n,y_{n+1})\big) < \psi(M(x_n,x_{n+1}))$ (since $\phi(t) < \psi(t), \forall t > 0$). (7)
$\Rightarrow d(fy_n,y_{n+1}) < M(x_n,x_{n+1}), \forall n \in \mathbb{N}$ (since ψ is nondecreasing). (8)
where $M(x_n,x_{n+1}) =$

$$\xi\left(max\left\{d(fy_{n-1},y_n),\frac{1}{2}d(fy_{n-1},y_{n+1}),ad(fy_{n-1},y_n),bd(fy_n,y_{n+1}),\frac{p[q+d(fy_{n-1},y_n)]d(fy_n,y_{n+1})}{q+d(fy_{n-1},y_n)}\right\}\right)$$

$$\le \xi\left(max\left\{d(fy_{n-1},y_n),\tfrac{1}{2}(d(fy_{n-1},y_n)+d(fy_n,y_{n+1})),bd(fy_n,y_{n+1}),pd(fy_n,y_{n+1})\right\}\right). \quad (9)$$

(since ξ is nondecreasing and $0 \le a \le 1$)

If $d(fy_{n-1},y_n) < d(fy_n,y_{n+1})$ for some $n \in \mathbb{N}$, then from (9) we get
$M(x_n,x_{n+1}) \le \xi\big(d(fy_n,y_{n+1}),bd(fy_n,y_{n+1}),pd(fy_n,y_{n+1})\big) = \xi(d(fy_n,y_{n+1}))$. (10)
(since ξ is nondecreasing, $fy_n \ne y_{n+1}$ and $0 \le b,p \le 1$).
From (8) and (10) we get $d(fy_n,y_{n+1}) < \xi\big(d(fy_n,y_{n+1})\big) < d(fy_n,y_{n+1})$, a contradiction.
(since$\xi(t) < t$ for all $t > 0$).
Therefore $d(fy_n,y_{n+1}) \le d(fy_{n-1},y_n), \forall n \in \mathbb{N}$. (11). Then $M(x_n,x_{n+1}) \le \xi\big(d(fy_{n-1},y_n)\big)$. (12)
Therefore $\{d(fy_{n-1},y_n)\}$ is a decreasing sequence of nonnegative real numbers so that it converges to some nonnegative real number r. Let $r > 0$. From (12) we get

