ON THE CONNECTED HUB POLYNOMIAL OF GRAPHS

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Abstract:

The connected hub polynomial of a graph G of order n is the polynomial, $H_c(G,x) = \sum_{i=\hbar_c(G)}^{n} h_c(G,i)x^i$, where $h_c(G,i)$ is the number of connected hub sets with cardinality i and $\hbar_c(G)$ is the connected hub number of G. In this paper, we obtain the connected hub polynomials of path graph, cycle graph, complete graph, bi-star graph, barbell graph, triangular cactus graph, helm graph, tadpole graph and star graph.

Keywords:

Connected Hub Number, Connected Hub Sets, Connected Hub Polynomial.

AMS Subject Classification:

05C40, 05C99

Introduction:

In the theory of graphs, a graph polynomial is a graph invariant whose values are considered as a polynomial. Invariants of this type are studied in algebraic graph theory. In the literature, many polynomials are connected with graphs, like domination polynomial, hosoya polynomial, clique polynomial and characteristic polynomial.

The concept of hub set and hub number of a graph G was introduced by M.Walsh in 2006. In 2020, Veettil and Ramakrishnan have introduced the concept of hub polynomial of a graph G. In 2021, B. Basavanagoud and Mahammadsadiq Sayyed also established the concept of hub polynomial of some special graphs.

In this paper we introduce the connected hub polynomial of a graph G of order n as, $H_c(G,x) = \sum_{i=\hbar_c(G)}^{n} h_c(G,i)x^i$, where $h_c(G,i)$ denotes the number of connected hub sets of cardinality i.

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Definition:1.1

Let $G = (V, E)$ be a connected graph. A subset H of V is called a hub set of G if for any two distinct vertices, $u, v \in V - H$, there exists a $u - v$ path P in G such that all the internal vertices of P are in H. The cardinality of a minimum hub set of G is called the hub number of G and is denoted by $\hbar(G)$.

Definition:1.2

A hub set H of G is called a connected hub set if the induced subgraph $< H >$ is connected. The minimum cardinality of connected hub set of G is called the hub number of G and is denoted by $\hbar_c(G)$.

Definition:1.3

A simple path is a path in which all its internal vertices have degree two and the end vertices have degree one and is denoted by P_n.

Definition:1.4

A closed path is called a cycle of G. A cycle on n vertices is denoted by C_n.

Definition:1.5

The complete graph on n vertices, denoted by K_n is the simple graph contains exactly one edge between each pair of distinct vertices.

Definition:1.6

A bi-star graph is a tree obtained from the graph K_2 with two vertices u and v by attaching m pendant edges in u and n pendant edges in V and it is denoted by $B_{m,n}$.

Definition:1.7

The barbell graph is the simple graph obtained by connecting two copies of complete graph by a bridge and it is denoted by B_n.

Definition:1.8

A cactus graph is a connected graph in which no edges lies in more than one cycle. A triangular cactus is a graph whose blocks are triangles.

Definition:1.9

A vertex shared by two or more triangles is called a cut-vertex. If each length of a triangular cactus G has atmost two cut-vertex and each cut-vertex is shared by exactly two triangles, we say that G is a chain triangular cactus and it is denoted by T_n. The number of triangles in G is called the length of the chain. A chain triangular cactus of length n has $2n + 1$ vertices and $3n$ edges.

Definition:1.10

The helm graph H_n is the graph obtained from wheel with central vertex x , by attaching a pendant edge at each vertex of outer circle.

Definition:1.11

The tadpole graph is the graph obtained by joining the cycle graph C_m to a path graph P_n with a bridge and it is denoted by $T_{m,n}$.

Definition:1.12

The star graph of order n, denoted by $K_{1,n}$ is a simple graph with n vertices with the following properties: One distinguished vertex is of degree $n - 1$. The remaining vertices are all degree 1 and are adjacent only to the distinguished vertex.

2.CONNECTED HUB POLYNOMIAL OF GRAPHS.

Definition:2.1

Let $H_c(G)$ be the family of connected hub sets of G with cardinality i.

Then the polynomial, $H_c(G,x) = \sum_{i=\hbar_c(G)}^{n} h_c(G,i)x^i$, where $h_c(G,i)$ is the number of connected hub sets of G of cardinality i.