$\lim_{n\to\infty} M(x_n, x_{n+1}) \leq \xi(r)$ (By continuity of ξ). (13)
Taking the limit as $n \to \infty$ in (6), using continuity of ψ, ϕ we get
$\psi(r) \leq \phi(\xi(r))$ (since ϕ is nondecreasing) $< \psi(\xi(r))$ (since $\phi(t) < \psi(t), \forall t > 0$).
$\Longrightarrow r < \xi(r)$ (since ψ is nondecreasing) $< r$ (since$\xi(t) < t$ for all $t > 0$), a contradiction.
Therefore $r = 0$. Thus $\lim_{n\to\infty} d(fy_{n-1}, y_n) = 0$. (14)
We claim that $\{y_{n-1}\}$ is I-cauchy in X. If not, then $\exists \epsilon > 0$, two strictly increasing sequences of positive integers $\{m_k\}$ and $\{n_k\}$ such that for every $k \in \mathbb{N}$, m_k is the smallest positive integer for which
$m_k > n_k > k$ and $d(fy_{n_k}, y_{m_k}) \geq \epsilon$. (15). Then $d(fy_{n_k}, y_{m_k-1}) < \epsilon$. (16)
By (14), (15), (16) and triangle inequality, subsequently we can easily prove that
$\lim_{k\to\infty} d(fy_{n_k}, y_{m_k}) = \epsilon$ (17) $\lim_{k\to\infty} d(fy_{n_k}, y_{m_k-1}) = \epsilon$ (18)
$\lim_{k\to\infty} d(fy_{n_k-1}, y_{m_k-1}) = \epsilon$ (19) $\lim_{k\to\infty} d(fy_{n_k-1}, y_{m_k}) = \epsilon$ (20)
Now we shall prove that $\forall n \in \mathbb{N} \cup \{0\}, \forall p \in \mathbb{N}$, $\alpha(y_n, y_{n+p}) \geq \eta(y_n, y_{n+p})$, $\beta(y_n, y_{n+p}) \geq \eta(y_n, y_{n+p})$, $\gamma(y_n, y_{n+p}) \leq \theta(y_n, y_{n+p})$, $\delta(y_n, y_{n+p}) \leq \theta(y_n, y_{n+p})$ by induction on p.
For $p = 1$, the result holds by (1) and (2).
Let $p = 2$. By (1), we have $\alpha(y_n, y_{n+1}) \geq \eta(y_n, y_{n+1})$and $\alpha(y_{n+1}, y_{n+2}) \geq \eta(y_{n+1}, y_{n+2})$.
This implies that $\alpha(y_n, y_{n+2}) \geq \eta(y_n, y_{n+2})$ (since S is a triangular (α, β)-η-T-cyclic admissible).
Similarly, we shall get $\beta(y_n, y_{n+2}) \geq \eta(y_n, y_{n+2}), \gamma(y_n, y_{n+2}) \leq \theta(y_n, y_{n+2})$ and
$\delta(y_n, y_{n+2}) \leq \theta(y_n, y_{n+2})$. Therefore the result holds for $p = 2$.
Let the result hold for any positive integer $p \geq 2$.
Then $\alpha(y_n, y_{n+p}) \geq \eta(y_n, y_{n+p})$, $\beta(y_n, y_{n+p}) \geq \eta(y_n, y_{n+p})$, $\gamma(y_n, y_{n+p}) \leq \theta(y_n, y_{n+p})$, $\delta(y_n, y_{n+p}) \leq \theta(y_n, y_{n+p})$.
Now by (1) we have $\alpha(y_{n+p}, y_{n+p+1}) \geq \eta(y_{n+p}, y_{n+p+1})$. Also $\alpha(y_n, y_{n+p}) \geq \eta(y_n, y_{n+p})$.
Since S is a triangular (α, β)-η-T-cyclic admissible, hence $\alpha(y_n, y_{n+p+1}) \geq \eta(y_n, y_{n+p+1})$.
Similarly, we shall get $\beta(y_n, y_{n+p+1}) \geq \eta(y_n, y_{n+p+1}), \gamma(y_n, y_{n+p+1}) \leq \theta(y_n, y_{n+p+1})$ and
$\delta(y_n, y_{n+p+1}) \leq \theta(y_n, y_{n+p+1})$.
Therefore by mathematical induction, we have the result.
Therefore we have $\alpha(y_{n_k-1}, y_{m_k-1}) \geq \eta(y_{n_k-1}, y_{m_k-1})$ (21)
$\beta(y_{n_k-1}, y_{m_k-1}) \geq \eta(y_{n_k-1}, y_{m_k-1})$ (22) $\gamma(y_{n_k-1}, y_{m_k-1}) \leq \theta(y_{n_k-1}, y_{m_k-1})$ (23)
$\delta(y_{n_k-1}, y_{m_k-1}) \leq \theta(y_{n_k-1}, y_{m_k-1})$ (24). From (ii) we get (taking $x = x_{n_k}, y = x_{m_k}$)
$H(\eta(y_{n_k-1}, y_{m_k-1}), \eta(y_{n_k-1}, y_{m_k-1}), \psi(d(fy_{n_k}, y_{m_k})))$
$= H(\eta(Tx_{n_k}, Tx_{m_k}), \eta(Tx_{n_k}, Tx_{m_k}), \psi(d((fS)x_{n_k}, Sx_{m_k})))$
$\leq H(\alpha(Tx_{n_k}, Tx_{m_k}), \beta(Tx_{n_k}, Tx_{m_k}), \psi(d((fS)x_{n_k}, Sx_{m_k})))$ (By (21), (22) and property of H).
$\leq F(\gamma(Tx_{n_k}, Tx_{m_k})\delta(Tx_{n_k}, Tx_{m_k}), \phi(M(x_{n_k}, x_{m_k})))$
$\leq F(\theta^2(Tx_{n_k}, Tx_{m_k}), \phi(M(x_{n_k}, x_{m_k})))$ (By (23), (24) and property of F).
$= F(\theta^2(y_{n_k-1}, y_{m_k-1}), \phi(M(x_{n_k}, x_{m_k})))$.
$\Longrightarrow \eta^2(y_{n_k-1}, y_{m_k-1})\, \psi\left(d(fy_{n_k}, y_{m_k})\right) \leq \theta^2(y_{n_k-1}, y_{m_k-1})\phi(M(x_{n_k}, x_{m_k}))$ (25)
$\Longrightarrow \eta^2(y_{n_k-1}, y_{m_k-1})\, \psi\left(d(fy_{n_k}, y_{m_k})\right) \leq \eta^2(y_{n_k-1}, y_{m_k-1})\phi(M(x_{n_k}, x_{m_k}))$ (26)
(since $\theta(x, y) \leq \eta(x, y)$).$\Longrightarrow \psi\left(d(fy_{n_k}, y_{m_k})\right) \leq \phi(M(x_{n_k}, x_{m_k}))$. (27)
$\Longrightarrow \psi\left(d(fy_{n_k}, y_{m_k})\right) < \psi(M(x_{n_k}, x_{m_k}))$ (since $\phi(t) < \psi(t), \forall t > 0$) (28)
$\Longrightarrow d(fy_{n_k}, y_{m_k}) < M(x_{n_k}, x_{m_k})$ (sinceψ is nodecreasing) (29)

where$M(x_{n_k}, x_{m_k}) =$

$$\xi\left(max\left\{d(fy_{n_k-1}, y_{m_k-1}), \frac{1}{2}d(fy_{n_k-1}, y_{m_k}), d(fy_{m_k-1}, y_{n_k}), ad(fy_{n_k-1}, y_{n_k}),\right.\right.$$

$\left.\left. bd(fy_{m_k-1}, y_{m_k}), \frac{p[q+d(fy_{n_k-1},y_{n_k})]d(fy_{m_k-1},y_{m_k})}{q+d(fy_{n_k-1},y_{m_k-1})}\right\}\right).$

Therefore $\lim_{k\to\infty} M(x_{n_k}, x_{m_k}) = \xi\left(max\left\{\epsilon, \frac{\epsilon}{2}, \epsilon, 0,0,0\right\}\right) = \xi(\epsilon)$ (30)

(By (14), (18), (19), (20) and continuity of ξ).

Taking the limit as $k \to \infty$ in (27), and using continuity of ψ, ϕ and by (17), (30) we get
$\psi(\epsilon) \leq \phi(\xi(\epsilon)) < \psi(\xi(\epsilon))$ (since $\epsilon > 0 \Longrightarrow \xi(\epsilon) > 0$ and $\phi(t) < \psi(t), \forall t > 0$).
$\Longrightarrow \epsilon < \xi(\epsilon) < \epsilon$ (since ψ is nondecreasing and $\xi(\epsilon) < \epsilon$, as $\epsilon > 0$), a contradiction.
Therefore $\{y_{n-1}\}$ is I-cauchy in X.
Now since $(X, d, f)$is either admissibly (α, β)-η I-complete or subadmissibly (γ, δ)-θ I-complete, hence$\{y_{n-1}\}$ I-converges to some point $u \in X$, *i.e.*, $\{Sx_{n-1}\} = \{Tx_n\} = \{y_{n-1}\}$ I-converges to u.
Since $S(X)$ or $T(X)$ is I-closed, hence $u \in S(X) \cup T(X) = T(X)$ (since $S(X) \subseteq T(X)$).
Therefore $u = Tv$ for some $v \in X$. (31). Now by (vi), (1) and (2) we get
$\alpha(y_{n-1}, u) \geq \eta(y_{n-1}, u), \beta(y_{n-1}, u) \geq \eta(y_{n-1}, u), \gamma(y_{n-1}, u) \leq \theta(y_{n-1}, u),$
$\delta(y_{n-1}, u) \leq \theta(y_{n-1}, u), \forall n \in \mathbb{N}.$ (32)
i.e., $\alpha(Tx_n, Tv) \geq \eta(Tx_n, Tv), \beta(Tx_n, Tv) \geq \eta(Tx_n, Tv), \gamma(Tx_n, Tv) \leq \theta(Tx_n, Tv),$
$\delta(Tx_n, Tv) \leq \theta(Tx_n, Tv), \forall\, n \in \mathbb{N}.$ (33)
Now by (ii) we get (taking $= x_n$, $y = v$)$H\left(\eta(y_{n-1}, u), \eta(y_{n-1}, u), \psi\left(d(fy_n, Sv)\right)\right)$
$\leq H(\alpha(y_{n-1}, u), \beta(y_{n-1}, u), \psi(d(fy_n, Sv)))$(By property of H and (32)).$=$
$H(\alpha(Tx_n, Tv), \beta(Tx_n, Tv), \psi(d((fS)x_n, Sv))) \leq F\left(\gamma(Tx_n, Tv)\delta(Tx_n, Tv), \phi\left(M(x_n, v)\right)\right)$
$\leq F\left(\theta^2(Tx_n, Tv), \phi\left(M(x_n, v)\right)\right)$ (By property of F and by (33)) $= F\left(\theta^2(y_{n-1}, u), \phi\left(M(x_n, v)\right)\right)$.
$\Longrightarrow \eta^2(y_{n-1}, u)\psi\left(d(fy_n, Sv)\right) \leq \theta^2(y_{n-1}, u)\, \phi\left(M(x_n, v)\right) \leq \eta^2(y_{n-1}, u)\, \phi\left(M(x_n, v)\right)$
(since $\theta(x, y) \leq \eta(x, y), \forall x, y \in X$).$\Longrightarrow \psi\left(d(fy_n, Sv)\right) \leq \phi\left(M(x_n, v)\right)$. (34)
$\Longrightarrow \psi\left(d(fy_n, Sv)\right) < \psi\left(M(x_n, v)\right)$ (since $\phi(t) < \psi(t), \forall t > 0$). (35)
$\Longrightarrow d(fy_n, Sv) < M(x_n, v)$ (since ψ is nondecreasing). (36)
where$M(x_n, v) =$
$\xi\left(max\left\{d(fy_{n-1}, u), \frac{1}{2}d(fy_{n-1}, Sv), d(fu, y_n), ad(fy_{n-1}, y_n), bd(fu, Sv), \frac{p[q+d(fy_{n-1},y_n)]d(fu,Sv)}{q+d(fy_{n-1},u)}\right\}\right).$
Therefore $\lim_{n\to\infty} M(x_n, v) = \xi\left(max\left\{0, \frac{1}{2}d(fu, Sv), 0,0, bd(fu, Sv), \frac{p[q+0)]d(fu,Sv)}{q+0}\right\}\right)$