Theroem:2.2

The connected hub polynomial of the path graph P_n with n vertices is,
$H_C(P_n,x) = 3x^{n-2} + 2x^{n-1} + x^n$.

Proof:

Let $V(P_n) = \{v_1, v_2, \dots, v_n\}$. Clearly $\{v_1, v_2, \dots, v_{n-2}\}$, $\{v_2, v_3 \dots, v_{n-1}\}$ and $\{v_3, v_4 \dots, v_n\}$ are the three connected hub sets of P_n with cardinality $n-2$.

Therefore $h_c(P_n, n-2) = 3$.

There are only two connected hub sets of cardinality $n-1$, namely $\{v_1, v_2, \dots, v_{n-1}\}$ and $\{v_2, v_3 \dots, v_n\}$.

Therefore, $h_c(P_n, n-1) = 2$ and there is only one connected hub set of cardinality n.

Therefore $h_c(P_n, n) = 1$.

Also, for any set H of cardinality less than $n-2$ is not a connected hub set. Therefore, the connected hub number of P_n is $n-2$.

Hence, $H_C(P_n, x) = \sum_{i=n-2}^{n} h_c(P_n, i)x^i$

$$= h_c(P_n, n-2)x^{n-2} + h_c(P_n, n-1)x^{n-1} + h_c(P_n, n)x^n$$

$$H_C(P_n, x) = 3x^{n-2} + 2x^{n-1} + x^n$$

Theroem:2.3

The connected hub polynomial of the cycle graph C_n with n vertices is,

$H_C(C_n, x) = nx^{n-3} + nx^{n-2} + nx^{n-1} + x^n$.

Proof:

Let $V(C_n) = \{v_1, v_2, \dots, v_n\}$. Clearly there are n connected hub sets of cardinality $n-3, n-2$ and $n-1$. Therefore $h_c(C_n, n-3) = n$, $h_c(C_n, n-2) = n$ and $h_c(C_n, n-1) = n$.

There is only one connected hub set of cardinality n. Therefore $h_c(C_n, n) = 1$.
Also, for any set H of cardinality less than $n-3$ is not a connected hub set.
Therefore, the connected hub number of C_n is $n-3$.

Hence, $H_C(C_n, x) = \sum_{i=n-3}^{n} h_c(C_n, i)x^i$

$$= h_c(C_n, n-3)x^{n-3} + h_c(C_n, n-2)x^{n-2} + h_c(C_n, n-1)x^{n-1}$$

$$+ h_c(C_n, n)x^n.$$

$$H_C(C_n, x) = nx^{n-3} + nx^{n-2} + nx^{n-1} + x^n.$$

Theroem:2.4

For a complete graph K_n with n vertices, $H_C(K_n, x) = (1 + x)^n - 1$.

Proof:

Let $V(K_n) = \{v_1, v_2, \dots, v_n\}$. Since any two vertices of K_n are adjacent, there is a path between any two vertices of K_n.

Therefore, $\hbar_c(K_n) = 1$.

For any1 $\leq i \leq n$, it is easy to see that $h_c(K_n, i) = \binom{n}{i}$

Therefore, $H_C(K_n, x) = \sum_{i=\hbar_c(K_n)}^{|V(K_n)|} h_c(K_n, i)x^i$.

$$= \sum_{i=1}^{n} \binom{n}{i} x^i.$$

$$= \binom{n}{1} x + \binom{n}{2} x^2 + \binom{n}{3} x^3 + \cdots + \binom{n}{n} x^n.$$

$$= \left[\sum_{i=0}^{n} \binom{n}{i} x^i\right] - 1.$$

$$H_C(K_n, x) = (1 + x)^n - 1.$$

Theroem:2.5

Let $B_{m,n}$ be a bi-star graph with $m + n + 2$ vertices. Then the connected hub polynomial of $B_{m,n}$ is $H_C(B_{m,n}, x) = x^2(1 + x)^{m+n}$

Proof:

Let $B_{m,n}$ be a bi-star graph with $m + n + 2$ vertices. Label the vertices of $B_{m,n}$ as $v_1, v_2, v_3, \dots, v_m, v_{m+1}, v_{m+2}, \dots, v_{m+n+2}$ as given in Figure 1.

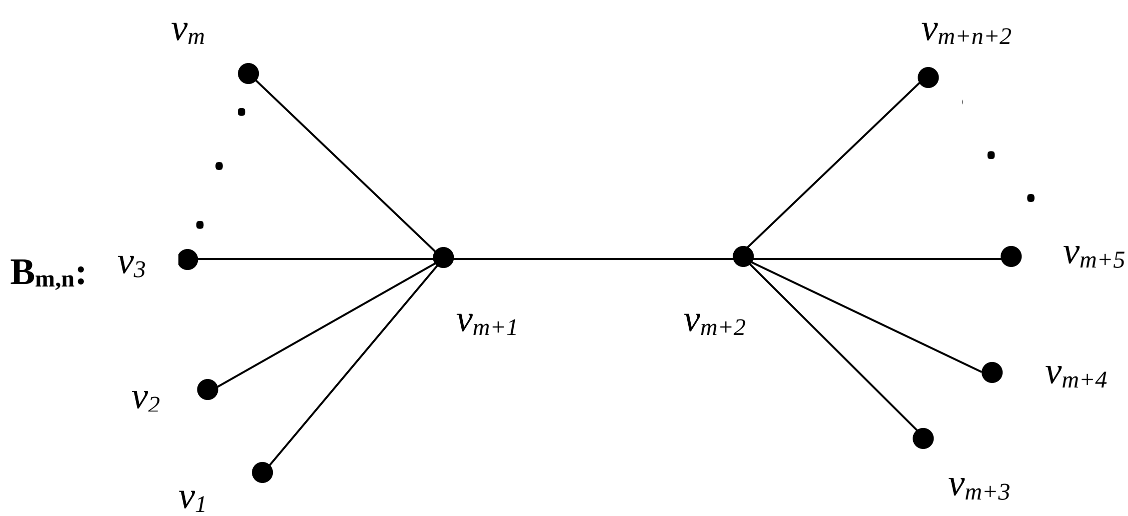


Figure.1

Then the set $\{v_{m+1}, v_{m+2}\}$ is the only unique minimum connected hub set of cardinality 2.

Therefore, $\hbar_c(B_{m,n}) = 2$ and $h_c(B_{m,n}, 2) = 1$.

It is obvious that, the other connected hub sets must contain the two vertices v_{m+1} and v_{m+2}.

Hence there are $\binom{m+n}{1}$ connected hub sets of cardinality three and $\binom{m+n}{2}$ connected hub sets of cardinality four. Proceeding like this, we obtain $\binom{m+n}{m+n}$ connected hub sets of cardinality $m+n+2$.

Therefore, $H_C(B_{m,n}, x) = \sum_{i=\hbar_c(B_{m,n})}^{|V(B_{m,n})|} h_c(B_{m,n}, i)x^i$.

$$= x^2 + \binom{m+n}{1}x^3 + \binom{m+n}{2}x^4 \ldots + \binom{m+n}{m+n}x^{m+n+2}.$$

$$= x^2\left[1 + \binom{m+n}{1}x + \binom{m+n}{2}x^2 + \cdots + \binom{m+n}{m+n}x^{m+n}\right]$$

$H_C(B_{m,n}, x) = x^2(1+x)^{m+n}$

Corollary:2.6

Let $B_{n,n}$ be a bi-star graph with $2n+2$ vertices. Then the connected hub polynomial of $B_{n,n}$ is $H_C(B_{n,n}, x) = x^2(1+x)^{2n}$