(since$\{y_n\}$ I-converges to u and ξ is continuous).

$= \xi\left(max\left\{\frac{1}{2}d(fu, Sv), bd(fu, Sv), pd(fu, Sv)\right\}\right) \leq \xi(d(fu, Sv)).$
(since$0 \leq b, p \leq 1$ and ξ is nondecreasing). $\Longrightarrow \lim_{n\to\infty} M(x_n, v) \leq \xi(d(fu, Sv))$ (37)
Taking the limit as $n \to \infty$ in (36), by (37) and I-convergence of $\{y_n\}$ to $u$we get
$d(fu, Sv) \leq \xi(d(fu, Sv)).$ (38)
Let $d(fu, Sv) > 0$, *i.e.*, $(fS)v \neq fu$. Then $\xi\left(d(fu, Sv)\right) > 0$ and from (38) we get
$d(fu, Sv) \leq \xi\left(d(fu, Sv)\right) < d(fu, Sv)$, a contradiction. Therefore $(fS)v = fu = (fT)v$(39)
Therefore u is a point of I-coincidence of S and T.

Let w be a point of I-coincidence of S and T such that $fw \neq fu$, *i.e.*, $d(fu, w) > 0$. Then $\exists z \in X$ with $(fS)z = (fT)z = fw$. (40). By (vii), (39), (40) we get

$\alpha(Tv,Tz) \geq \eta(Tv,Tz), \beta(Tv,Tz) \geq \eta(Tv,Tz), \gamma(Tv,Tz) \leq \theta(Tv,Tz), \delta(Tv,Tz) \leq \theta(Tv,Tz)$.
Therefore from (ii) we get (taking $x = v, y = z$)

$$H(\eta(Tv,Tz), \eta(Tv,Tz), \psi(d((fS)v,Sz))) \leq H(\alpha(Tv,Tz), \beta(Tv,Tz), \psi(d((fS)v,Sz)))$$
$$\leq F\left(\gamma(Tv,Tz)\delta(Tv,Tz), \phi\big(M(v,z)\big)\right) \leq F(\theta^2(Tv,Tz), \phi(M(v,z)))$$

$$\Rightarrow \eta^2(Tv,Tz)\psi\left(d\big((fS)v,Sz\big)\right) \leq \theta^2(Tv,Tz)\phi\big(M(v,z)\big) \leq \eta^2(Tv,Tz)\phi(M(v,z))$$