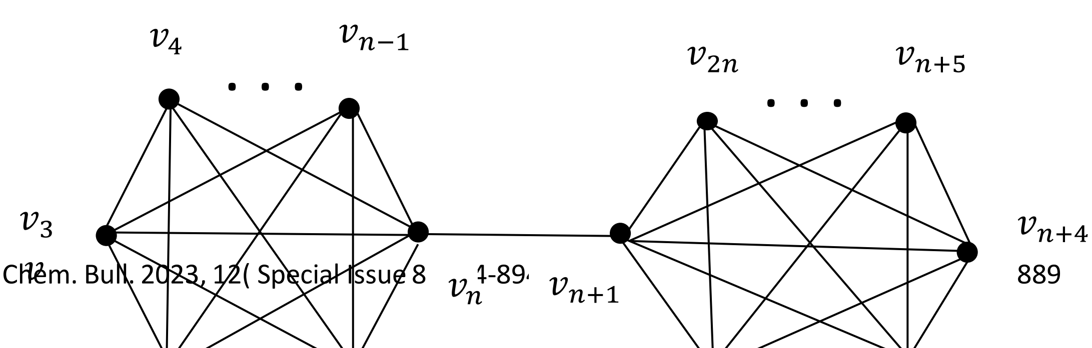
Theroem:2.7

For a barbell graph B_n with $2n$ vertices, the connected hub polynomial is $H_C(B_n, x) = x^2(1+x)^{2n-2} + 2x^n$.

Proof:

Let B_n be a barbell graph with $2n$ vertices. Label the vertices of B_n as $v_1, v_2, v_3, \ldots, v_n, v_{n+1}, \ldots, v_{2n}$ as given in Figure 2.

Bn:



Then the set $\{v_n, v_{n+1}\}$ is the only unique minimum connected hub set of cardinality 2.

Therefore, $\hbar_c(B_n) = 2$ and $h_c(B_{n,2}) = 1$.

It is obvious that, the other connected hub sets must contain the two vertices v_n and v_{n+1} .

Hence, there are $\binom{2n-2}{1}$ connected hub sets of cardinality three and $\binom{2n-2}{2}$ connected hub sets of cardinality four. Proceeding like this, we obtain $\binom{2n-2}{2n-2}$ connected hub sets of cardinality $2n$.

When the cardinality is n, the sets $\{v_1, v_2, \dots, v_n\}$ and $\{v_{n+1}, v_{n+2}, \dots, v_{2n}\}$ are also connected hub sets.

Hence, two more sets are connected hub sets when the cardinality is n.

Therefore,
$$H_C(B_n, x) = x^2 + \binom{2n-2}{1}x^3 + \binom{2n-2}{2}x^4 + \binom{2n-2}{3}x^5 + \cdots + \binom{2n-2}{2n-2}x^{2n} + 2x^n.$$
$$= x^2\left[1 + \binom{2n-2}{1}x + \binom{2n-2}{2}x^2 + \cdots + \binom{2n-2}{2n-2}x^{2n-2}\right] + 2x^n.$$
$$H_C(B_n, x) = x^2(1+x)^{2n-2} + 2x^n.$$

Theroem:2.8

Let T_n be chain triangular cactus graph with $2n+1$ vertices. Then the connected hub polynomial of T_n is $H_C(T_n, x) = x^{n-1}(1+x)^{n+2} + 2x^{n+2}$.

Proof:

Let T_n be a chain triangular cactus with $2n+1$ vertices. Label the vertices of T_n as given in Figure 3.

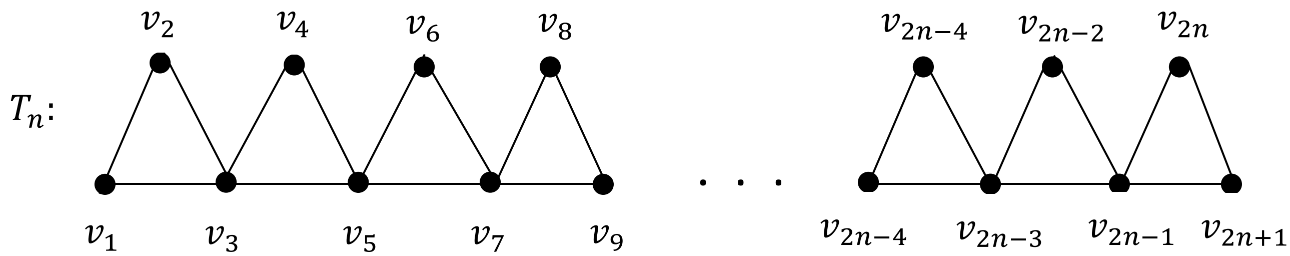


Figure.3

Clearly, every connected hub sets of ntain the vertex set $\{v_3, v_5, v_7, \dots, v_{2n-3}, v_{2n-1}\}$.
Hence, $\{v_3, v_5, v_7, \dots, v_{2n-3}, v_{2n-1}\}$ is the minimum connected hub set of T_n.
Therefore, $\hbar_c(T_n) = n - 1$.

It is obvious that, there is only one connected hub sets of T_n of cardinality $n - 1 + i$.

When the cardinality is $2n - 2$ two more sets are connected hub sets.

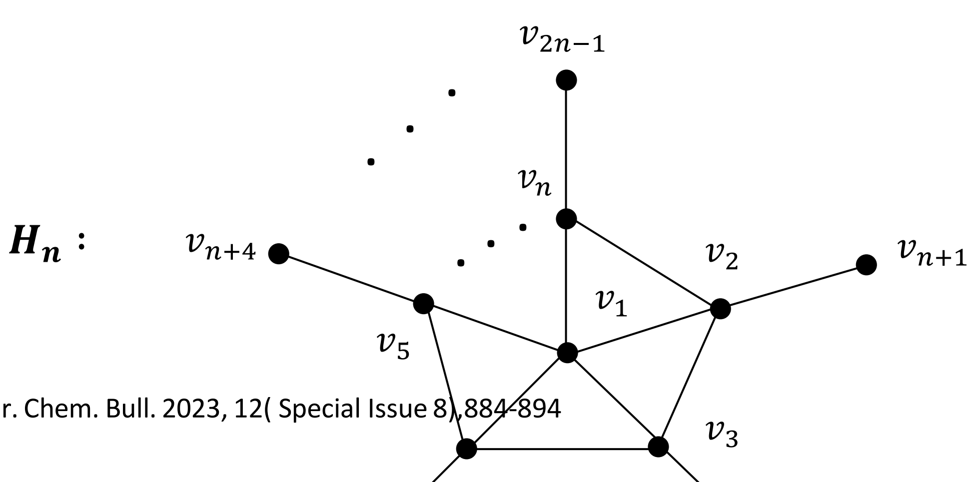
$$
\begin{aligned}
H_C(T_n, x) &= \sum_{i=\hbar_c(T_n)}^{|V(T_n)|} h_c(T_n, i)x^i. \\
&= x^{n-1} + \binom{n+2}{1} x^n + \binom{n+2}{2} x^{n+1} \dots + \binom{n+2}{n+2} x^{2n+1} + 2x^{n+2}. \\
&= x^{n-1}\left[1 + \binom{n+2}{1} x + \binom{n+2}{2} x^2 + \dots + \binom{n+2}{n+2} x^{n+2}\right] + 2x^{n+2}.
\end{aligned}
$$
$$H_C(T_n, x) = x^{n-1}(1+x)^{n+2} + 2x^{n+2}.$$

Theroem:2.9

Let H_n be the helm graph with $2n - 1$ vertices. Then the connected hub polynomial of H_n is $H_C(H_n, x) = x^{n-1}(1+x)^n$.

Proof:

Let H_n be the helm graph with $2n - 1$ vertices. Label the vertices of T_n as given in Figure 4.



Clearly, every connected hub sets of H_n must contain the vertex set $\{v_3, v_5, v_7, \ldots, v_{2n-1}\}$.

Hence, $\{v_3, v_5, v_7, \ldots, v_{2n-1}\}$ is the minimum connected hub set of H_n.

Therefore, $\hbar_c(H_n) = n - 1$.

It is obvious that, there is only one connected hub sets of H_n of cardinality $n-1$ and there are $\binom{n}{i}$ possibilities of connected hub sets of H_n of cardinality $n-1+i$.