$$\Rightarrow \psi\left(d\big((fS)v,Sz\big)\right) \leq \phi(M(v,z)). \quad (41)$$

$$\Rightarrow \psi\left(d\big((fS)v,Sz\big)\right) < \psi(M(v,z)). \quad (42). \Rightarrow d\big((fS)v,Sz\big) < M(v,z). \quad (43)$$

where$M(v,z) = \xi\left(max\left\{d(fu,w), \frac{1}{2}d(fu,w), d(fw,u), 0,0,0\right\}\right) = \xi(d(fu,w))$ (44)

Using (44) in (43) we get $(fu,w) < \xi\big(d(fu,w)\big) < d(fu,w)$, a contradiction.
Therefore $fw = fu$. Therefore S and T have an I-unique point of I-coincidence.

Let S and T are weakly I-compatible. Then from (39) we get $(fSf)u = (fTf)u = c$ (say).
$\Rightarrow (fS)(fu) = (fT)(fu) = f(c)$. $\Rightarrow c$is a point of I-coincidence of S and T, so that $fc = fu$. Therefore $(fS)(fu) = (fT)(fu) = fu = f(fu)$. $\Rightarrow fu$is a common I-fixed point of S and T. Since a common I-fixed point of S and T is a point of I-coincidence of S and T also, and S and T have an I-unique point of I-coincidence, hence S and T have an I-unique common I-fixed point in X.

Corollary(3.24) Let (X,d) be a metric space, and $S,T : X \to X$ satisfy(i) $S(X) \subseteq T(X)$.
(ii) $S$is$\big((\alpha,\beta,\eta),(\gamma,\delta,\theta),H,F\big)$-T-cyclic rational contractive map with $\theta(x,y) \leq \eta(x,y), \forall x,y \in X$, and $\theta(x,y) > 0$whenever $x \neq y$.(iii) $S(X)$or$T(X)$ is closed in X.
(iv) $\exists x_\circ \in X$such that $\alpha(Tx_\circ,Sx_\circ) \geq \eta(Tx_\circ,Sx_\circ), \beta(Tx_\circ,Sx_\circ) \geq \eta(Tx_\circ,Sx_\circ)$, $\gamma(Tx_\circ,Sx_\circ) \leq \theta(Tx_\circ,Sx_\circ), \delta(Tx_\circ,Sx_\circ) \leq \theta(Tx_\circ,Sx_\circ)$.
(v) (X,d) is admissibly (α,β)-η complete or subadmissibly (γ,δ)-θ complete.
(vi) If $\{y_n\}_{n\geq 0}$ be a sequence converging to u in X, and $\alpha(y_n,y_{n+1}) \geq \eta(y_n,y_{n+1})$, $\beta(y_n,y_{n+1}) \geq \eta(y_n,y_{n+1}), \forall n \in \mathbb{N}$, then $\alpha(y_n,u) \geq \eta(y_n,u)$, $\beta(y_n,u) \geq \eta(y_n,u)$; and if $\gamma(y_n,y_{n+1}) \leq \theta(y_n,y_{n+1})$, $\delta(y_n,y_{n+1}) \leq \theta(y_n,y_{n+1}), \forall n \in \mathbb{N}$, then $\gamma(y_n,u) \leq \theta(y_n,u)$, $\delta(y_n,u) \leq \theta(y_n,u)$.
(vii) $\alpha(Tu,Tv) \geq \eta(Tu,Tv), \beta(Tu,Tv) \geq \eta(Tu,Tv), \gamma(Tu,Tv) \leq \theta(Tu,Tv)$, $\delta(Tu,Tv) \leq \theta(Tu,Tv)$whenever$Su = Tu$, $Sv = Tv$.
Then S and T have a unique point of coincidence in X.
In addition, if S and T are weakly compatible, then S and T have a unique common fixed point in X.
Proof: Replacing f by the identity map on X in Theorem(3.23) we shall get the result.

Note(3.25) Let $\eta(x,y) = 1 = \theta(x,y), \forall x \text{ and } y \in X$. In this case S will be called $\big((\alpha,\beta),(\gamma,\delta),H,F\big)$-$T$-cyclic rational I-contractive map in Definition(3.22); and if f be the identity map on X in Definition(3.22), then S will be called $\big((\alpha,\beta),(\gamma,\delta),H,F\big)$-$T$-cyclic rational contractive map.Now we have the following corollaries immediately come from Theorem(3.23).