$$H_C(H_n, x) = \sum_{i=\hbar_c(H_n)}^{|V(H_n)|} h_c(H_n, i) x^i.$$

$$= x^{n-1} + \binom{n}{1} x^n + \binom{n}{2} x^{n+1} + \binom{n}{3} x^{n+1} + \cdots + \binom{n}{n} x^{2n-1}.$$

$$= x^{n-1}\left[1 + \binom{n}{1} x + \binom{n}{2} x^2 + \cdots + \binom{n}{n} x^n\right]$$

$$H_C(H_n, x) = x^{n-1}(1+x)^n.$$

Theroem:2.10

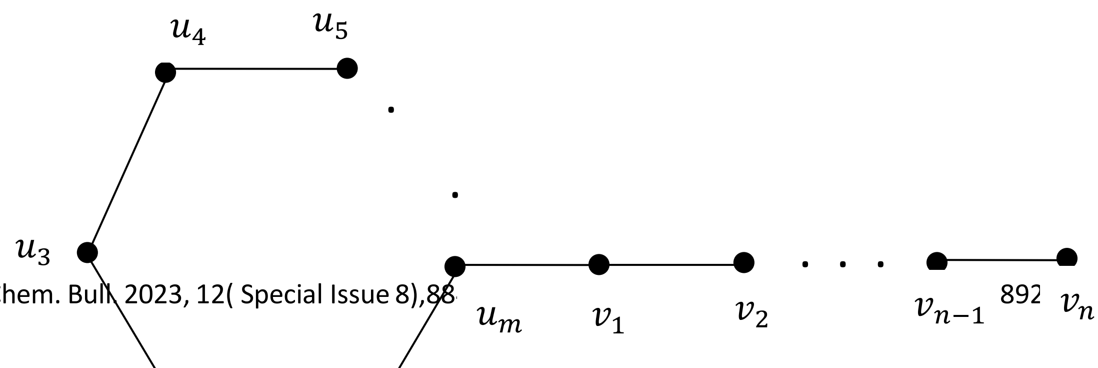
The connected hub polynomial of the tadpole graph $T_{m,n}$ is,

$$H_C(T_{m,n}, x) = (2n-5)\, x^{m+n-3} + (2n-2)\, x^{m+n-2} + n\, x^{m+n-1} + x^{m+n}.$$

Proof:

Let $\{u_1, u_2, u_3, \ldots, u_m, v_1, v_2, \ldots, v_n\}$ be the vertex set of $T_{m,n}$. Label the vertices of $T_{m,n}$ as $u_1, u_2, u_3, \ldots, u_m, v_1, v_2, \ldots, v_n$ as given in Figure 5.

Tm.n:



There is no connected hub sets of cardinality less than $m+n-3$.
Clearly, there are $(2n-5)$ connected hub sets of cardinality $m+n-3$, $(2n-2)$ connected hub sets of cardinality $m+n-2$, n connected hub sets of cardinality $m+n-1$ and only one connected hub set of cardinality $m+n$.
Since, the minimum cardinality is $m+n-3$, $\hslash_c(T_{m,n}) = m+n-3$.
Hence, $H_C(T_{m,n},x) = (2n-5)\,x^{m+n-3} + (2n-2)\,x^{m+n-2} + n\,x^{m+n-1} + x^{m+n}$.

Theroem:2.11

The connected hub polynomial of the star graph $K_{1,n}$ is,
$H_C(K_{1,n},x) = x(1+x)^n$

Proof:

Let v be the central vertex of $K_{1,n}$ and $\{v_1, v_2, \dots, v_n\}$ are the pendent vertices. Then v is the only connected hub set of cardinality 1.
Therefore, $\hslash_c(K_{1,n}) = 1$.
Since every connected hub sets must include the central vertex v there are $\binom{n}{i-1}$ connected hub sets of cardinality i. Hence, $H_C(K_{1,n},x) = \sum_{i=\hslash_c(K_{1,n})}^{|V(K_{1,n})|} h_c(K_{1,n},i)x^i$.

$$= x + \binom{n}{1}x^2 + \binom{n}{2}x^3 + \cdots + \binom{n}{n}x^{n+1}.$$

$$= x\left[1 + \binom{n}{1}x + \binom{n}{2}x^2 + \cdots + \binom{n}{n}x^n\right]$$

$$H_C(K_{1,n},x) = x(1+x)^n$$

CONCLUSION:

In this paper, we obtained connected hub polynomials of path, cycle, complete, bi-star, barbell, triangular cactus, helm, tadpole and star graphs. One can obtain connected hub polynomial of some graph operations.

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