Corollary(3.26) Let (X,d,f) be an I-metric space, and $S,T : X \to X$ satisfy(i) $S(X) \subseteq T(X)$.
(ii) $S$is$\big((\alpha,\beta),(\gamma,\delta),H,F\big)$-$T$-cyclic rational I-contractive map.(iii) $S(X)$or $T(X)$ is I-closed in X.
(iv) $\exists x_\circ \in X$ such that $\alpha(Tx_\circ,Sx_\circ) \geq 1, \beta(Tx_\circ,Sx_\circ) \geq 1, \gamma(Tx_\circ,Sx_\circ) \leq 1, \delta(Tx_\circ,Sx_\circ) \leq 1$, where $x_1 \in X$ such that $Sx_\circ = Tx_1$. (since $Sx_\circ \in S(X) \subseteq T(X), Sx_\circ = Tx_1$ for some $x_1 \in X$).
(v) (X,d,f) is admissibly (α,β) I-complete or subadmissibly (γ,δ) I-complete.
(vi) if$\{y_n\}_{n\geq 0}$ be a sequence I-converging to u in X, and $\alpha(y_n,y_{n+1}) \geq 1$, $\beta(y_n,y_{n+1}) \geq 1, \forall n \in \mathbb{N}$,

then $(y_n, u) \geq 1, \beta(y_n, u) \geq 1$; if $\gamma(y_n, y_{n+1}) \leq 1$, $\delta(y_n, y_{n+1}) \leq 1, \forall n \in \mathbb{N}$, then
$\gamma(y_n, u) \leq 1$, $\delta(y_n, u) \leq 1$.
(vii) $(Tp, Tq) \geq 1, \beta(Tp, Tq) \geq 1, \gamma(Tp, Tq) \leq 1, \delta(Tp, Tq) \leq 1$, whenever $(fS)p = (fT)p$, $(fS)q = (fT)q$.
Then S and T have an I-unique point of I-coincidence in X.
In addition, if S and T are weakly I-compatible, then S and T have an I-unique common I-fixed point in X.
Corollary(3.27) Let (X, d) be a metric space, and $S, T : X \to X$ satisfy(i) $S(X) \subseteq T(X)$.
(ii) $S$is$\big((\alpha, \beta), (\gamma, \delta), H, F\big)$-$T$-cyclic rational contractive map.(iii) $S(X)$or$T(X)$ is closed in X.
(iv) $\exists x_\circ \in X$such that $\alpha(Tx_\circ, Sx_\circ) \geq 1, \beta(Tx_\circ, Sx_\circ) \geq 1, \gamma(Tx_\circ, Sx_\circ) \leq 1, \delta(Tx_\circ, Sx_\circ) \leq 1$, where $x_1 \in X$ such that $Sx_\circ = Tx_1$. (since $Sx_\circ \in S(X) \subseteq T(X), Sx_\circ = Tx_1$ for some $x_1 \in X$).
(v) (X, d) is admissibly (α, β) complete or subadmissibly (γ, δ) complete.
(vi) if$\{y_n\}_{n\geq 0}$ be a sequence converging to u in X, and $\alpha(y_n, y_{n+1}) \geq 1$, $\beta(y_n, y_{n+1}) \geq 1, \forall n \in \mathbb{N}$, then$\alpha(y_n, u) \geq 1$, $\beta(y_n, u) \geq 1$; and if $\gamma(y_n, y_{n+1}) \leq 1$, $\delta(y_n, y_{n+1}) \leq 1, \forall n \in \mathbb{N}$, then
$\gamma(y_n, u) \leq 1$, $\delta(y_n, u) \leq 1$.
(vii) $\alpha(Tp, Tq) \geq 1, \beta(Tp, Tq) \geq 1, \gamma(Tp, Tq) \leq 1, \delta(Tp, Tq) \leq 1$, whenever $Sp = Tp, = Tq$.
Then S and T have a unique point of coincidence in X.
In addition, if S and T are weakly compatible, then S and T have a unique common fixed point in X.

Conclusion

Our results obviously generalized many results regarding fixed-point in I-metric spaces and metric spaces. Following our results, further study may go ahead for more new extended, and generalized fixed-point results.

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